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# Multiple Positive Solutions for Quasilinear Elliptic Problems in Expanding Domains

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**Abstract**
In this paper we prove the existence of multiple positive solutions for a quasilinear elliptic problem with unbalanced growth in expanding domains by using variational methods and the Lusternik–Schnirelmann category theory. Based on the properties of the category, we introduce suitable maps between the expanding domains and the critical levels of the energy functional related to the problem, which allow us to estimate the number of positive solutions by the shape of the domain.

**Keywords** Expanding domains · Lusternik–Schnirelmann category · Musielak–Orlicz spaces · Positive solutions · Problems with unbalanced growth

**Mathematics Subject Classification** 35J15 · 35J60 · 35J62

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# 1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $\Omega_\lambda := \lambda\Omega$ be an expanding domain, where $\lambda$ is a positive number. In this paper we consider quasilinear elliptic problems with unbalanced growth of the form

$$\begin{aligned} A(u) + |u|^{p-2}u + a(x)|u|^{q-2}u &= f(u) \qquad && \text{in } \Omega_\lambda, \\ u &= 0 && \text{on } \partial\Omega_\lambda, \end{aligned} \tag{1.1}$$

where $A(u)$ is the negative double phase operator given by

$$A(u) = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)$$

and we suppose the following hypotheses on the data:

(H1) $1 < p < N$, $p < q < p^* = \frac{Np}{N-p}$ and $0 \leq a(\cdot) \in L^\infty(\Omega_\lambda)$ is radially symmetric, that is, $a(x) = a(|x|)$ for a.a. $x \in \Omega_\lambda$.

(H2) The function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous with primitive $F(t) = \int_0^t f(s)\,\mathrm{d}s$ and $f(t) = 0$ for all $t \leq 0$ such that the following hold:

(i) there exist $r \in (q, p^*)$ and $C > 0$ such that

$$|f(t)| \leq C\left(1 + |t|^{r-1}\right)$$

for all $t \in \mathbb{R}$;

(ii)

$$\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = 0;$$

(iii)

$$\lim_{t\to\infty} \frac{F(t)}{t^q} = \infty;$$

(iv) $\frac{f(t)}{t^{q-1}}$ is strictly increasing on $(0, \infty)$.

We denote the energy functional associated to Problem 1.1 by

$$J_\lambda(u) = \int_{\Omega_\lambda} \left(\frac{1}{p}\left(|\nabla u|^p + |u|^p\right) + \frac{a(x)}{q}\left(|\nabla u|^q + |u|^q\right) - F(u)\right) \mathrm{d}x. \tag{1.2}$$

For simplification, we write $J_\lambda(u) = I_\lambda(u) - K_\lambda(u)$, where

$$I_\lambda(u) = \int_{\Omega_\lambda} \left(\frac{1}{p}\left(|\nabla u|^p + |u|^p\right) + \frac{a(x)}{q}\left(|\nabla u|^q + |u|^q\right)\right) \mathrm{d}x \tag{1.3}$$

and

$$K_\lambda(u) = \int_{\Omega_\lambda} F(u)\,\mathrm{d}x. \tag{1.4}$$

The solutions of Problem 1.1 are understood in the weak sense. Under the assumptions on $f$, it is easy to see that $J_\lambda$ is well-defined and of class $C^1$ and the solutions of Problem 1.1 are the critical points of $J_\lambda$.

Problem 1.1 arises in the study of some non-Newtonian fluids whereby $|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}$ is the viscosity coefficient of the fluid and $f(u) - |u|^{p-2}u - a|u|^{q-2}u$ is the divergence of shear stress. The solution of Problem 1.1 denotes the speed of the fluid, see Liu–Dai [30] for more details. Problem 1.1 is called double phase problem and belongs to the class of problems with unbalanced growth whose ellipticity rate changes according to the point in the domain. Such type of problems can be used to characterize the hardening properties of strongly anisotropic materials, see for instance Zhikov [39]. Research on double phase problems also helps us to understand and advance research on the prescribed mean curvature equation and the Born–Infeld equation, we refer to papers of Azzollini–d'Avenia–Pomponio [5] and Pomponio–Watanabe [36] for more related results.

Differential models involving the double phase operator defined by

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)$$

have their origin in the study of the related energy functional given by

$$\mathcal{F}(u) = \int \left(|\nabla u|^p + a(x)|\nabla u|^q\right)\,\mathrm{d}x. \tag{1.5}$$

The functional $\mathcal{F}$ in 1.5 belongs to the class of the integral functionals with nonstandard growth condition according to Marcellini's terminology [33, 34]. The regularity theory for minimizers of $\mathcal{F}$ have been considered and achieved sharp results for $q > p$ and $a(\cdot) \geq 0$ by Baroni–Colombo–Mingione [6–8] and Colombo–Mingione [16, 17] in a series of remarkable papers, which gave new impulse on the studies of the double phase problems.

Based on several different methods many authors achieved existence and multiplicity results for double phase problems of the form

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) &= f(x,u) \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where $\Omega \subseteq \mathbb{R}^N (N \geq 2)$ is a domain (bounded or unbounded) and the right-hand side of 1.6 consists of a Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ that satisfies appropriate conditions near the origin and/or at infinity. We refer, for example to the works of Biagi–Esposito–Vecchi [11], Colasuonno–Squassina [15], Farkas–Winkert [19], Fiscella [23], Gasiński–Papageorgiou [24], Gasiński–Winkert [25], Liu–Dai [28, 30, 31], Liu–Papageorgiou [32], Perera–Squassina [35], Zeng–Bai–Gasiński–Winkert [38] and the references therein.

The main objective of this paper is to investigate Problem 1.1 concerning multiple positive solutions on the expanding domain $\Omega_\lambda$ by applying the method of the Nehari manifold and the Lusternik–Schnirelmann category theory. The methods used here were first introduced by Benci–Cerami [9, 10], where they discussed the semilinear case given in the form

$$-\Delta u + \lambda u = u^{p-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $2 < p < 2^* = \frac{2N}{N-2}$. These works were the starting point to apply these techniques to other kind of problems in order to obtain multiple solutions. We mention the papers of Cingolani–Lazzo [13, 14] for nonlinear Schrödinger equations, Alves–Figueiredo–Furtado [4], Cingolani–Clapp [12] for nonlinear magnetic Schrödinger equations, Alves [2], Alves–Ding [3] for $p$-Laplace equations, Figueiredo–Siciliano [20], Liu–Dai [29] for fractional Schrödinger equations and Figueiredo–Molica Bisci–Servadei [21], Figueiredo–Pimenta–Siciliano [22] for fractional Laplacian equations, see also the references therein. In Ait–Mahiout–Alves [1] the authors studied problems in Orlicz Sobolev spaces of the form

$$-\Delta_\Phi u + \Phi(|u|)u = f(u) \quad \text{in } \Omega_\lambda, \qquad u = 0 \quad \text{on } \partial\Omega_\lambda,$$

where $\Phi(t) = \int_0^t \phi(s)s \, \mathrm{d}s$ is a $N$-function and $\Delta_\Phi$ denotes the $\Phi$-Laplacian. We point out that our $\Phi$-function (called $\mathcal{H}$, see Sect. 2) depends also on the domain, so on $x$ which leads to generalized Orlicz Sobolev spaces, also known as Musielak–Orlicz Sobolev spaces.

In all the above mentioned works the nonlinear terms are of class $C^1$. Then the related energy functional $J_\lambda$ belongs to class $C^2$ and so the Nehari manifold $\mathcal{N}_\lambda$ is a $C^1$-manifold of codimension 1. In that case one can use the method of the Lagrange multipliers to show that a point $u$ is a nonzero critical point of $J_\lambda$ if and only if $u \in \mathcal{N}_\lambda$ and $u$ is a critical point for the restriction of $J_\lambda$ to $\mathcal{N}_\lambda$. However, in this paper, we only assume that the nonlinearity $f$ is continuous, which cannot guarantee that $\mathcal{N}_\lambda$ is a $C^1$-manifold, and so we need a new treatment in order to get similar results. In order to overcome the non-differentiability of $\mathcal{N}_\lambda$, we set up a one-to-one correspondence $m$ between the unit sphere $\mathcal{S}_+$ and $\mathcal{N}_\lambda$ (see Lemma 2.2), that is, $m\colon \mathcal{S}_+ \to \mathcal{N}_\lambda$ is defined by $w \mapsto m(w)$. Since $\mathcal{S}_+$ is a $C^1$-submanifold, we can find critical points $w$ for the restriction of $\hat{\Psi}$ to $\mathcal{S}_+$, where $\hat{\Psi}$ is the energy functional associated with $J_\lambda$. Finally, we can show that $m(w)$ is a critical point for the restriction of $J_\lambda$ to $\mathcal{N}_\lambda$ (see Lemma 2.3 for more details). This idea is due to Szulkin–Weth [37], who studied the existence of solutions for a class of Laplace type equations. All in all, our paper extends the results of the papers mentioned above not only for a more general operator but also on much more general classes of functions on the right-hand side which do not have to be differentiable (so no condition on the derivative is assumed) and also do not satisfy the usual Ambrosetti–Rabinowitz condition as it was supposed, for example, in Ait–Mahiout–Alves [1] or Alves [2].

In order to state our main result we first recall the definition of the category. We denote by $\mathrm{cat}_B(A)$ the category of $A$ with respect to $B$, namely the least integer $k$ such that $A \subseteq A_1 \cup \cdots \cup A_k$ with $A_i$ $(i = 1, \cdots, k)$ being closed and contractible in $B$. We set $\mathrm{cat}_B(\emptyset) = 0$ and $\mathrm{cat}_B(A) = +\infty$ if there is no integer with the above property. Furthermore, we set $\mathrm{cat}(B) := \mathrm{cat}_B(B)$.

The following theorem is our main result.

**Theorem 1.1** *Let* (H1) *and* (H2) *be satisfied and suppose that the domain* $\Omega_\lambda$ *is topologically nontrivial, i.e., it is not contractible. Then there exists* $\lambda^* > 0$ *such that, for any* $\lambda \geq \lambda^*$*, Problem* 1.1 *admits at least* $\mathrm{cat}(\Omega_\lambda) + 1$ *positive solutions.*

***Remark 1.2*** The weight function $a(\cdot)$ is the essential characteristic of the double phase operator. It is the distinguishing feature of the double phase operator that is different from other operators like the $p$-Laplacian and the $(p, q)$-Laplacian. The two mappings we constructed when estimating the number of solutions are both affected by $a(\cdot)$. Indeed, $\Phi_\lambda$ is radially symmetric (see Sects. 4 and 5), which requires that $a(\cdot)$ is a radially symmetric function, since when $a(\cdot)$ is a radially symmetric function, the corresponding double phase problem has a radially symmetric solution. When estimating $\beta(\cdot)$, the weight function $a(\cdot)$ is also required to be radially symmetrical (see Lemmas 4.2 and 5.2). In addition, when we prove compactness, we also require $a(\cdot)$ to be radially symmetric (see Lemma 3.2). When $a(\cdot)$ is not a radially symmetric function, it is an interesting question whether the conclusion of this paper holds.

The proof of Theorem 1.1 is based on the Lusternik–Schnirelmann category theory. We first prove that $\hat{\Psi}$ has at least $\mathrm{cat}(\tilde{\mathcal{S}}_+)$ critical points on $\mathcal{S}_+$, where $\tilde{\mathcal{S}}_+$ is a subset of $\mathcal{S}_+$. Then we construct suitable maps between $\tilde{\mathcal{S}}_+$ and $\Omega_\lambda$. Using the properties of the category, we can show that $\mathrm{cat}(\tilde{\mathcal{S}}_+) \geq \mathrm{cat}(\Omega_\lambda)$, that is, $\mathcal{S}_+$ contains at least $\mathrm{cat}(\Omega_\lambda)$ critical points of $\hat{\Psi}$. After that, we prove that there exists another critical point of $\hat{\Psi}$. Finally, by the relationship between $\hat{\Psi}$ and $J_\lambda$, we know that $J_\lambda$ has at least $\mathrm{cat}(\Omega_\lambda)+1$ critical points, that is, Problem 1.1 admits at least $\mathrm{cat}(\Omega_\lambda) + 1$ positive solutions.

The rest of this paper is organized as follows. In Sect. 2, we will introduce the Musielak–Orlicz Sobolev spaces in which we will work and the mapping between $\mathcal{S}_+$ and $\mathcal{N}_\lambda$. Two compactness results are given in Sect. 3 and Sect. 4 is devoted to estimate the asymptotic behavior of some critical levels. The proof of Theorem 1.1 is provided in Sect. 5.

## 2 The Musielak–Orlicz Sobolev Spaces and the Mapping Between the Unit Sphere and the Nehari Manifold

First, we recall some fact about Musielak–Orlicz Sobolev spaces. To this end, let $\mathcal{D} \subseteq \mathbb{R}^N$ and let $\mathcal{H}\colon \mathcal{D} \times [0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{H}(x, t) = t^p + a(x)t^q \quad \text{for all } (x, t) \in \mathcal{D} \times [0, +\infty)$$

while we suppose (H1). The corresponding $\mathcal{H}$-modular is then given by

$$\rho_{\mathcal{H}}(u) = \int_{\mathcal{D}} \mathcal{H}(x, |u|)\, \mathrm{d}x = \int_{\mathcal{D}} \left(|u|^p + a(x)|u|^q\right)\, \mathrm{d}x.$$

Denoting by $M(\mathcal{D})$ the set of all measurable functions $u\colon \mathcal{D} \to \mathbb{R}$, the Musielak–Orlicz space $L^{\mathcal{H}}(\mathcal{D})$ is defined by

$$L^{\mathcal{H}}(\mathcal{D}) = \{u \in M(\mathcal{D}) \, : \, \rho_{\mathcal{H}}(u) < \infty\},$$

equipped with the norm

$$\|u\|_{\mathcal{H}} = \inf \left\{\mu > 0 \, : \, \rho_{\mathcal{H}}\left(\frac{u}{\mu}\right) \le 1\right\}.$$

The space $W^{1,\mathcal{H}}(\mathcal{D})$ is defined by

$$W^{1,\mathcal{H}}(\mathcal{D}) = \left\{u \in L^{\mathcal{H}}(\mathcal{D}) \, : \, |\nabla u| \in L^{\mathcal{H}}(\mathcal{D})\right\},$$

endowed with the norm

$$\|u\| = \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}$$

with $\|\nabla u\|_{\mathcal{H}} = \| \, |\nabla u| \, \|_{\mathcal{H}}$. We denote by $W_0^{1,\mathcal{H}}(\mathcal{D})$ the completion of $C_0^\infty(\mathcal{D})$ in $W^{1,\mathcal{H}}(\mathcal{D})$. From now on, for any $u \in W_0^{1,\mathcal{H}}(\mathcal{D})$ ($\mathcal{D} \subseteq \mathbb{R}^N$), we denote with the same symbol its extension to $\mathbb{R}^N$ obtained by setting $u \equiv 0$ outside of $\mathcal{D}$.

Next, let

$$W_r^{1,\mathcal{H}}(\mathbb{R}^N) = \left\{u \in W^{1,\mathcal{H}}(\mathbb{R}^N) \, : \, u \text{ is radially symmetric}\right\}.$$

The first two authors [30] proved that

$$\begin{aligned} &W_r^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^{\gamma}(\mathbb{R}^N) \quad \text{continuously for all } \gamma \in [p, p^*]; \\ &W_r^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^{\gamma}(\mathbb{R}^N) \quad \text{compactly for all } \gamma \in (p, p^*). \end{aligned}$$

For more details on the spaces, we refer to Liu–Dai [28, 30] and Perera–Squassina [35].

We denote the positive part of a function $u$ by $u^+ = \max\{u, 0\}$ and write $X := W_0^{1,\mathcal{H}}(\Omega_\lambda)$. We define

$$\mathcal{S}_+ = \left\{u \in X \, : \, \|u\| = 1 \text{ and } u^+ \neq 0\right\}.$$

The Nehari manifold related to Problem 1.1 is defined by

$$\mathcal{N}_\lambda = \left\{u \in X \setminus \{0\} \, : \, \langle J_\lambda'(u), u\rangle = 0\right\}.$$

Then we have the following properties.

**Lemma 2.1** *Let* (H1) *and* (H2) *be satisfied. Then the following hold:*

(i) *For each* $w \in X$ *with* $w^+ \neq 0$*, set* $h_w(t) = J_\lambda(tw)$*, there exists a unique* $t_w > 0$ *such that* $h'_w(t) > 0$ *if* $0 < t < t_w$ *and* $h'_w(t) < 0$ *if* $t > t_w$*, that is,* $\max_{t\in[0,+\infty)} h_w(t)$ *is achieved at* $t = t_w$ *and* $t_w w \in \mathcal{N}_\lambda$*.*

(ii) *There exists* $\delta > 0$ *such that* $t_w \geq \delta$ *for* $w \in \mathcal{S}_+$ *and for each compact subset* $\mathcal{W} \subseteq \mathcal{S}_+$ *there exists a constant* $C_{\mathcal{W}}$ *such that* $t_w \leq C_{\mathcal{W}}$ *for all* $w \in \mathcal{W}$*.*

***Proof*** (i) Let $w \in X$ with $w^+ \neq 0$ be fixed and define $h_w(t) = J_\lambda(tw)$ on $[0, \infty)$. It is clear that $h_w(0) = 0$. From (H2)(i), (ii) we deduce that

$$\begin{aligned}
h_w(t) &= \int_{\Omega_\lambda} \left( \frac{t^p}{p} \left( |\nabla w|^p + |w|^p \right) + \frac{a(x)t^q}{q} \left( |\nabla w|^q + |w|^q \right) \right) \mathrm{d}x - \int_{\Omega_\lambda} F(tw)\, \mathrm{d}x \\
&\geq \int_{\Omega_\lambda} \left( \frac{t^p}{p} \left( |\nabla w|^p + |w|^p \right) + \frac{a(x)t^q}{q} \left( |\nabla w|^q + |w|^q \right) \right) \mathrm{d}x \\
&\quad - \int_{\Omega_\lambda} \left( \varepsilon t^p |w|^p + C_\varepsilon t^r |w|^r \right) \mathrm{d}x \\
&\geq \frac{t^p}{2p} \int_{\Omega_\lambda} \left( |\nabla w|^p + |w|^p \right) \mathrm{d}x + \frac{t^q}{q} \int_{\Omega_\lambda} a(x) \left( |\nabla w|^q + |w|^q \right) \mathrm{d}x \\
&\quad - C_\varepsilon t^r \int_{\Omega_\lambda} |w|^r \, \mathrm{d}x \\
&= C_1 t^p + C_2 t^q - C_3 t^r \quad \text{for } 0 < \varepsilon < \frac{1}{2p},
\end{aligned}$$

which implies that $h_w(t) > 0$ for $t$ small enough. It follows from (H2)(iii) that, for any $M > 0$, there exists $T_M > 0$ such that $F(t) \geq M t^q$ for $t > T_M$. Using this gives

$$\begin{aligned}
h_w(t) &\leq \int_{\Omega_\lambda} \left( \frac{t^p}{p} \left( |\nabla w|^p + |w|^p \right) + \frac{a(x)t^q}{q} \left( |\nabla w|^q + |w|^q \right) \right) \mathrm{d}x \\
&\quad - M \int_{\Omega_\lambda} t^q |w|^q \, \mathrm{d}x - C \\
&= C_1 t^p + C_2 t^q - C_3 M t^q - C \\
&\leq C_1 t^p - C_2 t^q - C \quad \text{for } M \geq \frac{2C_2}{C_3},
\end{aligned}$$

which implies that $h_w(t) < 0$ for $t$ large enough. Hence we can find $t_w > 0$ such that $h'_w(t_w) = 0$. Moreover, from

$$\begin{aligned}
0 = h'_w(t) &= \int_{\Omega_\lambda} \left( t^{p-1} \left( |\nabla w|^p + |w|^p \right) + a(x) t^{q-1} \left( |\nabla w|^q + |w|^q \right) \right) \mathrm{d}x \\
&\quad - \int_{\Omega_\lambda} f(tw) w \, \mathrm{d}x,
\end{aligned}$$

we get $tw \in \mathcal{N}_\lambda$ and

$$\begin{aligned}&\int_{\Omega_\lambda} a(x)\left(|\nabla w|^q + |w|^q\right) \mathrm{d}x \\ &= \int_{\Omega_\lambda} \frac{f(tw)w}{t^{q-1}} \,\mathrm{d}x - \frac{1}{t^{q-p}} \int_{\Omega_\lambda} \left(|\nabla w|^p + |w|^p\right) \mathrm{d}x.\end{aligned} \tag{2.1}$$

Taking (H2)(iv) into account, the right-hand side of the last equality is a strictly increasing function in $t$. This implies that $h_w(\cdot)$ has a unique critical point. Thus $\max_{t\in[0,+\infty)} h_w(t)$ is achieved at the unique point $t = t_w > 0$ so that $h'_w(t_w) = 0$ and $t_w w \in \mathcal{N}_\lambda$.

(ii) First, we show that there exists $\delta > 0$ such that $t_w > \delta$ for $w \in \mathcal{S}_+$. If $t_w \geq 1$ we are done. If $t_w < 1$, we deduce from $t_w w \in \mathcal{N}_\lambda$, (H2)(i) and (H2)(ii) that

$$\begin{aligned}&\int_{\Omega_\lambda} \left(t_w^p \left(|\nabla w|^p + |w|^p\right) + a(x) t_w^q \left(|\nabla w|^q + |w|^q\right)\right) \mathrm{d}x \\ &= \int_{\Omega_\lambda} f(t_w w) t_w w \,\mathrm{d}x \\ &\leq \varepsilon t_w^p \int_{\Omega_\lambda} |w|^p \,\mathrm{d}x + C_\varepsilon t_w^r \int_{\Omega_\lambda} |w|^r \,\mathrm{d}x\end{aligned}$$

or

$$\frac{1}{2} t_w^q \leq \frac{1}{2} \int_{\Omega_\lambda} \left(t_w^p \left(|\nabla w|^p + |w|^p\right) + a(x) t_w^q \left(|\nabla w|^q + |w|^q\right)\right) \mathrm{d}x \leq C_2 t_w^r.$$

Obviously we can take $\delta = \left(\frac{1}{2C_2}\right)^{\frac{1}{r-q}} > 0$ in this case.

Now, let $\mathcal{W} \subseteq \mathcal{S}_+$ be compact and suppose by contradiction that there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{W}$ with $t_n := t_{w_n} \to +\infty$. By (i), we see that $J_\lambda(t_n w_n) = \max_{t\in[0,+\infty)} J_\lambda(t w_n) \geq 0$. On the other hand, by (H2)(iii), we deduce that

$$0 \leq \frac{J_\lambda(t_n w_n)}{t_n^q} \leq \frac{1}{p} - \int_{\Omega_\lambda} \frac{F(t_n w_n)}{t_n^q} \,\mathrm{d}x \to -\infty \quad \text{as } n \to \infty,$$

which is a contradiction. Thus there exists $C_W$ such that $t_w \leq C_W$. □

Let

$$\hat{m}\colon X \setminus \{0\} \to \mathcal{N}_\lambda, \quad w \mapsto \hat{m}(w) := t_w w,$$

where $t_w$ is defined in Lemma 2.1 and let $m := \hat{m}|_{\mathcal{S}_+}$. Then we can show that $m$ is a one-to-one correspondence between $\mathcal{S}_+$ and $\mathcal{N}_\lambda$.

**Lemma 2.2** *Let* (H1) *and* (H2) *be satisfied. Then the following hold:*

(i) *The mapping* $\hat{m}$ *is continuous.*

(ii) *The mapping $m$ is a homeomorphism between $\mathcal{S}_+$ and $\mathcal{N}_\lambda$, and the inverse of $m$ is given by $m^{-1}(w) = \frac{w}{\|w\|}$ for all $w \in \mathcal{N}_\lambda$.*

***Proof*** (i) Suppose that $w_n \to w$ in $X \setminus \{0\}$. Taking Lemma 2.1 (ii) into account, we see that $\{t_{w_n}\}_{n\in\mathbb{N}}$ is uniformly bounded in $n$. Therefore, we can find a subsequence of $\{t_{w_n}\}_{n\in\mathbb{N}}$ (which we still denote by $\{t_{w_n}\}_{n\in\mathbb{N}}$) converging to a limit, say $t_0$. It follows from 2.1 that $t_0 = t_w$. But then $t_{w_n} \to t_w$. Hence $\hat{m}$ is continuous.

(ii) By (i), we can easily see that $m(\mathcal{S}_+)$ is a bounded set in $X$ and for any $w \in m(\mathcal{S}_+)$, there exists $\delta > 0$ such that $\|w\| \geq \delta$. If $\|w\| \geq 1$ we are done. If $\|w\| < 1$ then we have

$$\begin{aligned}
&\int_{\Omega_\lambda} \left(|\nabla w|^p + |w|^p + a(x)\left(|\nabla w|^q + |w|^q\right)\right) \mathrm{d}x \\
&= \int_{\Omega_\lambda} f(w)w \,\mathrm{d}x \leq \varepsilon \int_{\Omega_\lambda} |w|^p \,\mathrm{d}x + C_\varepsilon \int_{\Omega_\lambda} |w|^r \,\mathrm{d}x
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{2}\|w\|^q &\leq \frac{1}{2} \int_{\Omega_\lambda} \left(|\nabla w|^p + |w|^p + a(x)\left(|\nabla w|^q + |w|^q\right)\right) \mathrm{d}x \\
&\leq C_\varepsilon \int_{\Omega_\lambda} |w|^r \,\mathrm{d}x \leq C\|w\|^r.
\end{aligned}$$

Choosing $\delta = \left(\frac{1}{2C}\right)^{\frac{1}{r-q}} > 0$ gives $\|w\| \geq \delta$. The continuity of $\hat{m}$ and its definition imply that the mapping $m\colon \mathcal{S}_+ \to \mathcal{N}_\lambda$ is continuous and one-to-one. Clearly, the inverse function of $m$ is $m^{-1}(w) = \frac{w}{\|w\|}$ for any $w \in \mathcal{N}_\lambda$. We only have to show that $m^{-1}$ is continuous. One has

$$\begin{aligned}
\left\|m^{-1}(w) - m^{-1}(v)\right\| &= \left\|\frac{w}{\|w\|} - \frac{v}{\|v\|}\right\| = \left\|\frac{w-v}{\|w\|} + \frac{v\left(\|v\| - \|w\|\right)}{\|w\|\|v\|}\right\| \\
&\leq \frac{2\|w-v\|}{\|w\|} \leq \frac{2}{\delta}\|w-v\|,
\end{aligned}$$

which shows that $m^{-1}$ is Lipschitz continuous. □

Recall that $J$ satisfies the (PS)-condition on $\mathcal{N}_\lambda$, if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\lambda$ such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \tag{2.2}$$

for any $c \in \mathbb{R}$, admits a convergent subsequence. Any sequence satisfying 2.2 is called a $(\mathrm{PS})_c$-sequence or (PS)-sequence. Defining $\hat{\Psi}(w) := J_\lambda(\hat{m}(w))$, we show next that the problem of finding critical points of $\hat{\Psi}|_{\mathcal{S}_+}$ is equivalent to the problem of finding critical points of $J_\lambda|_{\mathcal{N}_\lambda}$.

**Lemma 2.3** *Let* (H1) *and* (H2) *be satisfied. Then the following hold:*

(i) $\hat{\Psi} \in C^1(X \setminus \{0\}, \mathbb{R})$ *and*

$$\langle \hat{\Psi}'(w), z\rangle = \langle J_\lambda'(m(w)), \|m(w)\| z\rangle$$

*for all* $w \in \mathcal{S}_+$ *and for all* $z \in T_w(\mathcal{S}_+)$, *where* $T_w(\mathcal{S}_+)$ *denotes the tangent space to* $\mathcal{S}_+$ *at* $w$.

(ii) *If* $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ *is a* (PS)*-sequence for* $\hat{\Psi}$, *then* $\{m(w_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\lambda$ *is a* (PS)*-sequence for* $J_\lambda$. *If* $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\lambda$ *is a bounded* (PS)*-sequence for* $J_\lambda$, *then* $\{m^{-1}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ *is a* (PS)*-sequence for* $\hat{\Psi}$.

(iii) $w \in \mathcal{S}_+$ *is a critical point of* $\hat{\Psi}$ *if and only if* $m(w) \in \mathcal{N}_\lambda$ *is a nontrivial critical point of* $J_\lambda$. *Moreover,* $\inf_{\mathcal{S}_+} \hat{\Psi} = \inf_{\mathcal{N}_\lambda} J_\lambda$.

***Proof*** The lemma is a direct consequence of Proposition 9 and Corollary 10 in Szulkin–Weth [37] and Lemmas 2.1 and 2.2. □

***Remark 2.4*** Obviously, from Lemmas 2.1 and 2.2, we see that the infimum of $J_\lambda$ over $\mathcal{N}_\lambda$ has the following minimax characterization:

$$c(\Omega_\lambda) := \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) = \inf_{w\in X\setminus\{0\}} \max_{s>0} J_\lambda(sw) = \inf_{w\in\mathcal{S}_+} \max_{s>0} J_\lambda(sw) = \inf_{w\in\mathcal{S}_+} \hat{\Psi}(w).$$

Note that $c(\Omega_\lambda) > 0$. In fact, for any $w \in X \setminus \{0\}$, we know from the proof of Lemma 2.1 (i) that $\max_{s>0} J_\lambda(sw) \geq \delta_w$ for some $\delta_w > 0$. From Lemma 2.1 (ii) we can find $\delta > 0$ such that $\delta_w \geq \delta$ uniformly for $w \in X \setminus \{0\}$. Hence

$$c(\Omega_\lambda) = \inf_{w\in X\setminus\{0\}} \max_{s>0} J_\lambda(sw) \geq \delta > 0.$$

## 3 Two Compactness Results

First, we prove that $\hat{\Psi}$ satisfies the (PS)-condition on $\mathcal{S}_+$.

**Lemma 3.1** *Let* (H1) *and* (H2) *be satisfied. Then the following hold:*

(i) *If* $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ *is a sequence such that* $\operatorname{dist}(w_n, \partial\mathcal{S}_+) \to 0$ *as* $n \to +\infty$. *Then* $\|m(w_n)\| \to +\infty$ *and* $\hat{\Psi}(w_n) \to +\infty$ *as* $n \to +\infty$.

(ii) $\hat{\Psi}$ *satisfies the* (PS)*-condition on* $\mathcal{S}_+$, *that is, every sequence* $\{w_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{S}_+$ *such that, for any* $c > 0$, $\hat{\Psi}(w_n) \to c$ *and* $\hat{\Psi}'(w_n) \to 0$ *as* $n \to +\infty$ *contains a subsequence which converges strongly to some* $w \in \mathcal{S}_+$ *and* $\operatorname{dist}(w, \partial\mathcal{S}_+) > 0$.

***Proof*** (i) Let $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ be a sequence such that $\operatorname{dist}(w_n, \partial\mathcal{S}_+) \to 0$ as $n \to +\infty$. Then, for any $v \in \partial\mathcal{S}_+$ and $n \in \mathbb{N}$, we have $w_n^+ \leq |w_n - v|$ a.e. in $\Omega_\lambda$. For any $\gamma \in [1, p^*]$, by the embedding theorem, we have

$$\|w_n^+\|_{L^\gamma(\Omega_\lambda)} \leq \inf_{v\in\partial\mathcal{S}_+} \|w_n - v\|_{L^\gamma(\Omega_\lambda)} \leq C_\gamma \inf_{v\in\partial\mathcal{S}_+} \|w_n - v\| = C_\gamma \operatorname{dist}(w_n, \partial\mathcal{S}_+)$$

for all $n \in \mathbb{N}$. Recall that $f(t) \equiv 0$ for $t \le 0$. For every $t > 0$, it follows from (H2)(i) and (H2)(ii) that

$$
\begin{aligned}
|K(tw_n)| &= \left| \int_{\Omega_\lambda^{\le}} F(tw_n)\,\mathrm{d}x + \int_{\Omega_\lambda^{>}} F(tw_n)\,\mathrm{d}x \right| = \left| \int_{\Omega_\lambda} F(tw_n^+)\,\mathrm{d}x \right| \\
&\le \varepsilon t^p \int_{\Omega_\lambda} |w_n^+|^p\,\mathrm{d}x + C_\varepsilon t^r \int_{\Omega_\lambda} |w_n^+|^r\,\mathrm{d}x \\
&\le C\left(t^p \operatorname{dist}^p(w_n, \partial\mathcal{S}_+) + t^r \operatorname{dist}^r(w_n,\ \partial\mathcal{S}_+)\right) \to 0 \quad \text{as } n \to +\infty,
\end{aligned}
$$

where $\Omega_\lambda^{>} = \{x \in \Omega_\lambda : \ w_n(x) > 0\}$ and $\Omega_\lambda^{\le} = \Omega_\lambda \setminus \Omega_\lambda^{>}$. Note that for any $t > 1$,

$$
\frac{1}{p}\|tw_n\|^q + |K(tw_n)| \ge J_\lambda(tw_n) \ge \frac{1}{q}\|tw_n\|^p - |K(tw_n)| = \frac{t^p}{q} - |K(tw_n)|.
$$

Consequently

$$
\liminf_{n\to+\infty} \frac{1}{p}\|m(w_n)\|^q \ge \liminf_{n\to+\infty} \hat{\Psi}(w_n) \ge \liminf_{n\to+\infty} J_\lambda(tw_n) \ge \frac{t^p}{q}, \quad \text{for every } t > 1,
$$

and thus $\|m(w_n)\| \to +\infty$ and $\hat{\Psi}(w_n) \to +\infty$ as $n \to +\infty$.

(ii) For any $c > 0$, let $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ be a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$. It follows from Lemma 2.3 that $\{u_n := m(w_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\lambda$ is a $(\mathrm{PS})_c$-sequence for $J_\lambda$. First we will prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Suppose this is not the case, then there exists a subsequence (still denoted by $u_n$) such that $\|u_n\| \to +\infty$. Set $v_n = u_n/\|u_n\|$, then $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Thus, after passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v$ in $X$ as $n \to +\infty$. If $v = 0$, then it follows from Lemma 2.1 and Remark 2.4 that

$$
c + o(1) \ge J_\lambda(u_n) = J_\lambda(t_{v_n} v_n) \ge J_\lambda(tv_n) \quad \text{for all } t > 0.
$$

Recall that $K$ is weakly continuous. If $t > 1$, then we have that

$$
J_\lambda(tv_n) \ge \frac{1}{q}t^p - \int_{\Omega_\lambda} F(tv_n)\,\mathrm{d}x \to \frac{1}{q}t^p.
$$

This yields a contradiction by choosing $t > \max\{1,\ 2(qc)^{\frac{1}{p}}\}$. If $v \neq 0$, then we know from (H2)(iii) that

$$
0 \le \frac{J_\lambda(u_n)}{\|u_n\|^q} \le \frac{1}{p} - \int_{\Omega_\lambda} \frac{F(\|u_n\|v_n)}{\|u_n\|^q}\,\mathrm{d}x \to -\infty
$$

as $n \to \infty$, again a contradiction. Hence $\{u_n\}_{n\in\mathbb{N}}$ is bounded and so there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (still denoted by $u_n$) such that $u_n \rightharpoonup u$ in $X$. Recall that $K_\lambda'(u_n) \to K_\lambda'(u)$. Since $J_\lambda'(u_n) = L_\lambda(u_n) - K_\lambda'(u_n) \to 0$, one has that $L_\lambda(u_n) \to$

$K'_\lambda(u)$ as $n \to +\infty$, where $L_\lambda$ denotes the derivative operator of $I_\lambda$ in the weak sense. Therefore, we conclude that $u_n \to u$ in $X$ as $n \to +\infty$, since $L_\lambda$ is a mapping of type $(S_+)$. Consequently $m^{-1}(u_n) \to m^{-1}(u)$ by Lemma 2.2, that is, $w_n \to w$. Therefore, $\hat{\Psi}$ satisfies the (PS)-condition on $\mathcal{S}_+$. □

Our second compactness result is related to the limiting functional associated to $J_\lambda$, which is defined in the whole space. More precisely,

$$J_\infty(u) := \int_{\mathbb{R}^N} \left( \frac{1}{p} \left( |\nabla u|^p + |u|^p \right) + \frac{a(x)}{q} \left( |\nabla u|^q + |u|^q \right) - F(u) \right) \mathrm{d}x.$$

The corresponding Nehari manifold is defined by

$$\mathcal{N}_\infty := \left\{ u \in W_r^{1,\mathcal{H}}(\mathbb{R}^N) \setminus \{0\} \,:\, \langle J'_\infty(u), u \rangle = 0 \right\}$$

while the least energy level is given by

$$0 < c(\mathbb{R}^N) := \inf_{u \in \mathcal{N}_\infty} J_\infty(u).$$

In Liu–Dai [30, Theorem 1.9], it is proved that $c(\mathbb{R}^N)$ is achieved by a positive radially symmetric function. Moreover, we can show the following compactness lemma. The idea used here is due to Alves [2], who studied the $p$-Laplacian equation.

**Lemma 3.2** *Let* (H1) *and* (H2) *be satisfied and let* $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\infty$ *be such that* $J_\infty(w_n) \to c(\mathbb{R}^N)$. *Then either* $\{w_n\}_{n\in\mathbb{N}}$ *has a strongly convergent subsequence or there exists a sequence* $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ *with* $|y_n| \to +\infty$ *such that* $\{\tilde{w}_n(x) := w_n(x + y_n)\}_{n\in\mathbb{N}}$ *has a strongly convergent subsequence.*

***Proof*** Let $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\infty$ be such that $J_\infty(w_n) \to c(\mathbb{R}^N)$. As in the proof of Lemma 3.1 (ii), we can show that $w_n \rightharpoonup w$, up to a subsequence if necessary. If $w \neq 0$, then, as in Liu–Dai [30, Sect. 6], we can prove that $K'_\infty$ is completely continuous, that is, $K'_\infty(w_n) \to K'_\infty(w)$. Since $J'_\infty(w_n) = L_\infty(w_n) - K'_\infty(w_n) \to 0$, one has that $L_\infty(w_n) \to K'_\infty(w)$ as $n \to +\infty$, where $L_\infty$ denotes the derivative operator of $I_\infty$ in the weak sense, $K_\infty$ and $I_\infty$ are defined as in 1.4 and 1.3, just replacing $\Omega_\lambda$ by $\mathbb{R}^N$, respectively. Then we conclude that $w_n \to w$ as $n \to +\infty$ since $L_\infty$ is a mapping of type $(S_+)$ (see Liu–Dai [30] or Crespo–Blanco–Gasiński–Harjulehto–Winkert [18] for more details). It is clear that $w \in \mathcal{N}_\infty$ and $J_\infty(w) = c(\mathbb{R}^N)$ since $J_\infty \in C^1$.

If $w = 0$, then we claim that there exist $\overline{R}, \delta > 0$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ such that

$$\limsup_{n\to+\infty} \int_{B_{\overline{R}}(y_n)} |w_n|^p \,\mathrm{d}x \geq \delta. \tag{3.1}$$

Let us suppose this is not the case. Then we have

$$\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_{\overline{R}}(y)} |w_n|^p \,\mathrm{d}x = 0.$$

It follows from Lions [26, 27, Lemma I.1] that $w_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (p, p^*)$. Consequently, as in the proof of Lemma 2.1, we have

$$\int_{\mathbb{R}^N} f(w_n)w_n \, dx \to 0$$

by (H2)(i) and (H2)(ii). Recalling that $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\infty$, we have $\|w_n\| \to 0$ as $n \to +\infty$. Consequently $J_\infty(w_n) \to 0$, a contradiction to $c(\mathbb{R}^N) > 0$.

Next, we claim that $|y_n| \to +\infty$. Assume this is not true, then there exists $M > 0$ such that $|y_n| \le M$. Set $\hat{R} := M + \overline{R} + 1$. Since the embedding $W^{1,\mathcal{H}}(B_{\hat{R}}(0)) \hookrightarrow L^p(B_{\hat{R}}(0))$ is compact (see Liu–Dai [30]), we deduce that $\|w_n\|_{L^p(B_{\hat{R}}(0))} \to 0$, which contradicts 3.1, since $\|w_n\|_{L^p(B_{\hat{R}}(0))} \ge \|w_n\|_{L^p(B_{\overline{R}}(y_n))}$. Now, we define $\tilde{w}_n(x) = w_n(x + y_n)$ and apply the argument used in the case $w \ne 0$ again, with $w_n$ replaced by $\tilde{w}_n$, to obtain $\tilde{w}_n \to \tilde{w} \in \mathcal{N}_\infty$ with $J_\infty(\tilde{w}) = c(\mathbb{R}^N)$. This finishes the proof. □

## 4 Some Estimates

For $\tilde{R} > R > 0$, we need two auxiliary functionals which are defined as in 1.2, just replacing $\Omega_\lambda$ by $B_R := B_R(0)$ and $A_{\tilde{R},R}(\tilde{x}) := B_{\tilde{R}}(\tilde{x}) \setminus \overline{B_R(\tilde{x})}$ for all $\tilde{x} \in \mathbb{R}^N$, respectively, that is,

$$J_R(u) = \int_{B_R} \left( \frac{1}{p} \left(|\nabla u|^p + |u|^p\right) + \frac{a(x)}{q} \left(|\nabla u|^q + |u|^q\right) - F(u) \right) dx,$$
$$J_{\lambda,\tilde{x}}(u) = \int_{A_{\lambda\tilde{R},\lambda R}(\tilde{x})} \left( \frac{1}{p} \left(|\nabla u|^p + |u|^p\right) + \frac{a(x)}{q} \left(|\nabla u|^q + |u|^q\right) - F(u) \right) dx.$$

The corresponding Nehari manifolds are given by

$$\mathcal{N}_R := \left\{ u \in W_0^{1,\mathcal{H}}(B_R) \setminus \{0\} \,:\, \langle J_R'(u), u\rangle = 0 \right\}$$
$$\mathcal{N}_{\lambda,\tilde{x}} := \left\{ u \in W_0^{1,\mathcal{H}}(A_{\lambda\tilde{R},\lambda R}(\tilde{x})) \setminus \{0\} \,:\, \langle J_{\lambda,\tilde{x}}'(u), u\rangle = 0 \right\}.$$

We set

$$c(B_R) := \inf_{u\in\mathcal{N}_R} J_R(u). \tag{4.1}$$

Then $c(B_R)$ is achieved by a positive radially symmetric function $\Psi_R$. Indeed, similar to the proof of Liu–Dai [30, Theorem 1.4], we can show that $c(B_R)$ is attained by a positive function $v \in W_0^{1,\mathcal{H}}(B_R)$. Let $v^*$ be the Schwartz symmetrization of $v$, then we have that $v^* \in W_0^{1,\mathcal{H}}(B_R)$ and it satisfies

$$\int_{B_R} \left( \frac{1}{p} |\nabla v^*|^p + \frac{a(x)}{q} |\nabla v^*|^q \right) \mathrm{d}x \le \int_{B_R} \left( \frac{1}{p} |\nabla v|^p + \frac{a(x)}{q} |\nabla v|^q \right) \mathrm{d}x,$$
$$\int_{B_R} \left( \frac{1}{p} |v^*|^p + \frac{a(x)}{q} |v^*|^q \right) \mathrm{d}x = \int_{B_R} \left( \frac{1}{p} |v|^p + \frac{a(x)}{q} |v|^q \right) \mathrm{d}x$$

and

$$\int_{B_R} F(v^*)\, \mathrm{d}x = \int_{B_R} F(v)\, \mathrm{d}x.$$

As in Lemma 2.1, we can show that there exists a unique $t_{v^*} > 0$ such that $t_{v^*} v^* \in \mathcal{N}_R$. Moreover, it holds

$$c(B_R) \le J_R(t_{v^*} v^*) \le J_R(t_{v^*} v) \le \max_{t \ge 0} J_R(tv) = J_R(v) = c(B_R).$$

Set $\Psi_R := t_{v^*} v^*$.Then $\Psi_R$ has all the required properties. Furthermore, we can study the asymptotic behavior of $c(B_R)$.

**Lemma 4.1** *Let* (H1) *and* (H2) *be satisfied and let* $c(\Omega_\lambda)$ *and* $c(B_R)$ *be defined as in Remarks* 2.4 *and* 4.1*, respectively. Then*

$$\lim_{\lambda \to +\infty} c(\Omega_\lambda) = c(\mathbb{R}^N) \quad \textit{and} \quad \lim_{R \to +\infty} c(B_R) = c(\mathbb{R}^N).$$

***Proof*** We only show the first assertion, the second one can be done in a similar way. To this end, let $\tilde{\lambda} > 0$ and $R > 0$ be fixed such that $B_R \subseteq \Omega_{\tilde{\lambda}}$. Let $\eta_R \colon [0, +\infty) \to \mathbb{R}$ be a smooth nonincreasing cut-off function defined by

$$\eta_R(t) = 1 \text{ if } 0 \le t \le R/2, \quad \eta_R(t) = 0 \text{ if } t \ge R, \quad 0 \le \eta_R \le 1 \quad \text{and} \quad \left|\eta_R'(t)\right| \le 2.$$

We write $w_R(x) = \eta_R(x) w(x)$, where $w \in \mathcal{N}_\infty$ is such that $J_\infty(w) = c(\mathbb{R}^N)$. In addition, let $t_R > 0$ be such that $t_R w_R \in \mathcal{N}_\lambda$. Then we have

$$c(\Omega_\lambda) \le J_\lambda(t_R w_R) \quad \text{for all } \lambda > \tilde{\lambda}.$$

Passing to the limit as $\lambda \to +\infty$ we obtain

$$\limsup_{\lambda \to +\infty} c(\Omega_\lambda) \le J_\infty(t_R w_R).$$

As in the proof of Lemma 2.1 we can show that $t_R \to 1$ as $R \to +\infty$. Then $J_\infty(t_R w_R) \to J_\infty(w) = c(\mathbb{R}^N)$ as $R \to +\infty$. Consequently, we get

$$\limsup_{\lambda \to +\infty} c(\Omega_\lambda) \le c(\mathbb{R}^N). \tag{4.2}$$

On the other hand, from the definition of $c(\Omega_\lambda)$ and $c(\mathbb{R}^N)$, it follows that $c(\mathbb{R}^N) \leq c(\Omega_\lambda)$ for all $\lambda > 0$, which implies that

$$c(\mathbb{R}^N) \leq \liminf_{\lambda \to +\infty} c(\Omega_\lambda). \tag{4.3}$$

From 4.2 and 4.3 the assertion follows. □

Now we choose $\tilde{R} \geq \mathrm{diam}(\Omega)$. For $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with compact support in $B_{\tilde{R}}(0)$, we define the barycenter map $\beta \colon W^{1,\mathcal{H}}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta(u) = \frac{\displaystyle\int_{\mathbb{R}^N} x|u(x)|^p \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^p \,\mathrm{d}x}. \tag{4.4}$$

Using the map $\beta$ we can show the following estimate.

**Lemma 4.2** *Let* (H1) *and* (H2) *be satisfied and let*

$$\tilde{a}(\lambda, R, \tilde{R}, \tilde{x}) := \inf \left\{ J_{\lambda,\tilde{x}}(u) \,:\, u \in \mathcal{N}_{\lambda,\tilde{x}} \text{ and } \beta(u) = \tilde{x} \right\}.$$

*Then it holds*

$$\liminf_{\lambda \to +\infty} \tilde{a}(\lambda, R, \tilde{R}, \tilde{x}) > c(\mathbb{R}^N)$$

*and in particular,*

$$\liminf_{\lambda \to +\infty} \tilde{a}(\lambda, R, \tilde{R}, 0) > c(\mathbb{R}^N).$$

***Proof*** By the translation invariance of the Lebesgue integral and the symmetry of $u$, we know that

$$\tilde{a}(\lambda_n, R, \tilde{R}, 0) = \tilde{a}(\lambda_n, R, \tilde{R}, \tilde{x}).$$

Hence it suffices to prove that

$$\liminf_{\lambda \to +\infty} \tilde{a}(\lambda, R, \tilde{R}, 0) > c(\mathbb{R}^N).$$

It is clear from the definitions of $\tilde{a}(\lambda, R, \tilde{R}, 0)$ and $c(\mathbb{R}^N)$ that

$$\tilde{a}(\lambda, R, \tilde{R}, 0) \geq c(\mathbb{R}^N).$$

We only need to show that

$$\liminf_{\lambda \to +\infty} \tilde{a}(\lambda, R, \tilde{R}, 0) \neq c(\mathbb{R}^N).$$

Suppose this is not true, then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that

$$\tilde{a}(\lambda_n, R, \tilde{R}, 0) \to c(\mathbb{R}^N) \quad \text{as } n \to +\infty.$$

Here we set $\lambda_n \to +\infty$ as $n \to +\infty$. As in the proof of Lemma 3.1, we can show that $J_{\lambda_n,0}$ satisfies the (PS)-condition on $A_{\lambda_n \tilde{R},\lambda_n R}(0)$, and so $\tilde{a}(\lambda_n, R, \tilde{R}, 0)$ are attained by positive radially symmetric functions. Therefore, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_{\lambda_n,0}$ such that

$$\beta_{\lambda_n}(u_n) = 0 \quad \text{and} \quad J_{\lambda_n,0}(u_n) = \tilde{a}(\lambda_n, R, \tilde{R}, 0).$$

Note that $\operatorname{supp} u_n \subseteq B_{\lambda_n \tilde{R}}(0) \setminus \overline{B_{\lambda_n R}(0)}$. Thus $u_n \rightharpoonup 0$ as $n \to +\infty$. It is clear that $u_n \not\to 0$ since $c(\mathbb{R}^N) > 0$. Thus it follows from Lemma 3.2 that there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ with $|y_n| \to +\infty$ such that

$$u_n(x) = v_n(x - y_n) + \Psi(x - y_n),$$

where $\{v_n\}_{n\in\mathbb{N}}$ is a sequence converging strongly to 0 in $X$ while $\Psi$ is a nonnegative function satisfying $J_\infty(\Psi) = c(\mathbb{R}^N)$ and $\Psi \in \mathcal{N}_\infty$. We write

$$M := \int_{\mathbb{R}^N} |\Psi(x)|^p \,\mathrm{d}x > 0.$$

From $v_n \to 0$ in $X$, we conclude that

$$\int_{B_{\lambda_n R/2}(y_n)} |v_n(x - y_n) + \Psi(x - y_n)|^p \,\mathrm{d}x = \int_{B_{\lambda_n R/2}(0)} |v_n(x) + \Psi(x)|^p \,\mathrm{d}x \to M,$$

which implies

$$\int_{\Xi_n} |u_n|^p \,\mathrm{d}x \to M, \quad \text{where} \quad \Xi_n := B_{\lambda_n R/2}(y_n) \cap A_{\lambda_n \tilde{R},\lambda_n R}(0).$$

Thus,

$$\int_{\tilde{\Xi}_n} |u_n|^p \,\mathrm{d}x \to 0, \quad \text{where} \quad \tilde{\Xi}_n := A_{\lambda_n \tilde{R},\lambda_n R}(0) \setminus B_{\lambda_n R/2}(y_n). \tag{4.5}$$

Since $J_\infty$ is rotation invariant, we may assume that $y_n = (y_n^1, 0, \cdots, 0)$ with $y_n^1 < 0$. Hence, it follows from $\beta_{\lambda_n}(u_n) = 0$ that

$$0 = \int_{A_{\lambda_n \tilde{R},\lambda_n R}(0)} x^1 |u_n|^p \,\mathrm{d}x = \int_{\Xi_n} x^1 |u_n|^p \,\mathrm{d}x + \int_{\tilde{\Xi}_n} x^1 |u_n|^p \,\mathrm{d}x,$$

where $x^1$ is the first coordinate of $x$. It is easy to check that

$$\int_{\Xi_n} x^1 |u_n|^p \,\mathrm{d}x \le -\frac{\lambda_n R}{2}(M + o_n(1))$$

and

$$\int_{\tilde{\Xi}_n} x^1 |u_n|^p \,\mathrm{d}x \le \lambda_n \tilde{R} \int_{\tilde{\Xi}_n} |u_n|^p \,\mathrm{d}x,$$

where $o_n(1) \to 0$ as $n \to +\infty$. Therefore, we have

$$0 \le -\frac{\lambda_n R}{2}(M + o_n(1)) + \lambda_n \tilde{R} \int_{\tilde{\Xi}_n} |u_n|^p \,\mathrm{d}x,$$

that is,

$$\int_{\tilde{\Xi}_n} |u_n|^p \,\mathrm{d}x \ge \frac{R}{2\tilde{R}}(M + o_n(1)),$$

which contradicts 4.5. □

## 5 Proof of Theorem 1.1

In what follows, without any loss of generality, we shall assume that $0 \in \Omega$. Moreover, we fix real numbers $\tilde{R} > R > 0$ such that $B_R(0) \subseteq \Omega \subseteq B_{\tilde{R}}(0)$ and the sets

$$\Omega_R^+ := \left\{ x \in \mathbb{R}^N \,:\, \operatorname{dist}(x, \Omega) \le R \right\} \quad \text{and} \quad \Omega_R^- := \{ x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \ge R \}$$

are homotopically equivalent to $\Omega$. For $\lambda > 0$, let $\Psi_{\lambda R} \in \mathcal{N}_{\lambda R}$ be given as in Sect. 4 satisfying $J_{\lambda R}(\Psi_{\lambda R}) = c(B_{\lambda R})$. We define $\Phi_\lambda \colon \lambda\Omega_R^- \to \mathcal{N}_\lambda$ by

$$[\Phi_\lambda(\xi)](x) = \begin{cases} t_\lambda \Psi_{\lambda R}(|x - \xi|), & \text{if } x \in B_{\lambda R}(\xi), \\ 0, & \text{if } x \in \Omega_\lambda \setminus B_{\lambda R}(\xi), \end{cases}$$

where $t_\lambda > 0$ is such that $\Phi_\lambda(\xi) \in \mathcal{N}_\lambda$. Then we have the following lemma.

**Lemma 5.1** *Let* (H1) *and* (H2) *be satisfied. Then we have*

$$\lim_{\lambda \to +\infty} J_\lambda(\Phi_\lambda(\xi)) = c(\mathbb{R}^N)$$

*uniformly in* $\xi \in \lambda\Omega_R^-$.

***Proof*** Following the same arguments as in the proof of Lemma 2.1 and Remark 2.4, it is easy to see that

$$\begin{aligned} c(\Omega_\lambda) &\leq J_\lambda(\Phi_\lambda(\xi)) = J_\lambda(t_\lambda \Psi_{\lambda R}(|x-\xi|)) \\ &= J_\lambda(t_\lambda \Psi_{\lambda R}(|x|)) \leq J_\lambda(\Psi_{\lambda R}(|x|)) = c(B_{\lambda R}). \end{aligned}$$

Here we have used the translation invariance of the Lebesgue integral in the second equality. Lemma 4.1 gives that $\lim_{\lambda\to+\infty} c(B_{\lambda R}) = \lim_{\lambda\to+\infty} c(\Omega_\lambda) = c(\mathbb{R}^N)$. Hence our conclusion holds. □

Given $\xi \in \lambda\Omega_R^-$, we set $h(\lambda) := |J_\lambda(\Phi_\lambda(\xi)) - c(\mathbb{R}^N)|$. We conclude from Lemma 5.1 that $h(\lambda) \to 0$ as $\lambda \to +\infty$. Next, we define the sublevel set

$$\tilde{\mathcal{N}_\lambda} = \left\{ u \in \mathcal{N}_\lambda \;:\; J_\lambda(u) \leq c(\mathbb{R}^N) + h(\lambda) \right\}.$$

It is clear that $\Phi_\lambda(\xi) \in \tilde{\mathcal{N}_\lambda}$, which implies $\tilde{\mathcal{N}_\lambda} \neq \emptyset$ for any $\lambda > 0$. Furthermore, we have the following result.

**Lemma 5.2** *Let* (H1) *and* (H2) *be satisfied. Then there exists* $\lambda^* > 0$ *such that, for any* $\lambda \geq \lambda^*$, *if* $u \in \tilde{\mathcal{N}_\lambda}$, *then* $\beta(u) \in \lambda\Omega_R^+$, *where* $\beta$ *is defined in* (4.4).

***Proof*** We argue by contradiction and assume that there exist $\lambda_n \to +\infty$ as $n \to +\infty$ and $u_n \in \tilde{\mathcal{N}}_{\lambda_n}$ such that $\beta(u_n) =: x_n \notin \lambda_n\Omega_R^+$. We can choose $\tilde{R} > \operatorname{diam}(\Omega)$ such that

$$\Omega_{\lambda_n} \subseteq A_{\lambda_n \tilde{R}, \lambda_n R}(x_n). \tag{5.1}$$

Indeed, if $x \in \Omega_{\lambda_n}$ then $x/\lambda_n \in \Omega$. Note that $x_n/\lambda_n \notin \Omega_R^+$ since $x_n \notin \lambda_n\Omega_R^+$. Hence $|x/\lambda_n - x_n/\lambda_n| > R$, that is, $|x - x_n| > \lambda_n R$. Consequently, $x \notin B_{\lambda_n R}(x_n)$. Since $x \in \Omega_{\lambda_n}$, there exists $y \in \Omega$ such that $x = \lambda_n y$. Therefore, it follows that

$$|x - x_n| = \left| x - \frac{\int_{\Omega_{\lambda_n}} z|u(z)|^p \,\mathrm{d}z}{\int_{\Omega_{\lambda_n}} |u(z)|^p \,\mathrm{d}z} \right| \leq \lambda_n \left| \frac{\int_{\Omega_{\lambda_n}} \left| y - \frac{z}{\lambda_n} \right| |u(z)|^p \,\mathrm{d}z}{\int_{\Omega_{\lambda_n}} |u(z)|^p \,\mathrm{d}z} \right|.$$

Note that $z/\lambda_n \in \Omega$, $y \in \Omega$ and $\Omega \subseteq B_{\tilde{R}}(0)$. Hence $|y - z/\lambda_n| \leq \tilde{R}$. Thus,

$$|x - x_n| \leq \lambda_n \tilde{R},$$

that is, $x \in B_{\lambda_n \tilde{R}}(x_n)$, and so 5.1 holds. From 5.1 it follows that

$$\tilde{a}(\lambda_n, R, \tilde{R}, 0) = \tilde{a}(\lambda_n, R, \tilde{R}, x_n) \leq J_{\lambda_n}(u_n) \leq c(\mathbb{R}^N) + h(\lambda_n).$$

Letting $n \to +\infty$, Lemma 5.1 implies that $h(\lambda_n) \to 0$ and so

$$\limsup_{n\to+\infty} \tilde{a}(\lambda_n, R, \tilde{R}, 0) \le c(\mathbb{R}^N),$$

which contradicts Lemma 4.2. □

Now, we are ready to prove Theorem 1.1.

***Proof of Theorem 1.1*** We conclude from Lemmas 5.1 and 2.3 that

$$\lim_{\lambda\to+\infty} \hat{\Psi}(m^{-1}(\Phi_\lambda(\xi))) = \lim_{\lambda\to+\infty} J_\lambda(\Phi_\lambda(\xi)) = c(\mathbb{R}^N)$$

uniformly in $\xi \in \lambda\Omega_R^-$. We write

$$\tilde{\mathcal{S}}_+ = \left\{u \in \mathcal{S}_+ \,:\, \hat{\Psi}(u) \le c(\mathbb{R}^N) + h(\lambda)\right\},$$

where $h$ is given in the definition of $\tilde{\mathcal{N}_\lambda}$. It is clear that $\tilde{\mathcal{S}}_+ \neq \emptyset$, since $m^{-1}(\Phi_\lambda(\xi)) \in \tilde{\mathcal{S}}_+$. From Lemma 3.1 and the Lusternik–Schnirelmann theory (see Szulkin–Weth [37, Theorem 27]), it follows that $\hat{\Psi}$ has at least $\text{cat}(\tilde{\mathcal{S}}_+)$ critical points on $\tilde{\mathcal{S}}_+$. Lemmas 5.1, 5.2 and 2.3 imply that there exists $\lambda^* > 0$ such that, for any $\lambda \ge \lambda^*$, the diagram

$$\lambda\Omega_R^- \xrightarrow{\Phi_\lambda} \tilde{\mathcal{N}_\lambda} \xrightarrow{m^{-1}} \tilde{\mathcal{S}}_+ \xrightarrow{m} \tilde{\mathcal{N}_\lambda} \xrightarrow{\beta} \lambda\Omega_R^+$$

is well defined and $\beta \circ m \circ m^{-1} \circ \Phi_\lambda$ is homo topic to the inclusion $\text{id}\colon \lambda\Omega_R^- \to \lambda\Omega_R^+$. Hence

$$\text{cat}(\tilde{\mathcal{S}}_+) \ge \text{cat}_{\lambda\Omega_R^+}(\lambda\Omega_R^-) = \text{cat}(\Omega_\lambda). \tag{5.2}$$

Indeed, suppose that $\text{cat}(\tilde{\mathcal{S}}_+) = n$, that is, there exists a smallest positive integer $n$ such that

$$\tilde{\mathcal{S}}_+ \subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_n,$$

where $\mathcal{D}_i$, $i = 1, 2, \cdots, n$ are closed and contractible in $\tilde{\mathcal{S}}_+$, that is, there exist

$$h_i \in C([0, 1] \times \mathcal{D}_i, \tilde{\mathcal{S}}_+), \quad i = 1, 2, \cdots, n$$

such that

$$h_i(0, u) = u \quad \text{for all } u \in \mathcal{D}_i,$$
$$h_i(1, u) = \omega_i \in \tilde{\mathcal{S}}_+ \quad \text{for all } u \in \mathcal{D}_i.$$

We set

$$\mathcal{K}_i = \Phi_\lambda^{-1}(m(\mathcal{D}_i)).$$

Clearly $\mathcal{K}_i$ are closed subsets of $\lambda\Omega_R^-$ and $\lambda\Omega_R^- \subseteq \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_n$. Moreover $\mathcal{K}_i$, $i = 1, \cdots, n$ are contractible in $\lambda\Omega_R^+$ using the deformation $\mathfrak{h}_i : [0,1] \times \mathcal{K}_i \to \lambda\Omega_R^+$ defined by

$$\mathfrak{h}_i(t, x) = (\beta \circ m \circ h_i)(t, m^{-1}(\Phi_\lambda(x))).$$

We conclude from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
&\mathfrak{h}_i \in C([0,1] \times \mathcal{K}_i, \lambda\Omega_R^+),\\
&\mathfrak{h}_i(0, x) = (\beta \circ m \circ h_i)(0, m^{-1}(\Phi_\lambda(x))) = x \quad \text{for all } x \in \mathcal{K}_i,\\
&\mathfrak{h}_i(1, x) = (\beta \circ m \circ h_i)(1, m^{-1}(\Phi_\lambda(x))) = \beta(m(\omega_i)) = x_i \in \lambda\Omega_R^+ \quad \text{for all } x \in \mathcal{K}_i.
\end{aligned}$$

Hence

$$\operatorname{cat}_{\lambda\Omega_R^+}(\lambda\Omega_R^-) \le n,$$

that is, 5.2 holds, which implies that $\tilde{\mathcal{S}}_+$ contains at least $\operatorname{cat}(\Omega_\lambda)$ critical points of $\hat{\Psi}$.

Now we prove that there exists another critical point of $\hat{\Psi}$. Let

$$\mathcal{M} := \overline{m^{-1}(\Phi_\lambda(\lambda\Omega_R^-))}$$

and note that $\mathcal{M}$ is non-contractible in $\tilde{\mathcal{S}}_+$ since $\Omega_\lambda$ is not contractible. Next, we will prove that there exists an energy level $d > c(\mathbb{R}^N) + h(\lambda)$ such that $\mathcal{M}$ is contractible in $\mathcal{S}_+^d := \{u \in \mathcal{S}_+ : \hat{\Psi}(u) \le d\}$. In order to do that, we choose $u^* \in \mathcal{S}_+$ such that $\hat{\Psi}(u^*) > c(\mathbb{R}^N) + h(\lambda)$ and define the set

$$\mathfrak{C} := \left\{\theta u^* + (1-\theta)u \, : \, \theta \in [0,1], \; u \in \mathcal{M}\right\}.$$

It is clear that $\mathfrak{C}$ is compact and contractible, moreover $0 \notin \mathfrak{C}$. Then we can define a continuous projection $P_{\mathcal{S}_+}$ from $\mathfrak{C}$ onto $\mathcal{S}_+$ by

$$P_{\mathcal{S}_+}(\mathfrak{C}) = \left\{m^{-1}(\hat{m}(w)) \, : \, w \in \mathfrak{C}\right\}.$$

Set

$$d := \max_{w \in P_{\mathcal{S}_+}(\mathfrak{C})} \hat{\Psi}(w).$$

It is obvious that $d > c(\mathbb{R}^N) + h(\lambda)$ and $P_{\mathcal{S}_+}(\mathfrak{C})$ is contractible in $\mathcal{S}_+^d$. Note that $\mathcal{M} \subseteq P_{\mathcal{S}_+}(\mathfrak{C}) \subseteq \mathcal{S}_+$, and so $\mathcal{M}$ is also contractible in $\mathcal{S}_+^d$. Consequently, it follows from Szulkin–Weth [37, Theorem 27] that there exists another critical point of $\hat{\Psi}$ in

$\mathcal{S}_+^d \setminus \tilde{\mathcal{S}}_+$. Thus we conclude from Lemma 2.3 that there exist at least $\text{cat}(\Omega_\lambda) + 1$ critical points of $J_\lambda$, that is, Problem 1.1 has at least $\text{cat}(\Omega_\lambda) + 1$ positive solutions. □

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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