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# Infinitely many solutions to Kirchhoff double phase problems with variable exponents

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In this work we deal with elliptic equations driven by the variable exponent double phase operator with a Kirchhoff term and a right-hand side that is just locally defined in terms of very mild assumptions. Based on an abstract critical point result of Kajikiya (2005) and recent a priori bounds for generalized double phase problems by the authors (Ho and Winkert, 2022), we prove the existence of a sequence of nontrivial solutions whose $L^\infty$-norms converge to zero.



## 1. Introduction

In this paper we study multiplicity results for the following Kirchhoff-type problem

$$-M\left(\int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x\right)\operatorname{div} A(x,\nabla u)=f(x,u)\quad\text{in } \Omega,\qquad u=0\quad\text{on } \partial\Omega, \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$, $\mathcal{A}\colon\Omega\times\mathbb{R}^N\to\mathbb{R}$ and $A\colon\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ are given by

$$\mathcal{A}(x,\xi):=\frac{1}{p(x)}|\xi|^{p(x)}+\frac{\mu(x)}{q(x)}|\xi|^{q(x)},\qquad A(x,\xi):=\nabla_\xi\mathcal{A}(x,\xi)=|\xi|^{p(x)-2}\xi+\mu(x)|\xi|^{q(x)-2}\xi.$$

In the following, for $h\in C(\overline{\Omega})$ we denote $h^-:=\inf_{x\in\overline{\Omega}}h(x)$ and $h^+:=\sup_{x\in\overline{\Omega}}h(x)$.

We suppose the subsequent hypotheses:

($\mathrm{H}_1$) $p,q\in C^{0,1}(\overline{\Omega})$ such that $1<p(x)<q(x)<N$ for all $x\in\overline{\Omega}$, $\left(\frac{q}{p}\right)^+<1+\frac{1}{N}$ and $0\le\mu(\cdot)\in C^{0,1}(\overline{\Omega})$.

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($\mathrm{H}_2$) $M\colon[0,\infty)\to\mathbb{R}$ is a function and $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that the following conditions are satisfied:

(i) there exist positive constants $t_0, m_0$ such that $M\in C[0,t_0]$ and $m_0\le M(t)\le M(t_0)$ for all $t\in[0,t_0]$;

(ii) there exists $\varepsilon_0>0$ such that $f\colon\Omega\times[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}$ is odd with respect to the second variable and $\sup_{|t|\le\varepsilon_0}|f(\cdot,t)|\in L^\infty(\Omega)$;

(iii) there exists a nonempty open ball $B\subset\Omega$ such that

$$\lim_{t\to 0}\frac{F(x,t)}{|t|^{p_B^-}}=\infty \quad \text{uniformly for a. a. } x\in B,$$

where $F(x,t):=\int_0^t f(x,\tau)\,\mathrm{d}\tau$ and $p_B^-:=\inf_{x\in B}p(x)$.

We shall look for solutions to problem (1.1) in the Musielak-Orlicz Sobolev space $\left(W_0^{1,\mathcal{H}}(\Omega),\|\cdot\|\right)$, where $\mathcal{H}(x,t):=t^{p(x)}+\mu(x)t^{q(x)}$ for all $(x,t)\in\overline{\Omega}\times[0,\infty)$ (see Section 2 for the definitions). We call a function $u\in W_0^{1,\mathcal{H}}(\Omega)$ a solution of problem (1.1) if $f(\cdot,u)\in L^1_{\mathrm{loc}}(\Omega)$ and if

$$M\left(\int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x\right)\int_\Omega A(x,\nabla u)\cdot\nabla v\,\mathrm{d}x=\int_\Omega f(x,u)v\,\mathrm{d}x$$

is satisfied for all $v\in C_c^\infty(\Omega)$.

Our main result reads as follows.

**Theorem 1.1.** *Let hypotheses* ($\mathrm{H}_1$) *and* ($\mathrm{H}_2$) *be satisfied. Then, problem* (1.1) *admits a sequence of solutions* $\{u_n\}_{n\in\mathbb{N}}$ *with* $\|u_n\|+\|u_n\|_\infty\to 0$ *as* $n\to\infty$*, where* $\|\cdot\|_\infty$ *is the norm in* $L^\infty(\Omega)$.

The proof of Theorem 1.1 is based on an abstract critical point result of Kajikiya [1] (see also Theorem 2.2) and recent a priori bounds for generalized double phase problems by the authors [2] in which new embedding results of the form $W^{1,\mathcal{H}}(\Omega)\hookrightarrow L^\Psi(\Omega)$, with

$$\Psi(x,t):=t^{r(x)}+\mu(x)^{\frac{s(x)}{q(x)}}t^{s(x)} \quad \text{for } (x,t)\in\overline{\Omega}\times[0,\infty),$$

where $r,s\in C(\overline{\Omega})$ satisfy $1<r(x)\le\frac{Np(x)}{N-p(x)}=:p^*(x)$ and $1<s(x)\le\frac{Nq(x)}{N-q(x)}=:q^*(x)$ for all $x\in\overline{\Omega}$ are presented.

The novelty of our work is the fact that we combine the variable exponent double phase operator with a Kirchhoff term and a reaction term that are both locally defined. As far as we know, there is no other work dealing with a Kirchhoff term along with the variable exponent double phase operator. In case the exponents $p,q$ are constants, we refer to the work of Fiscella–Pinamonti [3] who considered the problem

$$-m\left[\int_\Omega\left(\frac{|\nabla u|^p}{p}+a(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\mathcal{L}^a_{p,q}(u)=f(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega, \tag{1.2}$$

where $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function that satisfies subcritical growth and the Ambrosetti–Rabinowitz condition and

$$\mathcal{L}^a_{p,q}(u):=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+a(x)|\nabla u|^{q-2}\nabla u\right),\quad u\in W_0^{1,\mathcal{H}}(\Omega). \tag{1.3}$$

By applying the mountain-pass theorem, the existence of a nontrivial weak solution of (1.2) is shown. Recently, Arora–Fiscella–Mukherjee–Winkert [4] studied singular Kirchhoff double phase problems given by

$$-m\left[\int_\Omega\left(\frac{|\nabla u|^p}{p}+a(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\mathcal{L}^a_{p,q}(u)=\lambda u^{-\gamma}+u^{r-1}\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega,$$

with $\mathcal{L}^a_{p,q}$ as in (1.3), where a suitable Nehari manifold decomposition provides the existence of two different solutions. Another interesting work in the context of Kirchhoff constant exponent double phase problems has been published in [5] with nonlinear boundary condition based on variational tools. All these works use different methods than in our paper.

It should be noted that the occurrence of a nonlocal Kirchhoff term was first introduced by Kirchhoff [6]. Such problems have a strong background in several applications in physics. Existence results on degenerate and nondegenerate Kirchhoff problems for different type of problems can be found, for example, in the works [7–12] and the references therein.

If $m(t) \equiv 1$ for all $t \geq 0$, problem (1.1) reduces to a double phase problem with variable exponents. In this case, only few and very recent results exist. We refer to the papers [13–18], see also the references therein. If $p$ and $q$ are constants, we point out that the double phase operator in (1.1) is associated to the functional

$$u \mapsto \int_\Omega \left( \frac{1}{p} |\nabla u|^p + \frac{\mu(x)}{q} |\nabla u|^q \right) \mathrm{d}x, \tag{1.4}$$

which occurred for the first time in the work of Zhikov [19]. Such functionals are used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. In the past decade, functionals of the form (1.4) have been studied by several authors concerning regularity properties of local minimizers, we refer to the papers [20–23], see also [24] for variable exponents and the recent paper [25] about nonautonomous integrals.

## 2. Preliminaries and notations

In this section we recall the main properties about Musielak-Orlicz Sobolev spaces and the double phase operator with variable exponents along with an abstract critical point result.

To this end, let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and let $M(\Omega)$ be the space of all measurable functions $u \colon \Omega \to \mathbb{R}$. We denote by $L^r(\Omega)$ the usual Lebesgue space endowed with the norm $\|\cdot\|_r$ for any $1 \leq r \leq \infty$. Suppose (H$_1$) and let $\mathcal{H} \colon \overline{\Omega} \times [0,\infty) \to [0,\infty)$ be the nonlinear function defined by

$$\mathcal{H}(x,t) := t^{p(x)} + \mu(x) t^{q(x)} \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty).$$

The corresponding modular to $\mathcal{H}$ is given by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left( |u|^{p(x)} + \mu(x)|u|^{q(x)} \right) \mathrm{d}x$$

and the associated Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is then defined by

$$L^{\mathcal{H}}(\Omega) = \{ u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty \}$$

endowed with the Luxemburg norm $\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1 \right\}$. When $\mu(\cdot) \equiv 0$, we write $L^{p(\cdot)}(\Omega)$ in place of $L^{\mathcal{H}}(\Omega)$. Similarly, the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

equipped with the norm $\|u\|_{1,\mathcal{H}} = \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}$, where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. Moreover, $W^{1,\mathcal{H}}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. We know that $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are reflexive Banach spaces and we can equip $W^{1,\mathcal{H}}_0(\Omega)$ with the equivalent norm $\|\cdot\| := \|\nabla \cdot\|_{\mathcal{H}}$, see [14].

Moreover, we have

$$\|u\|^{p^-} - 1 \leq \rho_{\mathcal{H}}(|\nabla u|) \leq \|u\|^{q^+} + 1 \quad \text{for all } u \in W^{1,\mathcal{H}}_0(\Omega), \tag{2.1}$$

$$\|u\|^{q^+} \leq \rho_{\mathcal{H}}(|\nabla u|) \leq \|u\|^{p^-} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega) \text{ with } \|u\| < 1, \tag{2.2}$$

and

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \tag{2.3}$$

is compact for $r \in C(\overline{\Omega})$ with $1 \leq r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, see [14, Propositions 2.13 and 2.16].

Let $B\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear map defined by

$$\langle B(u), v\rangle := \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x \tag{2.4}$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle \cdot\,, \cdot\rangle$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The operator $B\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ has the following properties, see [14, Theorem 3.3].

**Proposition 2.1.** *Let hypotheses* $(\mathrm{H}_1)$ *be satisfied. Then, the operator $B$ defined in* (2.4) *is bounded, continuous, strictly monotone and of type* $(\mathrm{S}_+)$*, that is, $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $\limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \leq 0$, imply $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.*

Let $X$ be a Banach space, let $X^*$ be its dual space and let $\varphi \in C^1(X, \mathbb{R})$. We say that $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ is a Palais–Smale sequence ((PS)-sequence for short) for $\varphi$ if $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\varphi'(u_n) \to 0 \quad \text{in } X^* \quad \text{as } n \to \infty$. We say that $\varphi$ satisfies the Palais–Smale condition ((PS)-condition for short) if any (PS)-sequence $\{u_n\}_{n\in\mathbb{N}}$ of $\varphi$ admits a convergent subsequence in $X$. The proof of Theorem 1.1 is based on the following abstract critical point result due to Kajikiya [1, Theorem 1].

**Theorem 2.2.** *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and $J \in C^1(X, \mathbb{R})$ such that the following two assumptions hold:*

(J1) *$J$ is even, bounded from below, $J(0) = 0$ and it satisfies the* (PS)*-condition.*

(J2) *For any $k \in \mathbb{N}$, there exist a $k$-dimensional subspace $X_k$ of $X$ and a number $r_k > 0$ such that $\sup_{X_k \cap S_{r_k}} J(u) < 0$, where $S_{r_k} = \{u \in X : \|u\| = r_k\}$.*

*Then, the functional $J$ admits a sequence of critical points $\{v_k\}_{k\in\mathbb{N}}$ satisfying $\|v_k\| \to 0$ as $k \to \infty$.*

## 3. Proof of the main result

In this section we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since the conditions on the Kirchhoff and reaction terms are given locally, the corresponding energy functional associated with problem (1.1) may not be well defined. In order to deal with this difficulty and get the symmetry of the associated energy functional, we first modify the functions $M$ and $f$ as follows: We define $M_0, \widehat{M}_0\colon [0, \infty) \to \mathbb{R}$ given by

$$M_0(t) := \begin{cases} M(t) & \text{if } 0 \leq t \leq t_0, \\ M(t_0) & \text{if } t > t_0, \end{cases} \quad \text{and} \quad \widehat{M}_0(t) := \int_0^t M_0(s)\,\mathrm{d}s.$$

It is clear that $M_0 \in C([0, \infty), \mathbb{R})$ and

$$m_0 \leq M_0(t) \leq M(t_0) \quad \text{for all } t \in [0, \infty), \tag{3.1}$$

$$m_0 t \leq \widehat{M}_0(t) \leq M(t_0)t \quad \text{for all } t \in [0, \infty). \tag{3.2}$$

Next, let $\eta \in C_c^\infty(\mathbb{R})$ be a function such that $0 \le \eta(t) = \eta(-t) \le 1$ for $t \in \mathbb{R}$ and

$$\eta(t) = 1 \quad \text{for } |t| \le \frac{\varepsilon_0}{2}, \qquad \eta(t) = 0 \quad \text{for } |t| \ge \varepsilon_0,$$

where $\varepsilon_0$ is given in (H$_2$)(ii). For $x \in \Omega$ we define

$$h(x,t) := \begin{cases} \eta(t) f(x,t) & \text{if } |t| \le \varepsilon_0, \\ 0 & \text{if } |t| \ge \varepsilon_0, \end{cases} \quad \text{and} \quad H(x,t) := \int_0^t h(x,s)\,\mathrm{d}s.$$

Obviously, we have

$$\sup_{t\in\mathbb{R}} |h(x,t)| \le \sup_{|t|\le\varepsilon_0} |f(x,t)| =: f_0(x) \quad \text{for a. a. } x \in \Omega. \tag{3.3}$$

Furthermore, $H$ is even with respect to the second variable and

$$\sup_{t\in\mathbb{R}} |H(x,t)| \le \varepsilon_0 f_0(x) \quad \text{for a. a. } x \in \Omega. \tag{3.4}$$

Note that $f_0 \in L^\infty(\Omega)$ by hypothesis (H$_2$)(ii).

Now, we consider the following modified problem

$$-M_0\left(\int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x\right) \operatorname{div} A(x,\nabla u) = h(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

We point out that if $\{v_k\}_{k\in\mathbb{N}}$ is a sequence of solutions to problem (3.5) satisfying $\|v_k\| + \|v_k\|_\infty \to 0$ as $n \to \infty$, then $\{v_k\}_{k\ge k_0}$ is a sequence of solutions to problem (1.1) for some $k_0 \in \mathbb{N}$. In order to derive the desired conclusion, we will apply Theorem 2.2 for $(X, \|\cdot\|) := (W_0^{1,\mathcal{H}}(\Omega), \|\nabla \cdot\|_{\mathcal{H}})$ and

$$J(u) := \widehat{M}_0\left(\int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x\right) - \int_\Omega H(x,u)\,\mathrm{d}x, \quad u \in X.$$

First, we see that $J\colon X \to \mathbb{R}$ is of class $C^1$ and its Fréchet derivative $J'\colon X \to X^*$ is given by

$$\langle J'(u), v\rangle = M_0\left(\int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x\right)\int_\Omega A(x,\nabla u)\cdot\nabla v\,\mathrm{d}x - \int_\Omega h(x,u)v\,\mathrm{d}x$$

for all $u, v \in X$. Clearly, any critical point of $J$ is a solution of problem (3.5). We will verify that $J$ satisfies conditions (J1) and (J2) of Theorem 2.2.

**Step 1:** $J$ fulfills (J1)

Clearly, $J$ is even and $J(0) = 0$. By (3.2), (2.1) and (3.4), we have

$$J(u) \ge m_0 \int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x - \varepsilon_0 \int_\Omega f_0(x)\,\mathrm{d}x \ge \frac{1}{q^+}\left(\|u\|^{p^-} - 1\right) - \varepsilon_0\|f_0\|_1 \quad \text{for all } u \in X.$$

This implies that $J$ is coercive and bounded from below on $X$ since $p^- > 1$. For verification of the (PS)-condition for $J$, let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS)-sequence for $J$, that is

$$J'(u_n) \to 0 \quad \text{in } X^* \tag{3.6}$$

and

$$\sup_{n\in\mathbb{N}} |J(u_n)| < \infty. \tag{3.7}$$

Then, the coercivity of $J$ and (3.7) guarantee the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $X$. Thus, up to a subsequence if necessary, we have

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad u_n \to u \quad \text{in } L^1(\Omega), \tag{3.8}$$

by (2.3). On the other hand, we have

$$M_0\left(\int_\Omega \mathcal{A}(x,\nabla u_n)\,\mathrm{d}x\right)\int_\Omega A(x,\nabla u_n)\cdot(\nabla u_n-\nabla u)\,\mathrm{d}x=\left\langle J'(u_n),u_n-u\right\rangle+\int_\Omega h(x,u_n)(u_n-u)\,\mathrm{d}x.$$

Combining this with (3.1) and (3.3) yields

$$m_0\left|\int_\Omega A(x,\nabla u_n)\cdot(\nabla u_n-\nabla u)\,\mathrm{d}x\right|\le\|J'(u_n)\|_{X^*}\|u_n-u\|+\varepsilon_0\|f_0\|_\infty\|u_n-u\|_1.$$

Invoking (3.6), (3.8) and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $X$, from the last inequality it follows that

$$\int_\Omega A(x,\nabla u_n)\cdot(\nabla u_n-\nabla u)\mathrm{d}x\to 0\quad\text{as } n\to\infty.$$

Hence, $u_n\to u$ in $X$ in view of Proposition 2.1. Thus, $J$ satisfies the (PS)-condition and so (J1) is fulfilled.

**Step 2:** $J$ fulfills (J2)

Let $k\in\mathbb{N}$ be given and set $X_k:=\operatorname{span}\{\varphi_1,\varphi_2,\dots,\varphi_k\}$, where $\varphi_n$ is an eigenfunction corresponding to the $n$th eigenvalue of the eigenvalue problem $-\Delta u=\lambda u$ in $B$, $u=0$ on $\partial B$, and it is extended on $\Omega$ by putting $\varphi_n(x)=0$ for $x\in\Omega\setminus B$. Since $X_k$ is finitely dimensional, all norms on $X_k$ are equivalent. Thus, we find positive constants $\alpha_k,\beta_k$ such that

$$\beta_k\|u\|_\infty\le\|u\|\le\alpha_k\|u\|_{p_B^-}\quad\text{for all } u\in X_k. \tag{3.9}$$

By condition (H$_2$)(iii) we can choose

$$M_k>\frac{M(t_0)\alpha_k^{p_B^-}}{p^-}\quad\text{and}\quad\delta_k\in(0,\varepsilon_0/2) \tag{3.10}$$

such that

$$H(x,t)=F(x,t)\ge M_k|t|^{p_B^-}, \tag{3.11}$$

for a. a. $x\in B$ and for all $|t|<\delta_k$.

Let $r_k\in\left(0,\min\{1,\beta_k^{-1}\delta_k\}\right)$. Then, from (3.9) we have

$$\|u\|<1\quad\text{and}\quad\|u\|_\infty\le\beta_k^{-1}r_k<\delta_k<\frac{\varepsilon_0}{2}\quad\text{for all } u\in X\cap S_{r_k}, \tag{3.12}$$

where $S_{r_k}=\{u\in X:\ \|u\|=r_k\}$. Utilizing (3.2), (3.11) and then (3.12) with noticing $\operatorname{supp}(u)\subset B$ we obtain

$$\begin{aligned}J(u)&=\widehat{M_0}\left(\int_\Omega\mathcal{A}(x,\nabla u)\,\mathrm{d}x\right)-\int_\Omega H(x,u)\,\mathrm{d}x\le M(t_0)\int_\Omega\mathcal{A}(x,\nabla u)\,\mathrm{d}x-\int_\Omega F(x,u)\,\mathrm{d}x\\&\le\frac{M(t_0)}{p^-}\int_B\left[|\nabla u|^{p(x)}+\mu(x)|\nabla u|^{q(x)}\right]\,\mathrm{d}x-M_k\int_B|u|^{p_B^-}\,\mathrm{d}x\end{aligned}$$

for all $u\in X_k\cap S_{r_k}$. Invoking (2.2) and (3.9) we infer from the last inequality that

$$J(u)\le\frac{M(t_0)}{p^-}\|u\|^{p_B^-}-M_k\|u\|_{p_B^-}^{p_B^-}\le\frac{M(t_0)}{p^-}\|u\|^{p_B^-}-M_k(\alpha_k^{-1}\|u\|)^{p_B^-}=\left(\frac{M(t_0)}{p^-}-M_k\alpha_k^{-p_B^-}\right)r_k^{p_B^-}.$$

Thus, we obtain $\sup_{X_k\cap S_{r_k}}J(u)<0$ due to (3.10). Hence, $J$ satisfies (J2).

Applying Theorem 2.2 we find a sequence of critical points $\{v_k\}_{k\in\mathbb{N}}$ of $J$ satisfying $J(v_k)<0$ for all $k\in\mathbb{N}$ and $\|v_k\|\to 0$ as $k\to\infty$. Therefore, $v_k$ are nontrivial solutions of problem (3.5), which can be rewritten as

$$-\operatorname{div}A(x,\nabla u)=g(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega,$$

where
$$g(x,u) := \frac{h(x,u)}{M_0\left(\displaystyle\int_\Omega \mathcal{A}(x,\nabla u)\mathrm{d}x\right)} \quad \text{with} \quad |g(x,t)| \le \frac{\|f_0\|_\infty}{m_0}$$
for a. a. $x \in \Omega$ and for all $t \in \mathbb{R}$. According to Theorem 4.2 and Proposition 3.7 of the authors [2], we also have that $\|v_k\|_\infty \to 0$ as $k \to \infty$. Hence $\{v_k\}_{k \ge k_0}$ for some $k_0 \in \mathbb{N}$ are solutions to our original problem (1.1) and satisfy $\|v_k\| + \|v_k\|_\infty \to 0$ as $k \to \infty$. This finishes the proof. $\square$

## Data availability

No data was used for the research described in the article.


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## References


[1] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, J. Funct. Anal. 225 (2) (2005) 352–370.
[2] K. Ho, P. Winkert, New embedding results for double phase problems with variable exponents and a priori bounds for corresponding generalized double phase problems, preprint, https://arxiv.org/abs/2208.00504.
[3] A. Fiscella, A. Pinamonti, Existence and multiplicity results for Kirchhoff type problems on a double phase setting, Mediterr. J. Math. 20 (1) (2023) Paper No. 33.
[4] R. Arora, A. Fiscella, T. Mukherjee, P. Winkert, On double phase Kirchhoff problems with singular nonlinearity, Adv. Nonlinear Anal. 12 (1) (2023) 24, Paper (20220312).
[5] A. Fiscella, G. Marino, A. Pinamonti, S. Verzellesi, Multiple solutions for nonlinear boundary value problems of Kirchhoff type on a double phase setting, Rev. Mat. Complut. (2023) http://dx.doi.org/10.1007/s13163-022-00453-y.
[6] G.R. Kirchhoff, Vorlesungen Über Mathematische Physik, Mechanik, Teubner, Leipzig, 1876.
[7] G. Autuori, P. Pucci, M. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, Arch. Ration. Mech. Anal. 196 (2) (2010) 489–516.
[8] A. Fiscella, A fractional Kirchhoff problem involving a singular term and a critical nonlinearity, Adv. Nonlinear Anal. 8 (1) (2019) 645–660.
[9] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, Nonlinear Anal. 94 (2014) 156–170.
[10] X. Mingqi, V.D. Rădulescu, B. Zhang, Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity, Calc. Var. Partial Differential Equations 58 (2) (2019) 27, Paper (57).
[11] P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional $p$-Laplacian in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations 54 (3) (2015) 2785–2806.
[12] M. Xiang, B. Zhang, V.D. Rădulescu, Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional $p$-Laplacian, Nonlinearity 29 (10) (2016) 3186–3205.
[13] A. Bahrouni, V.D. Rădulescu, P. Winkert, Double phase problems with variable growth and convection for the Baouendi-Grushin operator, Z. Angew. Math. Phys. 71 (6) (2020) 183, 14.
[14] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, A new class of double phase variable exponent problems: existence and uniqueness, J. Differential Equations 323 (2022) 182–228.
[15] I.H. Kim, Y.-H. Kim, M.W. Oh, S. Zeng, Existence and multiplicity of solutions to concave-convex-type double-phase problems with variable exponent, Nonlinear Anal. RWA 67 (2022) 25, Paper (103627).
[16] S. Leonardi, N.S. Papageorgiou, Anisotropic Dirichlet double phase problems with competing nonlinearities, Rev. Mat. Complut. 36 (2) (2023) 469–490.
[17] F. Vetro, P. Winkert, Constant sign solutions for double phase problems with variable exponents, Appl. Math. Lett. 135 (2023) 7, Paper (108404).
[18] S. Zeng, V.D. Rădulescu, P. Winkert, Double phase obstacle problems with variable exponent, Adv. Differential Equations 27 (9–10) (2022) 611–645.
[19] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Izv. Akad. Nauk SSSR Ser. Mat. 50 (4) (1986) 675–710.
[20] P. Baroni, M. Colombo, G. Mingione, Harnack inequalities for double phase functionals, Nonlinear Anal. 121 (2015) 206–222.

[21] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, Calc. Var. Partial Differential Equations 57 (2) (2018) 48, Art. 62.
[22] M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, Arch. Ration. Mech. Anal. 218 (1) (2015) 219–273.
[23] M. Colombo, G. Mingione, Regularity for double phase variational problems, Arch. Ration. Mech. Anal. 215 (2) (2015) 443–496.
[24] M.A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, Adv. Nonlinear Anal. 9 (1) (2020) 710–728.
[25] C. De Filippis, G. Mingione, Lipschitz bounds and nonautonomous integrals, Arch. Ration. Mech. Anal. 242 (2021) 973–1057.