Research article

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Convergence analysis for double phase obstacle problems with multivalued convection term

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Abstract: In the present paper, we introduce a family of the approximating problems corresponding to an elliptic obstacle problem with a double phase phenomena and a multivalued reaction convection term. Denoting by $\mathcal{S}$ the solution set of the obstacle problem and by $\mathcal{S}_n$ the solution sets of approximating problems, we prove the following convergence relation

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S},$$

where $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ and $s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ denote the weak and the strong Kuratowski upper limit of $\mathcal{S}_n$, respectively.

Keywords: Double phase problem, multivalued convection term, Kuratowski upper limit, Tychonov fixed point principle, obstacle problem

MSC: 35J20, 35J25, 35J60

1 Introduction

Recently, based on a surjectivity result for pseudomonotone operators obtained by Le [25], the authors [44] have studied the nonemptyness, boundedness and closedness of the set of weak solutions to the following double phase problem with a multivalued convection term and obstacle effect

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &\in f(x, u, \nabla u) && \text{in } \Omega,\\ u(x) &\leq \Phi(x) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $1 < p < q < N$, $\mu\colon \overline{\Omega} \to [0, \infty)$ is Lipschitz continuous, $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ is a multivalued function depending on the gradient of the solution and $\Phi\colon \Omega \to \mathbb{R}_+$ is a given function, see Section 3 for the precise assumptions.

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As the obstacle effect leads to various difficulties in obtaining the exact and numerical solutions, it is reasonable to consider some appropriate approximating methods to overcome/avoid the obstacle effect. In the present paper, we are going to propose a family of approximating problems corresponding to (1.1) and deliver an important convergence theorem which indicates that the solution set of the obstacle problem can be approximated by the solutions of perturbation problems. More precisely, let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_n \to 0$ as $n \to \infty$ and for each $n \in \mathbb{N}$, we consider the following problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + \frac{1}{\rho_n}\left(u(x) - \Phi(x)\right)^+ &\in f(x, u, \nabla u) \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Denoting by $\mathcal{S}$ and $\mathcal{S}_n$ the sets of solutions to problems (1.1) and (1.2), respectively, we shall establish the relations between the sets $\mathcal{S}$, w-$\limsup_{n\to\infty} \mathcal{S}_n$ (being the weak Kuratowski upper limit of $\mathcal{S}_n$) and s-$\limsup_{n\to\infty} \mathcal{S}_n$ (being the strong Kuratowski upper limit of $\mathcal{S}_n$), see Definition 2.2.

The introduction of so-called double phase operators goes back to Zhikov [46] who described models of strongly anisotropic materials by studying the functional

$$u \mapsto \int \left(|\nabla u|^p + \mu(x)|\nabla u|^q\right)\, dx. \tag{1.3}$$

The integral functional (1.3) is characterized by the fact that the energy density changes its ellipticity and growth properties according to the point in the domain. More precisely, its behavior depends on the values of the weight function $\mu(\cdot)$. Indeed, on the set $\{x \in \Omega : \mu(x) = 0\}$ it will be controlled by the gradient of order p and in the case $\{x \in \Omega : \mu(x) \neq 0\}$ it is the gradient of order q. This is the reason why it is called double phase.

Functionals of the expression (1.3) have been studied more intensively in the last five years. Concerning regularity results, we refer, for example, to the works of Baroni-Colombo-Mingione [4–6], Baroni-Kuusi-Mingione [7], Cupini-Marcellini-Mascolo [15], Colombo-Mingione [13], [14], Marcellini [28, 29] and the references therein.

Double phase differential operators and corresponding energy functionals appear in several physical applications. For example, in the elasticity theory, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents q and p, see Zhikov [47]. We also refer to other applications which can be found in the works of Bahrouni-Rădulescu-Repovš [1] on transonic flows, Benci-D'Avenia-Fortunato-Pisani [8] on quantum physics and Cherfils-Il′yasov [9] on reaction diffusion systems.

Existence and uniqueness results have been recently obtained by several authors. In the case of single-valued equations with or without convection term, we refer to Colasuonno-Squassina [12], Gasiński-Papageorgiou [16, 17], Gasiński-Winkert [19–21], Liu-Dai [27], Perera-Squassina [39], Papageorgiou-Vetro-Vetro [34, 35] and the references therein.

Finally, papers or monographs dealing with certain types of double phase problems or multivalued problems can be found in Bahrouni-Rădulescu-Repovš [1], Bahrouni-Rădulescu-Winkert [2], [3], Carl-Le-Motreanu [10], Cencelj-Rădulescu-Repovš [11], Clarke [22], Gasiński-Papageorgiou [18], Marino-Winkert [30], Papageorgiou-Rădulescu-Repovš [32, 33], Papageorgiou-Vetro-Vetro [37] , Rădulescu [40], Vetro [41], Vetro-Vetro [42], Zhang-Rădulescu [45], Zeng-Bai-Gasiński-Winkert [43] and the references therein.

The paper is organized as follows. In Section 2 we recall the definition of the Musielak-Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and its corresponding Sobolev spaces $W^{1,\mathcal{H}}(\Omega)$ and we recall the definition of the Kuratowski lower and upper limit, respectively. In Section 3 we present the full assumptions on the data of problem (1.2), give the definition of weak solutions for (1.1) as well as (1.2) and state and prove our main result, see Theorem 3.4.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $1 \leq r \leq \infty$. In what follows, we denote by $L^r(\Omega) := L^r(\Omega;\mathbb{R})$ and $L^r(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$. Moreover, $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ stand for the Sobolev spaces endowed with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0}$, respectively. For any $1 < r < \infty$ we denote by r' the conjugate of r, that is, $\frac{1}{r} + \frac{1}{r'} = 1$.

For the weight function μ and powers p, q we will assume that:
H(μ):$\mu\colon \overline{\Omega} \to \mathbb{R}_+ := [0,\infty)$ is Lipschitz continuous and $1 < p < q < N$ are chosen such that

$$\frac{q}{p} < 1 + \frac{1}{N}.$$

We consider the function $\mathcal{H}\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+.$$

Based on the definition of $\mathcal{H}$ we are able to introduce the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ given by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{ \tau > 0 \;\middle|\; \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1 \right\}.$$

We know that $L^{\mathcal{H}}(\Omega)$ is uniformly convex and so a reflexive Banach space. In addition, we introduce the seminormed function space

$$L^q_\mu(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mu(x)|u|^q\,dx < +\infty \right\},$$

which is equipped with the seminorm $\|\cdot\|_{q,\mu}$ given by

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q\,dx \right)^{\frac{1}{q}}.$$

It is known that the embeddings

$$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$$

are continuous, see Colasuonno-Squassina [12, Proposition 2.15 (i), (iv) and (v)]. Taking into account these embeddings we have the inequalities

$$\min\left\{ \|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}} \right\} \leq \|u\|^p_p + \|u\|^q_{q,\mu} \leq \max\left\{ \|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}} \right\} \tag{2.1}$$

for all $u \in L^{\mathcal{H}}(\Omega)$.

By $W^{1,\mathcal{H}}(\Omega)$ we denote the corresponding Sobolev space which is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \||\nabla u|\|_{\mathcal{H}}$.

By $W_0^{1,\mathcal{H}}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$, that is,

$$W_0^{1,\mathcal{H}}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\mathcal{H}}(\Omega)}.$$

Besides, from condition H(μ) and Colasuonno-Squassina [12, Proposition 2.18] we can see that

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega)$$

is an equivalent norm on $W_0^{1,\mathcal{H}}(\Omega)$. Now we are able to adapt (2.1) in terms of $W_0^{1,\mathcal{H}}(\Omega)$-norm as follows

$$\min\left\{\|u\|_{1,\mathcal{H},0}^p, \|u\|_{1,\mathcal{H},0}^q\right\} \le \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q \le \max\left\{\|u\|_{1,\mathcal{H},0}^p, \|u\|_{1,\mathcal{H},0}^q\right\} \tag{2.2}$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$. Since both spaces $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are uniformly convex, we know that they are reflexive Banach spaces.

Furthermore, we have the following compact embedding

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega) \tag{2.3}$$

for each $1 < r < p^*$, where p^* is the critical exponent to p given by

$$p^* := \frac{Np}{N-p}, \tag{2.4}$$

see Colasuonno-Squassina [12, Proposition 2.15].

Let us now consider the eigenvalue problem for the minus r-Laplacian with homogeneous Dirichlet boundary condition and $1 < r < \infty$ which is defined by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2} u \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

From Lê [26] we know that the set σ_r being the set of all eigenvalues of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ has a smallest element $\lambda_{1,r}$ which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,r} = \inf\left\{\frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right\}.$$

Now, let $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the operator defined by

$$\langle A(u), v\rangle_{\mathcal{H}} := \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla v\, dx, \tag{2.6}$$

for $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$.

The properties of the operator $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ can be summarized as follows, see Liu-Dai [27].

Proposition 2.1. *The operator A defined by* (2.6) *is bounded, continuous, monotone (hence maximal monotone) and of type* (S_+).

Throughout the paper the symbols "$\rightharpoonup$" and "$\to$" stand for the weak and the strong convergence, respectively. Let $(V, \|\cdot\|_V)$ be a Banach space with its dual V^* and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. We end this section by recalling the following definition, see, for example, Papageorgiou-Winkert [38, Definition 6.7.4].

Definition 2.2. *Let (X, τ) be a Hausdorff topological space and let $\{A_n\} \subset 2^X$ be a sequence of sets. We define the τ-Kuratowski lower limit of the sets A_n by*

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{n\to\infty} x_n,\ x_n \in A_n \text{ for all } n \ge 1 \right\},$$

and the τ-Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \ldots < n_k < \ldots \right\}.$$

If

$$A = \tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n,$$

then A is called τ-Kuratowski limit of the sets A_n.

3 Main results

We assume the following hypotheses on the data of problem (1.2).

H(f):The multivalued convection mapping $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ has nonempty, compact and convex values such that

(i) the multivalued mapping $x \mapsto f(x, s, \xi)$ has a measurable selection for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

(ii) the multivalued mapping $(s, \xi) \mapsto f(x, s, \xi)$ is upper semicontinuous for almost all (a. a.) $x \in \Omega$;

(iii) there exists $\alpha \in L^{\frac{q_1}{q_1-1}}(\Omega)$ and $a_1, a_2 \geq 0$ such that

$$|\eta| \leq a_1|\xi|^{p\frac{q_1-1}{q_1}} + a_2|s|^{q_1-1} + \alpha(x)$$

for all $\eta \in f(x, s, \xi)$, for a. a. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, where $1 < q_1 < p^*$ with the critical exponent p^* given in (2.4);

(iv) there exist $w \in L^1_+(\Omega)$ and $b_1, b_2 \geq 0$ such that

$$b_1 + b_2\lambda_{1,p}^{-1} < 1,$$

and

$$\eta s \leq b_1|\xi|^p + b_2|s|^p + w(x)$$

for all $\eta \in f(x, s, \xi)$, for a. a. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, where $\lambda_{1,p}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the p-Laplacian, see (2.5).

H(Φ):$\Phi\colon \Omega \to [0, \infty)$ is such that $\Phi \in L^{q_1'}(\Omega)$.

H(0):$\{\rho_n\}$ is a sequence with $\rho_n > 0$ for each $n \in \mathbb{N}$ such that $\rho_n \to 0$ as $n \to \infty$.

Let K be a subset of $W_0^{1,\mathcal{H}}(\Omega)$ defined by

$$K := \left\{ u \in W_0^{1,\mathcal{H}}(\Omega) \;\middle|\; u(x) \leq \Phi(x) \text{ for a. a. } x \in \Omega \right\}. \tag{3.1}$$

Remark 3.1.

(a) The set K is a nonempty, closed and convex subset of $W_0^{1,\mathcal{H}}(\Omega)$.

(b) From assumption H(Φ) *we see that $0 \in K$.*

The weak solutions for problems (1.1) and (1.2) are understood in the following way.

Definition 3.2.

(a) We say that $u \in K$ is a weak solution of problem (1.1) *if there exists $\eta \in L^{\frac{q_1}{q_1-1}}(\Omega)$ such that $\eta(x) \in f(x, u(x), \nabla u(x))$ for a. a. $x \in \Omega$ and*

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla(v-u)\right)\,dx = \int_\Omega \eta(x)(v-u)\,dx$$

for all $v \in K$, where K is given by (3.1).

(b) *We say that* $u \in W_0^{1,\mathcal{H}}(\Omega)$ *is a weak solution of problem* (1.2) *if there exists* $\eta \in L^{\frac{q_1}{q_1-1}}(\Omega)$ *such that* $\eta(x) \in f(x,u(x),\nabla u(x))$ *for a. a.* $x \in \Omega$ *and*

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla v\,dx + \frac{1}{\rho_n}\int_\Omega \left(u(x)-\Phi(x)\right)^+ v(x)\,dx = \int_\Omega \eta(x)v(x)\,dx$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$.

It is straightforward, to prove the following lemma.

Lemma 3.3. *If hypothesis* H(Φ) *holds, then the function* $B\colon L^{q_1}(\Omega) \to L^{q_1'}(\Omega)$ *given by*

$$\langle Bu, v\rangle_{q_1} = \int_\Omega \left(u(x)-\Phi(x)\right)^+ v(x)\,dx \quad \text{for all } u,v \in L^{q_1}(\Omega), \tag{3.2}$$

is bounded, demicontinuous and monotone, where $\langle\cdot,\cdot,\rangle_{q_1}$ *denotes the duality pairing between* $L^{q_1}(\Omega)$ *and its dual space* $L^{q_1'}(\Omega)$.

Now, we can state the main result of this paper.

Theorem 3.4. *If hypotheses* H(μ), H(f), H(Φ), *and* H(0) *hold, then*
(i) *for each* $n \in \mathbb{N}$, *the set* $\mathcal{S}_n$ *of solutions to problem* (1.2) *is nonempty, bounded and closed.*
(ii) *it holds*

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}.$$

(iii) *for each* $u \in s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ *and any sequence* $\{\widetilde{u}_n\}$ *with*

$$\widetilde{u}_n \in \mathcal{T}(\mathcal{S}_n, u) \quad \text{for each } n \in \mathbb{N},$$

there exists a subsequence of $\{\widetilde{u}_n\}$ *converging strongly to* u *in* $W_0^{1,\mathcal{H}}(\Omega)$, *where the set* $\mathcal{T}(\mathcal{S}_n,u)$ *is defined by*

$$\mathcal{T}(\mathcal{S}_n,u) := \left\{\widetilde{u} \in \mathcal{S}_n \mid \|u-\widetilde{u}\|_{1,\mathcal{H},0} \le \|u-v\|_{1,\mathcal{H},0} \text{ for all } v \in \mathcal{S}_n\right\}.$$

Proof. (i) Let $i\colon W_0^{1,\mathcal{H}}(\Omega) \to L^{q_1}(\Omega)$ be the embedding operator from $W_0^{1,\mathcal{H}}(\Omega)$ to $L^{q_1}(\Omega)$ with its adjoint operator $i^*\colon L^{q_1'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$. Since $1 < q_1 < p^*$ the embedding operator i is compact and so i^* as well. From hypotheses H(f)(i) and (iii), we see that the Nemytskij operator $\widetilde{N}_f\colon W_0^{1,\mathcal{H}}(\Omega) \subset L^{q_1}(\Omega) \to 2^{L^{q_1'}(\Omega)}$ associated to the multivalued mapping f given by

$$\widetilde{N}_f(u) := \left\{\eta \in L^{q_1'}(\Omega) \,\middle|\, \eta(x) \in f(x,u(x),\nabla u(x)) \text{ for a. a. } x \in \Omega\right\}$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$ is well-defined (see the proof of Proposition 3 in Papageorgiou-Vetro-Vetro [36]). The convexity and closedness of the values of f ensure that $\widetilde{N}_f$ has closed and convex values as well. Moreover, by hypothesis H(f)(iv) we have

$$\begin{aligned} \|\eta\|_{q_1'}^{q_1'} &= \int_\Omega |\eta(x)|^{q_1'}\,dx \\ &\le \int_\Omega \left(a_1|\nabla u(x)|^{\frac{p}{q_1'}} + a_2|u(x)|^{q_1-1} + \alpha(x)\right)^{q_1'} dx \\ &\le M_0 \int_\Omega |\nabla u(x)|^p + |u(x)|^{q_1} + \alpha(x)^{q_1'}\,dx \\ &= M_0\left(\|\nabla u\|_p^p + \|u\|_{q_1}^{q_1} + \|\alpha\|_{q_1'}^{q_1'}\right). \end{aligned} \tag{3.3}$$

Notice that the embeddings $W_0^{1,\mathcal{H}}(\Omega) \subset W_0^{1,p}(\Omega) \subset L^{q_1}(\Omega)$ are both continuous, so, $\widetilde{N}_f(u)$ is bounded in $L^{q_1'}(\Omega)$ for each $u \in W_0^{1,\mathcal{H}}(\Omega)$.

It is easy to see that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution of problem (1.2) (see Definition 3.2(b)), if and only if u solves the following inclusion:

Find $u \in W_0^{1,\mathcal{H}}(\Omega)$ and $\eta \in \widetilde{N}_f(u)$ such that

$$A(u) + \frac{1}{\rho_n} i^\star B(u) - i^\star \widetilde{N}_f(u) \ni 0,$$

where $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^\star$ and $B\colon L^{q_1}(\Omega) \to L^{q_1'}(\Omega)$ are given by (2.6) and (3.2), respectively.

Then, using the same arguments as in the proof of Zeng-Gasiński-Winkert-Bai [44, Theorem 3.3], we can conclude that for each $n \in \mathbb{N}$, the set $\mathcal{S}_n$ of solutions to problem (1.2) is nonempty, bounded and closed.

(ii) First, we prove that the set $w\text{-}\limsup\limits_{n\to\infty} \mathcal{S}_n$ is nonempty. Indeed, we have the following claims.

Claim 1. The set $\bigcup\limits_{n\in\mathbb{N}} \mathcal{S}_n$ is uniformly bounded in $W_0^{1,\mathcal{H}}(\Omega)$.

Arguing by contradiction, suppose that $\bigcup\limits_{n\in\mathbb{N}} \mathcal{S}_n$ is unbounded. Without any loss of generality (passing to a subsequence if necessary), we may assume that there exists a sequence $\{u_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $u_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$ such that

$$\|u_n\|_{1,\mathcal{H},0} \to \infty \quad \text{as } n \to \infty.$$

Hence, for each $n \in \mathbb{N}$, we are able to find $\eta_n \in \widetilde{N}_f(u_n)$ such that

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla v\,dx + \frac{1}{\rho_n}\int_\Omega \left(u_n(x) - \Phi(x)\right)^+ v(x)\,dx \\ &= \int_\Omega \eta_n(x)v(x)\,dx \end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Inserting $v = u_n$ into the inequality above, we get

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla u_n\,dx - \int_\Omega \eta_n(x)u_n(x)\,dx \\ &= -\frac{1}{\rho_n}\int_\Omega \left(u_n(x) - \Phi(x)\right)^+ u_n(x)\,dx. \end{aligned}$$

By the nonnegativity of Φ and the monotonicity of the function $s \mapsto s^+$, we have

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla u_n\,dx - \int_\Omega \eta_n(x)u_n(x)\,dx$$

$$= -\frac{1}{\rho_n}\int_\Omega \left[\left(u_n(x)-\Phi(x)\right)^+ - \left(0-\Phi(x)\right)^+\right] u_n(x)\,dx$$
$$\le 0,$$

thus

$$\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q - \int_\Omega \eta_n(x)u_n(x)\,dx \le 0. \tag{3.4}$$

However, by hypothesis H(f)(iv), we have

$$\int_\Omega \eta_n(x)u_n(x)\,dx \le b_1\|\nabla u_n\|_p^p + b_2\|u_n\|_p^p + \|w\|_1. \tag{3.5}$$

Applying (3.5) in (3.4), using the continuity of the embedding $W_0^{1,\mathcal{H}}(\Omega) \subseteq W_0^{1,p}(\Omega)$ as well as the estimate

$$\|u\|_p^p \le \lambda_{1,p}^{-1}\|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

we get

$$\begin{aligned}
0 &\ge \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q - \int_\Omega \eta_n(x)u_n(x)\,dx \\
&\ge \|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q - b_1\|\nabla u_n\|_p^p - b_2\|u_n\|_p^p - \|w\|_1 \\
&\ge \left(1-b_1-b_2\lambda_{1,p}^{-1}\right)\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q - \|w\|_1 \\
&\ge \left(1-b_1-b_2\lambda_{1,p}^{-1}\right)\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q\right) - \|w\|_1 \\
&\ge \left(1-b_1-b_2\lambda_{1,p}^{-1}\right)\min\left\{\|u_n\|_{1,\mathcal{H},0}^p, \|u_n\|_{1,\mathcal{H},0}^q\right\} - \|w\|_1,
\end{aligned}$$

where the last inequality is obtained by (2.2). Since $1 < p < q < N$ and $b_1 + b_2\lambda_{1,p}^{-1} < 1$, we can take $R_0 > 0$ large enough such that for all $R \ge R_0$ it holds

$$\left(1-b_1-b_2\lambda_{1,p}^{-1}\right)\min\left\{R^p, R^q\right\} - \|w\|_1 > 0.$$

Therefore, we are able to find $N_0 > 0$ large enough such that $\|u_n\|_{1,\mathcal{H},0} > R_0$ for all $n \ge N_0$ and

$$0 \ge \left(1-b_1-b_2\lambda_{1,p}^{-1}\right)\min\left\{\|u_n\|_{1,\mathcal{H},0}^p, \|u_n\|_{1,\mathcal{H},0}^q\right\} - \|w\|_1 > 0$$

for all $n \ge N_0$. This gives a contradiction, so Claim 1 is proved.

Let $\{u_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $u_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$ be an arbitrary sequence. Claim 1 indicates that $\{u_n\}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Then, we may assume that along a relabeled subsequence we have

$$u_n \rightharpoonup u \quad \text{as } n \to \infty \tag{3.6}$$

for some $u \in W_0^{1,\mathcal{H}}(\Omega)$. This guarantees that the set $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is nonempty.

Next, we are going to demonstrate that $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is a subset of $\mathcal{S}$. Let $u \in w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ be arbitrary. Without loss of generality, we may suppose that there exists a subsequence $\{u_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $u_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}$, satisfying (3.6). Our goal is to prove that $u \in \mathcal{S}$.

Claim 2. $u(x) \le \Phi(x)$ for a.a. $x \in \Omega$.

For every $n \in \mathbb{N}$, we have

$$\frac{1}{\rho_n}\int_\Omega \left(u_n(x)-\Phi(x)\right)^+ v(x)\,dx = \langle Au_n, -v\rangle_{\mathcal{H}} + \int_\Omega \eta_n(x)v(x)\,dx. \tag{3.7}$$

It follows from Hölder inequality and (3.3) that

$$\int_\Omega \eta_n(x)v(x)\,dx \le M_0^{\frac{1}{q_1'}} \left(\|\nabla u_n\|_p^p + \|u_n\|_{q_1}^{q_1} + \|\alpha\|_{q_1'}^{q_1'}\right)^{\frac{1}{q_1'}} \|v\|_{q_1}. \tag{3.8}$$

Putting (3.8) into (3.7), employing the boundedness of A (see Proposition 2.1), the convergence (3.6), and the embedding (2.3), we have

$$\begin{aligned}
&\frac{1}{\rho_n}\int_\Omega \left(u_n(x) - \Phi(x)\right)^+ v(x)\,dx \\
&\le \|Au_n\|_{1,\mathcal{H},0}\|v\|_{1,\mathcal{H},0} + M_0^{\frac{1}{q_1'}} \left(\|\nabla u_n\|_p^p + \|u_n\|_{q_1}^{q_1} + \|\alpha\|_{q_1'}^{q_1'}\right)^{\frac{1}{q_1'}} \|v\|_{q_1} \\
&\le M_1\|v\|_{1,\mathcal{H},0}
\end{aligned}$$

for some $M_1 > 0$, where $M_1 > 0$ is independent of n, that is

$$\int_\Omega \left(u_n(x) - \Phi(x)\right)^+ v(x)\,dx \le \rho_n M_1\|v\|_{1,\mathcal{H},0}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Passing to the limit in the above inequality, using convergence (3.6), the compact embedding (2.3), and the Lebesgue Dominated Convergence Theorem, we conclude that

$$\begin{aligned}
\int_\Omega \left(u(x) - \Phi(x)\right)^+ v(x)\,dx &= \int_\Omega \lim_{n\to\infty} \left(u_n(x) - \Phi(x)\right)^+ v(x)\,dx \\
&= \lim_{n\to\infty} \int_\Omega \left(u_n(x) - \Phi(x)\right)^+ v(x)\,dx \\
&\le \lim_{n\to\infty} \rho_n M_1\|v\|_{1,\mathcal{H},0} \\
&= 0
\end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Therefore, we have $\left(u(x) - \Phi(x)\right)^+ = 0$ for a.a. $x \in \Omega$, thus, $u(x) \le \Phi(x)$ for a.a. $x \in \Omega$.

Claim 3. $u \in \mathcal{S}$.

For each $n \in \mathbb{N}$, we have

$$\langle Au_n, u_n - v\rangle_{\mathcal{H}} = \frac{1}{\rho_n}\int_\Omega \left(u_n(x) - \Phi(x)\right)^+ (v(x) - u_n(x))\,dx + \int_\Omega \eta_n(x)(u_n(x) - v(x))\,dx$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. The latter combined with the monotonicity of $s \mapsto s^+$ gives

$$\langle Au_n, u_n - v\rangle_{\mathcal{H}} \le \frac{1}{\rho_n}\int_\Omega \left(v(x) - \Phi(x)\right)^+ (v(x) - u_n(x))\,dx + \int_\Omega \eta_n(x)(u_n(x) - v(x))\,dx$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Hence,

$$\langle Au_n, u_n - v\rangle_{\mathcal{H}} - \int_\Omega \eta_n(x)(u_n(x) - v(x))\,dx \le 0 \tag{3.9}$$

for all $v \in K$, where K is defined in (3.1).

Claim 2 indicates that $u \in K$, so, we put $v = u$ in (3.9) to obtain

$$\langle Au_n, u_n - u\rangle_{\mathcal{H}} - \int_\Omega \eta_n(x)(u_n(x) - u(x))\,dx \le 0,$$

that is,

$$\limsup_{n\to\infty}\langle Au_n - i^\star\eta_n, u_n - u\rangle_{\mathcal{H}} \le 0.$$

It follows from the proof of Theorem 3.3 in Zeng-Gasiński-Winkert-Bai [44] that the multivalued mapping $\mathcal{A} = A - i^\star\widetilde{N}_f$ is pseudomonotone. So, for each $v \in K$, there exists $u^\star \in \mathcal{A}u$ such that

$$\liminf_{n\to\infty}\langle Au_n - i^\star\eta_n, u_n - v\rangle_{\mathcal{H}} \ge \langle u^\star(v), u_n - v\rangle.$$

This means that for each $v \in K$, there is an element $\eta(v) \in \widetilde{N}_f(u)$ satisfying

$$u^\star(v) = Au - i^\star\eta(v).$$

For each $v \in K$, passing to the lower limit as $n \to \infty$ in inequality (3.9), we are able to find an element $\eta(v) \in \widetilde{N}_f(u)$ such that

$$\langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta(v)(x)(v(x) - u(x))\,dx \ge 0. \tag{3.10}$$

We shall prove that $u \in K$ is a weak solution to problem (1.1), namely, there exists an element $\eta^\star \in \widetilde{N}_f(u)$, which is independent of v, such that

$$\langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta^\star(x)(v(x) - u(x))\,dx \ge 0 \tag{3.11}$$

for all $v \in K$. Arguing by contradiction, suppose that for each $\eta \in \widetilde{N}_f(u)$, there is $v \in K$ such that

$$\langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta(x)(v(x) - u(x))\,dx < 0.$$

For any $v \in K$, let us consider the set $R_v \subset \widetilde{N}_f(u)$ defined by

$$R_v := \left\{\eta \in \widetilde{N}_f(u) \mid \langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta(x)(v(x) - u(x))\,dx < 0\right\}$$

for all $v \in K$. We now assert that for each $v \in K$, the set R_v is weakly open. Let $\{\eta_n\} \subset R_v^c$ be such that $\eta_n \rightharpoonup \eta$ for some $\eta \in L^{q_1'}(\Omega)$ as $n \to \infty$, where R_v^c denotes the complement of R_v. Hence,

$$\langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta_n(x)(v(x) - u(x))\,dx \ge 0$$

for all $n \in \mathbb{N}$. Passing to the limit in the above inequality, we obtain that $\eta \in R_v^c$. Therefore, for every $v \in K$, the set R_v is weakly open in $L^{q_1'}(\Omega)$. Besides, we observe that $\{R_v\}_{v\in K}$ is an open covering of $\widetilde{N}_f(u)$. The latter coupled with the facts that $L^{q_1}(\Omega)$ is reflexive and $\widetilde{N}_f(u)$ is weakly compact and convex in $L^{q_1'}(\Omega)$, ensures that $\{R_v\}_{v\in K}$ has a finite sub-covering of $\widetilde{N}_f(u)$, let us say $\{R_{v_1}, R_{v_2}, \dots, R_{v_n}\}$ for some points $\{v_1, v_2, \dots, v_n\} \subseteq K$. Let $\kappa_1, \kappa_2, \dots, \kappa_n$ be a partition of unity for $\widetilde{N}_f(u)$, where for each $i = 1, 2, \dots, n$, $\kappa_i\colon \widetilde{N}_f(u) \to [0, 1]$ is a weakly continuous function such that $\sum_{i=1}^n \kappa_i(\eta) = 1$ for all $\eta \in \widetilde{N}_f(u)$, see, for example, Granas-Dugundji [23, Lemma 7.3].

Also, we introduce a function $\mathcal{M}\colon \widetilde{N}_f(u) \to W_0^{1,\mathcal{H}}(\Omega)$ defined by

$$\mathcal{M}(\eta) = \sum_{i=1}^n \kappa_i(\eta)v_i \quad \text{for all } \eta \in \widetilde{N}_f(u).$$

Obviously, the function $\mathcal{M}$ is also weakly continuous due to the weak continuity of κ_i for $i = 1, 2, \ldots, n$. For any $\eta \in \widetilde{N}_f(u)$, we have

$$\begin{aligned}\langle Au - i^\star \eta, \mathcal{M}(\eta) - u\rangle_{\mathcal{H}} &= \Big\langle Au - i^\star \eta, \sum_{i=1}^{n} \kappa_i(\eta) v_i - u\Big\rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{n} \kappa_i(\eta)\langle Au - i^\star \eta, v_i - u\rangle_{\mathcal{H}} \\ &< 0\end{aligned} \tag{3.12}$$

for all $\eta \in \widetilde{N}_f(u)$, where the last inequality is obtained by the use of Lemma 7.3(ii) of Granas-Dugundji [23].

Let us define two multivalued functions $\Lambda\colon K \to 2^{\widetilde{N}_f(u)}$ and $\Psi\colon \widetilde{N}_f(u) \to 2^{\widetilde{N}_f(u)}$ by

$$\Lambda(v) := \Big\{\eta \in \widetilde{N}_f(u) \mid \langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta(x)(v(x) - u(x))\,dx \geq 0\Big\}$$

for all $v \in K$, and

$$\Psi(\eta) := \Lambda(\mathcal{M}(\eta)) \quad \text{for all } \eta \in \widetilde{N}_f(u).$$

Then, Ψ has nonempty, weakly compact and convex values (by (3.10) and because $\widetilde{N}_f(u)$ is bounded closed and convex in $L^{q_1'}(\Omega)$) and Λ is upper semicontinuous from the normal topology of K to weak topology of $L^{q_1'}(\Omega)$. From Migórski-Ochal-Sofonea [31, Proposition 3.8], it is enough to verify that for each weakly closed set D in $L^{q_1'}(\Omega)$, the set

$$\Lambda^-(D) := \{v \in K \mid \Lambda(v) \cap D \neq \emptyset\}$$

is closed in $W_0^{1,\mathcal{H}}(\Omega)$. Let $\{v_n\} \subset \Lambda^-(D)$ be a sequence such that $v_n \to v$ as $n \to \infty$. Then, for each $n \in \mathbb{N}$, we are able to find $\eta_n \in \widetilde{N}_f(u)$ satisfying

$$\langle Au, v_n - u\rangle_{\mathcal{H}} - \int_\Omega \eta_n(x)(v_n(x) - u(x))\,dx \geq 0. \tag{3.13}$$

From the weak compactness of $\widetilde{N}_f(u)$, without any loss of generality, we may suppose that $\eta_n \rightharpoonup \eta$ in $L^{q_1'}(\Omega)$, as $n \to \infty$, for some $\eta \in \widetilde{N}_f(u)$. Passing to the upper limit as $n \to \infty$ for (3.13), we have

$$\langle Au, v - u\rangle_{\mathcal{H}} - \int_\Omega \eta(x)(v(x) - u(x))\,dx \geq 0,$$

that is, $\eta \in \Lambda(v)$. But, the weak closedness of D implies that $\eta \in D$. Therefore, $\eta \in \Lambda(v) \cap D$ and so $v \in \Lambda^-(D)$. Applying Migórski-Ochal-Sofonea [31, Proposition 3.8] derives that Λ is strongly-weakly upper semicontinuous. On the other hand, the continuity of $\mathcal{M}$ and Theorem 1.2.8 of Kamenskii-Obukhovskii-Zecca [24] imply that Ψ is also strongly-weakly upper semicontinuous.

We are now in a position to employ Tychonov fixed point principle, (see, for example, Granas-Dugundji [23, Theorem 8.6]) for function Ψ, to conclude that there exists $\eta \in \widetilde{N}_f(u)$ such that

$$\langle Au, \mathcal{M}(\eta) - u\rangle_{\mathcal{H}} - \int_\Omega \eta(x)(\mathcal{M}(\eta)(x) - u(x))\,dx \geq 0.$$

This leads to a contraction with (3.12). Consequently, we infer that $u \in K$ solves problem (1.1) as well, that means, there exists $\eta \in \widetilde{N}_f(u)$, which is independent of v, such that (3.11) holds.

Consequently, we conclude that $\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.

Claim 4. It holds $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$.

Since $s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$, it is enough to verify the condition $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$. Let $u \in w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ be arbitrary. Without any loss of generality, there exists a sequence, still denoted by $\{u_n\}$ with $u_n \in \mathcal{S}_n$ such that $u_n \rightharpoonup u$ as $n \to \infty$. We claim that $u_n \to u$ as $n \to \infty$. For each $n \in \mathbb{N}$, it holds

$$\langle Au_n, u_n - v\rangle_{\mathcal{H}} = -\int_\Omega (u_n(x) - \Phi(x))^+(u_n(x) - v(x))\,dx + \int_\Omega \eta_n(x)(u_n(x) - v(x))\,dx$$

for some $\eta_n \in \widetilde{N}_f(u_n)$ and for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Inserting $v = u$ into the above inequality and passing to the upper limit as $n \to \infty$ for the resulting inequality, we can use the compact embedding (2.3) to get

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{\mathcal{H}} \le 0.$$

The latter combined with the convergence $u_n \rightharpoonup u$ as $n \to \infty$ and the fact that A is of type (S_+) (see Proposition 2.1) implies that $u_n \to u$ as $n \to \infty$. This means that $u \in s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$. Therefore $s\text{-}\limsup_{n\to\infty} \mathcal{S}_n = w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$.

(iii) Let $u \in s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ be arbitrary. Since $\mathcal{S}_n$ is nonempty, bounded and closed, so, the set $\mathcal{T}(\mathcal{S}_n, u)$ is nonempty. Let $\{\widetilde{u}_n\}$ be any sequence such that

$$\widetilde{u}_n \in \mathcal{T}(\mathcal{S}_n, u) \quad \text{for each } n \in \mathbb{N}.$$

It follows from Claim 1 that the sequence $\{\widetilde{u}_n\}$ is bounded. So, passing to a subsequence, we may assume, that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{as } n \to \infty$$

for some $\widetilde{u} \in W_0^{1,\mathcal{H}}(\Omega)$. Thus, using the same argument as the proof of Claim 2, we get that $\widetilde{u} \in K$. Then, for each $n \in \mathbb{N}$, we have

$$\langle A\widetilde{u}_n, \widetilde{u}_n - v\rangle_{\mathcal{H}} = \frac{1}{\rho_n}\int_\Omega \left(\widetilde{u}_n(x) - \Phi(x)\right)^+ (v(x) - \widetilde{u}_n(x))\,dx + \int_\Omega \eta_n(x)(\widetilde{u}_n(x) - v(x))\,dx$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Proceeding in the same way as in the proof of Claim 3, we conclude that $\widetilde{u}$ is a solution to problem (1.1) as well. Consequently, the desired conclusion is proved. □

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