Discrete and Continuous Dynamical Systems - Series S
Vol. 16, No. 11, November 2023, pp. 3333-3349
doi:10.3934/dcdss.2023095

NODAL SOLUTIONS FOR CRITICAL ROBIN DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENT

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Abstract. In this paper, we study a nonlinear double phase problem with variable exponent and critical growth on the boundary. The problem has in the reaction the combined effects of a Carathéodory perturbation defined only locally and of a critical term. The presence of the critical term does not permit to apply results of the critical point theory to the corresponding energy functional. Thus, we use appropriate cut-off functions and truncation techniques to work on an auxiliary coercive problem. In this way, we can use variational tools to get a sequence of sign changing solutions to our main problem. Further, we show that such a sequence converges to 0 in L^∞ and in the Musielak-Orlicz Sobolev space.

1. Introduction.

Given a bounded domain $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, we deal with critical Robin double phase problems of the form

$$\begin{aligned} -\operatorname{div} A(u) &= f(z,u) + |u|^{p^*-2}u \quad && \text{in } \Omega,\\ A(u)\cdot\nu + \beta(z)|u|^{p_*-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\operatorname{div} A(u)$ is the variable exponent double phase operator given by

$$\operatorname{div} A(u) := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(z)|\nabla u|^{q(z)-2}\nabla u\right),$$

$\nu(z)$ denotes the unit normal of Ω at the point $z \in \partial\Omega$ and

$$p^* := \frac{Np}{N-p} \quad \text{and} \quad p_* := \frac{(N-1)p}{N-p}$$

are the critical exponents corresponding to p. Here, we suppose that the exponents and the functions μ as well as β satisfy the following conditions:

(H1) $q \in C(\overline{\Omega})$ is such that $1 < p < N$, $p < q(z) < p^*$ for all $z \in \overline{\Omega}$. Moreover, $\mu \in L^\infty(\Omega)$ and $\beta \in L^\infty(\partial\Omega)$ are such that $\mu(z) > 0$ for a.a. $z \in \Omega$ and $\beta(z) \geq 0$ for a.a. $z \in \partial\Omega$ with $\beta \neq 0$.

For $r \in C(\overline{\Omega})$, we put

$$r^- = \min_{z\in\overline{\Omega}} r(z) \quad \text{and} \quad r^+ = \max_{z\in\overline{\Omega}} r(z).$$

We suppose the following assumptions on the data of problem (1.1).

2020 *Mathematics Subject Classification.* 35A01, 35D30, 35J60, 35J62, 35J66.

Key words and phrases. Critical problem, double phase operator, existence results, multiple solutions, sign changing solutions, variable exponent, critical growth on the boundary.

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(H2) $f\colon \Omega \times [-\eta_0, \eta_0] \to \mathbb{R}$ is a Carathéodory function with $\eta_0 > 0$ such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot)$ is odd and it holds

(i) there exists $a_0 \in L^\infty(\Omega)$ such that

$$|f(z,x)| \le a_0(z) \quad \text{for a.a. } z \in \Omega \text{ and for all } |x| \le \eta_0;$$

(ii) there exists $\tau \in \left(1, \min\left\{p, \frac{p^2}{N-p} + 1\right\}\right)$ such that

$$\lim_{x \to 0} \frac{f(z,x)}{|x|^{\tau-2}x} = 0 \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii)

$$\lim_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 1.1. We point out that f is defined only locally. Hence, on account of hypothesis (H2)(iii) we can assume, without loss of generality, that

$$\frac{f(z,x)}{|x|^{p-2}x} > 0 \qquad \text{for a.a. } z \in \Omega \text{ and for all } 0 < |x| \le \eta_0.$$

This guarantees that

$$f(z,x) > 0 \quad \text{for all } 0 < x \le \eta_0 \quad \text{and} \quad f(z,x) < 0 \quad \text{for all } -\eta_0 \le x < 0.$$

We say that a function $u \in W^{1,\mathcal{H}}(\Omega)$ (the Musielak-Orlicz Sobolev space, see Section 2) is a weak solution of problem (1.1) if

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(z)\,|\nabla u|^{q(z)-2}\nabla u\right) \cdot \nabla h \,\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^{p_*-2}uh \,\mathrm{d}\sigma \\
&= \int_\Omega \left(f(z,u) + |u|^{p^*-2}u\right) h \,\mathrm{d}z
\end{aligned}$$

is fulfilled for all $h \in W^{1,\mathcal{H}}(\Omega)$.

Then, the main result of the paper is the following.

Theorem 1.2. *Let hypotheses (H1) and (H2) be satisfied. Then, problem (1.1) has a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ of nodal (that is, sign changing) solutions such that $\|w_n\|_{1,\mathcal{H}} \to 0$ and $\|w_n\|_\infty \to 0$.*

We stress that our work is closely connected to a recent paper of Liu-Papageorgiou [29] where a similar double phase problem with constant exponents was studied in a Dirichlet setting. Precisely as in [29], we find in the right-hand side of problem (1.1) the combined effects of a Carathéodory perturbation $f(z,\cdot)$ which is defined only locally and of a critical term $u \to |u|^{p^*-2}u$. In contrast to [29], we allow a variable exponent $q \in C(\overline{\Omega})$ and critical growth on the boundary.

We point out that the presence of critical terms do not permit to apply results of the critical point theory to the corresponding energy functional. Therefore, we use suitable cut-off functions and truncation techniques to work with a coercive functional. In this way, we can act by using variational tools. Hence, we study a nonlinear auxiliary coercive problem. We establish the existence of extremal constant sign solutions for such a problem (see Section 3). Next, we use these extremal solutions and a generalized version of the symmetric mountain pass theorem due to Kajikiya [24, Theorem 1] in order to get a sequence of nodal solutions for problem (1.1). So, we extend the results of Liu-Papageorgiou [29] to the Robin double phase operator with one variable exponent and further we can skip hypothesis H1(iii) in [29]. In addition, we remark that our main result extends the recent work Papageorgiou-Vetro-Winkert [35] to Robin double phase problems with critical growth even on the boundary.

Functionals of type

$$\omega \mapsto \int_\Omega \left(|\nabla \omega|^p + \mu(z)|\nabla \omega|^q\right) \mathrm{d}z, \quad 1 < p < q < N,$$

were first considered by Zhikov [44] in connection with problems of homogenization and nonlinear elasticity. In addition, there are several other applications in the study of duality theory and of the Lavrentiev gap phenomenon, see Zhikov [45, 46]. A first mathematical framework for such type of functionals has been introduced by Baroni-Colombo-Mingione [8], see also the related works by the same authors in [9, 10] and of De Filippis-Mingione [15] about nonautonomous integrals.

However so far, there are only few results for problems involving the variable exponent double phase operator. We refer to the recent results of Aberqi-Bennouna-Benslimane-Ragusa [1] for existence results in complete manifolds, Albalawi-Alharthi-Vetro [2] for convection problems with $(p(\cdot), q(\cdot))$-Laplace type problems, Bahrouni-Rădulescu-Winkert [6] for double phase problems of Baouendi-Grushin type operator, Crespo-Blanco-Gasiński-Harjulehto-Winkert [13] for double phase convection problems, Kim-Kim-Oh-Zeng [25] for concave-convex-type double-phase problems, Leonardi-Papageorgiou [26] for concave-convex problems, Vetro-Winkert [40] for parametric problems involving superlinear nonlinearities and Zeng-Rădulescu-Winkert [43] for multivalued problems, see also the references therein.

Furthermore, we mention the works of Ambrosio-Isernia [3] for (p, q)-Schrödinger-Kirchhoff type equations, Ambrosio-Rădulescu [4] for concentrating solutions for (p, q)-Schrödinger equations, Ambrosio-Repovš [5] for multiplicity and concentration results for (p, q)-Laplacian problems in $\mathbb{R}^N$, Bai-Papageorgiou-Zeng [7] for singular eigenvalue problems for (p, q)-equations, Cen-Khan-Motreanu-Zeng [11] for inverse problems for generalized quasi-variational inequalities, Crespo-Blanco-Papageorgiou-Winkert [14] for double phase problems with singular term and critical growth on the boundary, Colasuonno-Squassina [12] for eigenvalue problems of double phase type, Farkas-Winkert [17] for Finsler double phase problems, Gasiński-Papageorgiou [18] for locally Lipschitz right-hand sides, Gasiński-Winkert [20, 19] for convection problems and constant sign-solutions, Liu-Dai [28] for a Nehari manifold approach, Papageorgiou-Vetro [33] for superlinear problems, Papageorgiou-Vetro-Vetro [34] for parametric Robin problems, Perera-Squassina [37] for Morse theoretical approach, Vetro-Winkert [39] for parametric convective problems, Zeng-Rădulescu-Winkert [42] for double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions, Zeng-Bai-Gasiński-Winkert [41] for implicit obstacle problems with multivalued operators.

2. **Preliminaries.** The function space framework for the analysis of problem (1.1) is provided by the so-called Musielak-Orlicz Sobolev spaces. Therefore, we devote this section to recall some elements from theory of such spaces. Also, we introduce some tools which we will need later.

We denote by $M(\Omega)$ the space of all functions $u\colon \Omega \to \mathbb{R}$ which are measurable. Let $r \in C(\overline{\Omega})$ be such that $r(z) > 1$ for all $z \in \overline{\Omega}$, then we denote by $L^{r(\cdot)}(\Omega)$ the usual variable exponent Lebesgue space defined by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) \,:\, \varrho_r(u) := \int_\Omega |u(z)|^{r(z)} \,\mathrm{d}z < +\infty \right\},$$

and endowed it with the Luxemburg norm

$$\|u\|_{r(\cdot)} := \inf\left\{\alpha > 0 \,:\, \varrho_r\left(\frac{u}{\alpha}\right) \leq 1\right\}.$$

In addition, we write $W^{1,r(\cdot)}(\Omega)$ for the corresponding Sobolev space equipped with the norm $\|\cdot\|_{1,r(\cdot)}$, see Diening-Harjulehto-Hästö-Růžička [16] or Harjulehto-Hästö [22].

Further, on $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$ and, using this measure, we define in the usual way the boundary Lebesgue spaces $L^{r(\cdot)}(\partial\Omega)$.

Next, we assume that (H1) holds. Then, we consider the nonlinear function $\mathcal{H}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{H}(z,x) = x^p + \mu(z)x^{q(z)} \quad \text{for all } z \in \Omega \text{ and for all } x \geq 0.$$

We write $\rho_{\mathcal{H}}$ for the corresponding modular function, that is,

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(z,|u|)\,\mathrm{d}z = \int_\Omega \left(|u|^p + \mu(z)\,|u|^{q(z)}\right)\mathrm{d}z.$$

Hypotheses (H1) guarantees that $\mathcal{H}$ is a generalized N-function satisfying the so-called Δ_2-condition, i.e., $\mathcal{H}(z,2x) \leq 2^{q^+}\mathcal{H}(z,x)$ for all $z \in \Omega$ and for all $x \geq 0$. Now we can define the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{\alpha > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\alpha}\right) \leq 1\right\}.$$

The modular $\rho_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$ are related by the following proposition, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [13, Proposition 2.13] or Harjulehto-Hästö [22].

Proposition 2.1. *Let hypotheses (H1) be satisfied. Then the following hold:*

(i) $\|u\|_{\mathcal{H}} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{\mathcal{H}}(u) < 1$ *(resp.* $> 1, = 1$*);*
(ii) *if* $\|u\|_{\mathcal{H}} < 1$ *then* $\|u\|_{\mathcal{H}}^{q^+} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{p}$*;*
(iii) *if* $\|u\|_{\mathcal{H}} > 1$ *then* $\|u\|_{\mathcal{H}}^{p} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{q^+}$*;*
(iv) $\|u\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(u) \to 0$*;*
(v) $\|u\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(u) \to +\infty$*.*

Now, using the Musielak-Orlicz space, we define the corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}.$$

We equip this space with the norm

$$\|u\|_{1,\mathcal{H}} := \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} := \|\,|\nabla u|\,\|_{\mathcal{H}}$. Note that the norm $\|\cdot\|_{\mathcal{H}}$ defined on $L^{\mathcal{H}}(\Omega)$ is uniformly convex and hence the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces, see [13, Proposition 2.12]. Furthermore, for $W^{1,\mathcal{H}}(\Omega)$ the following embedding results hold, see [13, Propositions 2.16 and 2.18].

Proposition 2.2. *Let hypotheses (H1) be satisfied. Then the following hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$ *are continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(z) \leq p$ *for all* $z \in \overline{\Omega}$*;*

(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is compact for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(z) < p^*$ *for all* $z \in \overline{\Omega}$;
(iii) $W^{1,H}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *is compact for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(z) < p_*$ *for all* $z \in \overline{\Omega}$;
(iv) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is compact;*
(v) $L^{q(\cdot)}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

Also, we denote by $C^1(\overline{\Omega})_+$ the positive cone of the ordered Banach space $C^1(\overline{\Omega})$ given by

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega}) \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\},$$

Next, we consider the eigenvalue problem for the p-Laplacian with Robin boundary condition given in the form

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\nabla u \cdot \nu + \beta(z)|u|^{p-2}u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of (2.1) if problem (2.1) has a nontrivial solution $u \in W^{1,p}(\Omega)$. Such a solution is called eigenfunction corresponding to the eigenvalue λ. It is known that there exists a smallest eigenvalue $\lambda_{1,p}$ of problem (2.1) which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,p} := \inf\left\{\frac{\|\nabla u\|_p^p + \int_{\partial\Omega}\beta(z)|u|^p \,\mathrm{d}\sigma}{\|u\|_p^p} \,:\, u \in W^{1,p}(\Omega),\ u \neq 0\right\},$$

see Papageorgiou-Rădulescu [30].

In the sequel, we write $u_{1,p}$ for the L^p-normalized (i.e., $\|u_{1,p}\|_p = 1$) positive eigenfunction corresponding to $\lambda_{1,p}$. From the nonlinear regularity theory and the nonlinear strong maximum principle we know that $u_{1,p} \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, see Lieberman [27] and Vázquez [38].

For any $s \in \mathbb{R}$ we put $s_\pm = \max\{\pm s, 0\}$, that means $s = s_+ - s_-$ and $|s| = s_+ + s_-$. Also, for any function $v\colon \Omega \to \mathbb{R}$ we put $v_\pm(\cdot) = [v(\cdot)]_\pm$.

Finally, given a Banach space X and its dual space X^*, we say that a functional $\Phi \in C^1(X)$ satisfies the Palais-Smale condition (PS-condition for short), if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\Phi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\Phi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence. Moreover, we denote by K_Φ the set of all critical points of Φ, that is,

$$K_\Phi = \left\{u \in X \,:\, \Phi'(u) = 0\right\}.$$

We also recall that a set $\mathcal{S} \subseteq X$ is said to be downward directed if for given $u_1, u_2 \in \mathcal{S}$ we can find $u \in \mathcal{S}$ such that $u \leq u_1$ and $u \leq u_2$. Analogously, $\mathcal{S} \subseteq X$ is said to be upward directed if for given $v_1, v_2 \in \mathcal{S}$ we can find $v \in \mathcal{S}$ such that $v_1 \leq v$ and $v_2 \leq v$.

3. **A nonlinear auxiliary problem.** In this section, we work with a nonlinear auxiliary problem and show the existence of extremal constant sign solutions for such a problem. This will help us in the next section to produce nodal (that is, sign changing) solutions for problem (1.1).

Before introducing such an auxiliary problem, we first prove a result which we will use later.

Lemma 3.1. *Let hypotheses (H1) be satisfied. Then, for some $a > 0$ and all $u \in W^{1,\mathcal{H}}(\Omega)$, we have*

$$\|u\|_{1,p} \leq a\left[\|\nabla u\|_p + \|u\|_{p_*,\beta,\partial\Omega}\right]$$

where $\|u\|_{p_*,\beta,\partial\Omega}$ is the seminorm

$$\|u\|_{p_*,\beta,\partial\Omega} := \int_{\partial\Omega} \beta(z)|u|^{p_*}\,\mathrm{d}\sigma.$$

Proof. The assertion of the lemma follows if we show that there exists $\hat{a} > 0$ such that

$$\|u\|_p \leq \hat{a}\left[\|\nabla u\|_p + \|u\|_{p_*,\beta,\partial\Omega}\right] \tag{3.1}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$. Indeed, since $\|u\|_{1,p} := \|\nabla u\|_p + \|u\|_p$, (3.1) gives

$$\|u\|_{1,p} \leq \|\nabla u\|_p + \hat{a}\left[\|\nabla u\|_p + \|u\|_{p_*,\beta,\partial\Omega}\right] \leq (\hat{a}+1)\left[\|\nabla u\|_p + \|u\|_{p_*,\beta,\partial\Omega}\right].$$

In order to prove the validity of (3.1), we argue indirectly. So, assume that (3.1) is not true. Then we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}}(\Omega)$ such that

$$\|u_n\|_p \geq n\left[\|\nabla u_n\|_p + \|u_n\|_{p_*,\beta,\partial\Omega}\right] \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

We put $y_n = \frac{u_n}{\|u_n\|_p}$ which gives $\|y_n\|_p = 1$. Then, from (3.2) we obtain

$$\frac{1}{n} \geq \|\nabla y_n\|_p + \|y_n\|_{p_*,\beta,\partial\Omega}. \tag{3.3}$$

Hence, we deduce that $\{y_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded (we recall that $\|y_n\|_{1,p} = \|\nabla y_n\|_p + \|y_n\|_p$). Therefore, we can assume that

$$y_n \rightharpoonup y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \rightharpoonup y \text{ in } \; L^{p^*}(\Omega) \text{ and } \; L^{p_*}(\partial\Omega). \tag{3.4}$$

Next, we know that $W^{1,p}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact (being $q^+ < p^*$) and further $L^{q(\cdot)}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous (see Proposition 2.2 (v)). On account of this, we conclude that $y_n \to y$ in $L^{\mathcal{H}}(\Omega)$. Also, from $\|y_n\|_p = 1$, we have $y \neq 0$.

Now, passing to limit in (3.3) as $n \to +\infty$ and using (3.4) along with the weak lower semicontinuity of the norm $\|\nabla \cdot\|_p$ and of the seminorm $\|\cdot\|_{p_*,\beta,\partial\Omega}$ we obtain that

$$0 \geq \|\nabla y\|_p + \|y\|_{p_*,\beta,\partial\Omega},$$

which gives $\nabla y = 0$. Hence, we deduce that $y = \hat{b}$ is a constant with $\hat{b} \neq 0$. Therefore, by (3.3) passing to limit as $n \to +\infty$ we get

$$0 \geq |\hat{b}|^{p_*}\int_{\partial\Omega} \beta(z)\,\mathrm{d}\sigma > 0$$

since $\beta(z) \geq 0$ for a.a. $z \in \Omega$ and $\beta \neq 0$. This gives a contradiction and so (3.1) holds true. □

Now, let $k\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$k(z,x) = \theta(x)[f(z,x) + |x|^{p^*-2}x] + (1-\theta(x))|x|^{\tau-2}x \tag{3.5}$$

for all $(z,x) \in \Omega \times \mathbb{R}$, where τ is given in (H2)(ii) and $\theta \in C^1(\mathbb{R})$ is an even cut-off function satisfying the following conditions:

$$\operatorname{supp}\theta \subseteq [-\eta_0,\eta_0], \quad \theta\big|_{[\frac{-\eta_0}{2},\frac{\eta_0}{2}]} \equiv 1 \quad \text{and} \quad 0 < \theta \le 1 \text{ on } (-\eta_0,\eta_0). \tag{3.6}$$

Note that (3.6) and (H2)(ii) ensure that

$$|k(z,x)| \le c\left(1+|x|^{\tau-1}\right) \tag{3.7}$$

for some $c>0$, for a.a. $z \in \Omega$ and for all $x \in \mathbb{R}$.

Then, we consider the following auxiliary Robin double phase problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(z)\,|\nabla u|^{q(z)-2}\nabla u\right) &= k(z,u) && \text{in } \Omega,\\ \left(|\nabla u|^{p-2}\nabla u + \mu(z)|\nabla u|^{q(z)-2}\nabla u\right)\cdot \nu &= -\beta(z)|u|^{p_*-2}u && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

We denote by $\mathcal{S}_+$ and $\mathcal{S}_-$ the set of positive and negative solutions of problem (3.8), respectively. We start showing that these sets are nonempty.

Proposition 3.2. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then $\mathcal{S}_+$ and $\mathcal{S}_-$ are nonempty subsets in $W^{1,\mathcal{H}}(\Omega)\cap L^\infty(\Omega)$.*

Proof. Our first aim is to prove that $\mathcal{S}_+ \neq \emptyset$. Thus, we consider the C^1-functional $\Phi_+\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \Phi_+(u) = &\int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(z)}{q(z)}|\nabla u|^{q(z)}\right]\mathrm{d}z + \int_{\partial\Omega}\frac{1}{p_*}\beta(z)|u|^{p_*}\,\mathrm{d}\sigma\\ &- \int_\Omega K(z,u_+)\,\mathrm{d}z, \end{aligned}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$, where $K(z,x) = \int_0^x k(z,s)\,\mathrm{d}s$. We know that

$$\begin{aligned} \Phi_+(u) \ge \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}z &+ \frac{1}{q^+}\int_\Omega \mu(z)|\nabla u|^{q(z)}\,\mathrm{d}z\\ &+ \frac{1}{p_*}\int_{\partial\Omega}\beta(z)|u|^{p_*}\,\mathrm{d}\sigma - \int_\Omega K(z,u_+)\,\mathrm{d}z. \end{aligned}$$

Then, using (3.7), hypothesis (H2)(ii) (which gives $\tau < p$) and hypothesis (H1) (which ensures $\beta(z) \ge 0$ for a.a. $z \in \partial\Omega$), we conclude that Φ_+ is coercive. Next, we recall that the embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ is compact for any $r \in C(\overline{\Omega})$ with $1 \le r(z) < p^*$ for all $z \in \overline{\Omega}$ (see Proposition 2.2(ii)). On account of this, we see that the functional Φ_+ is sequentially weakly lower semicontinuous. Therefore, there exists $u_0 \in W^{1,\mathcal{H}}(\Omega)$ such that

$$\Phi_+(u_0) = \inf\left[\Phi_+(u)\,:\, u \in W^{1,\mathcal{H}}(\Omega)\right].$$

Next, we show that u_0 is nontrivial. By hypothesis (H2)(iii) we can find for each $\eta > 0$ a number $\delta \in \left(0, \min\{\frac{\eta_0}{2},1\}\right)$ such that

$$F(z,x) = \int_0^x f(z,s)\,\mathrm{d}s \ge \frac{\eta}{p}|x|^p \quad \text{for all } |x| \le \delta. \tag{3.9}$$

Also, we can choose $t \in (0,1)$ small enough so that $tu_{1,p}(z) \in (0,\delta]$ for all $z \in \overline{\Omega}$, where $u_{1,p} \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ denotes the L^p-normalized positive eigenfunction corresponding to $\lambda_{1,p}$ (see Section 2). According of this, we have

$$\begin{aligned}
\Phi_+(tu_{1,p}) &= \int_\Omega \left[\frac{1}{p}|\nabla (tu_{1,p})|^p + \frac{\mu(z)}{q(z)}|\nabla (tu_{1,p})|^{q(z)}\right] \mathrm{d}z \\
&\quad + \int_{\partial\Omega} \frac{1}{p_*}\beta(z)(tu_{1,p})^{p_*}\,\mathrm{d}\sigma - \int_\Omega K\left(z, tu_{1,p}\right)\,\mathrm{d}z \\
&\leq \frac{t^p}{p}\int_\Omega |\nabla u_{1,p}|^p\,\mathrm{d}z + \frac{t^{q^-}}{q^-}\int_\Omega \mu(z)\,|\nabla u_{1,p}|^{q(z)}\,\mathrm{d}z \\
&\quad + \frac{t^{p_*}}{p_*}\int_{\partial\Omega}\beta(z)u_{1,p}^{p_*}\,\mathrm{d}\sigma - \int_\Omega K\left(z, tu_{1,p}\right)\,\mathrm{d}z \\
&= \frac{t^p}{p}\left[\lambda_{1,p}\int_\Omega u_{1,p}^p\,\mathrm{d}z - \int_{\partial\Omega}\beta(z)\,u_{1,p}^p\mathrm{d}\,\sigma\right] \\
&\quad + \frac{t^{q^-}}{q^-}\int_\Omega \mu(z)\,|\nabla u_{1,p}|^{q(z)}\,\mathrm{d}z + \frac{t^{p_*}}{p_*}\int_{\partial\Omega}\beta(z)u_{1,p}^{p_*}\,\mathrm{d}\sigma \\
&\quad - \int_\Omega K\left(z, tu_{1,p}\right)\,\mathrm{d}z \\
&\leq \frac{t^p}{p}\lambda_{1,p} + \frac{t^{q^-}}{q^-}\int_\Omega \mu(z)|\nabla u_{1,p}|^{q(z)}\,\mathrm{d}z + \frac{t^{p_*}}{p_*}\int_{\partial\Omega}\beta(z)u_{1,p}^{p_*}\,\mathrm{d}\sigma \\
&\quad - \int_\Omega K\left(z, tu_{1,p}\right)\,\mathrm{d}z.
\end{aligned}$$

Now, since $tu_{1,p} \in (0,\delta]$ and $\delta \in \left(0, \min\{\frac{\eta_0}{2}, 1\}\right)$, from (3.6) we know that

$$k(z, tu_{1,p}) = f(z, tu_{1,p}) + (tu_{1,p})^{p^*-2}tu_{1,p} \geq f(z, tu_{1,p}). \tag{3.10}$$

Then, using (3.9) and (3.10), we get

$$\begin{aligned}
\Phi_+(tu_{1,p}) &\leq \frac{t^p}{p}\lambda_{1,p} + \frac{t^{q^-}}{q^-}\int_\Omega \mu(z)|\nabla u_{1,p}|^{q(z)}\,\mathrm{d}z \\
&\quad + \frac{t^{p_*}}{p_*}\int_{\partial\Omega}\beta(z)u_{1,p}^{p_*}\,\mathrm{d}\sigma - \frac{t^p}{p}\,\eta \\
&= \frac{t^p}{p}(\lambda_{1,p} - \eta) + \frac{t^{q^-}}{q^-}\int_\Omega \mu(z)|\nabla u_{1,p}|^{q(z)}\,\mathrm{d}z \\
&\quad + \frac{t^{p_*}}{p_*}\int_{\partial\Omega}\beta(z)u_{1,p}^{p_*}\,\mathrm{d}\sigma.
\end{aligned}$$

Recall that η is arbitrary. Thus, if we choose $\eta > \lambda_{1,p}$ (which gives $\lambda_{1,p} - \eta < 0$), we have for $t > 0$ sufficiently small

$$\frac{t^p}{p}\left(\lambda_{1,p} - \eta\right) + \frac{t^{q^-}}{q^-}\int_\Omega \mu(z)|\nabla u_{1,p}|^{q(z)}\,\mathrm{d}z + \frac{t^{p_*}}{p_*}\int_{\partial\Omega}\beta(z)u_{1,p}^{p_*}\,\mathrm{d}\sigma < 0$$

since $p < q^-$ and $p < p_*$. This implies that $\Phi_+(tu_{1,p}) < 0 = \Phi_+(0)$ for $t \in (0,1)$ sufficiently small. Hence, $u_0 \neq 0$.

As u_0 is a global minimizer of Φ_+, we have $\Phi_+'(u_0) = 0$, that is,

$$\begin{aligned}\int_\Omega \left(|\nabla u_0|^{p-2}\nabla u_0 + \mu(z)|\nabla u_0|^{q(z)-2}\nabla u_0\right) \cdot \nabla h \, \mathrm{d}z \\ + \int_{\partial\Omega} \beta(z)|u_0|^{p_*-2}u_0 \cdot h \, \mathrm{d}\sigma = \int_\Omega k(z,(u_0)_+)\, h \, \mathrm{d}z\end{aligned} \tag{3.11}$$

for all $h \in W^{1,\mathcal{H}}(\Omega)$. Also, we know that $\pm u_\pm \in W^{1,\mathcal{H}}(\Omega)$ for any $u \in W^{1,\mathcal{H}}(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [13, Proposition 2.17]. So, if we take $h = -(u_0)_-$ in (3.11) we obtain that $(u_0)_- = 0$. This gives $u_0 \geq 0$. Taking into account that $u_0 \neq 0$, we conclude that u_0 is a positive weak solution of problem (3.8). Thus, $\mathcal{S}_+ \neq \emptyset$. Similar to the proof of Theorem 3.1 in Gasiński-Winkert [21] we can show that $u_0 \in L^\infty(\Omega)$.

Arguing in a similar way, we get the existence of a negative weak solution for problem (3.8). It is sufficient to work with the C^1-functional $\Phi_- \colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi_-(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{1}{q(z)}\mu(z)|\nabla u|^{q(z)}\right] \mathrm{d}z \\ + \int_{\partial\Omega} \frac{1}{p_*}\beta(z)|u|^{p_*} \, \mathrm{d}\sigma - \int_\Omega K(z,-u_-)\, \mathrm{d}z\end{aligned}$$

and show that the global minimizer of $\Phi_-(u)$ is nontrivial. □

Now, we are ready to prove the existence of a smallest positive solution $u_* \in \mathcal{S}_+$ and the existence of a largest negative solution $v_* \in \mathcal{S}_-$.

Proposition 3.3. *Let hypotheses (H1) and (H2) be satisfied. Then there exists $u_* \in \mathcal{S}_+$ such that $u_* \leq u$ for all $u \in \mathcal{S}_+$ and there exists $v_* \in \mathcal{S}_-$ such that $v_* \geq v$ for all $v \in \mathcal{S}_-$.*

Proof. We prove the existence of a smallest positive solution in $\mathcal{S}_+$. Arguing in a similar way, we also obtain the existence of a largest negative solution $v_* \in \mathcal{S}_-$.

Following the proof of Proposition 7 in Papageorgiou-Rădulescu-Repovš [31], we can easily check that $\mathcal{S}_+$ is a downward directed set. This guarantees, thanks to Hu-Papageorgiou [23, Lemma 3.10, p. 178], that we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf \mathcal{S}_+.$$

In addition, since $u_n \in \mathcal{S}_+$, we know that

$$\begin{aligned}\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(z)|\nabla u_n|^{q(z)-2}\nabla u_n\right) \cdot \nabla h \, \mathrm{d}z \\ + \int_{\partial\Omega} \beta(z)|u_n|^{p_*-2}u_n \cdot h \, \mathrm{d}\sigma = \int_\Omega k(z,u_n) h \, \mathrm{d}z\end{aligned} \tag{3.12}$$

for all $h \in W^{1,\mathcal{H}}(\Omega)$ and for all $n \in \mathbb{N}$.

Now, taking $h = u_n$ in (3.12), on account of (3.7) and using $0 \leq u_n \leq u_1$, we deduce that

$$\int_\Omega |\nabla u_n|^p \, \mathrm{d}z + \int_\Omega \mu(z)|\nabla u_n|^{q(z)} \, \mathrm{d}z + \int_{\partial\Omega} \beta(z)|u_n|^{p_*} \, \mathrm{d}\sigma < b_1 \tag{3.13}$$

for some $b_1 > 0$ and for all $n \in \mathbb{N}$. Thus, since $\beta(z) \geq 0$ for a.a. $z \in \partial\Omega$, we have

$$\rho_{\mathcal{H}}(\nabla u_n) = \int_\Omega |\nabla u_n|^p \, \mathrm{d}z + \int_\Omega \mu(z)|\nabla u_n|^{q(z)} \, \mathrm{d}z < b_1$$

for some $b_1 > 0$ and for all $n \in \mathbb{N}$. This shows that $\{\rho_{\mathcal{H}}(\nabla u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ is bounded. Hence, thanks to Proposition 2.1, we deduce that $\{\|\nabla u_n\|_{\mathcal{H}}\}_{n\in\mathbb{N}}$ is bounded in $L^{\mathcal{H}}(\Omega)$.

Next, we observe that (3.13) also gives

$$\int_\Omega |\nabla u_n|^p \,\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u_n|^{p_*} \,\mathrm{d}\sigma < b_1 \tag{3.14}$$

for some $b_1 > 0$ and for all $n \in \mathbb{N}$. Taking Lemma 3.1 into account yields

$$\|u_n\|_{1,p} \le a\,[\|\nabla u_n\|_p + \|u_n\|_{p_*,\beta,\partial\Omega}] \quad \text{for some} \quad a > 0 \text{ and for all } n \in \mathbb{N}.$$

Using this and (3.14) we infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. Finally, we recall that the embedding $W^{1,p}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact (due to $q^+ < p^*$) and further the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous (see Proposition 2.2 (v)). Therefore, we conclude that $\|u_n\|_{\mathcal{H}} \le b_2$ for some $b_2 > 0$ and for all $n \in \mathbb{N}$.

On account of this, since $\{\|\nabla u_n\|_{\mathcal{H}}\}_{n\in\mathbb{N}}$ is bounded in $L^{\mathcal{H}}(\Omega)$ and $\|u_n\|_{1,\mathcal{H}} = \|\nabla u_n\|_{\mathcal{H}} + \|u_n\|_{\mathcal{H}}$, we conclude that $\{u_n\} \subseteq W^{1,\mathcal{H}}(\Omega)$ is bounded.

Observe that $\tau < \frac{p^2}{N-p} + 1$ due to hypothesis (H2)(ii). Hence, we know that $\frac{N-1}{p}(\tau - 1) < p_*$. So, if we choose $s > \frac{N-1}{p}$ such that $s(\tau-1) < p_*$, taking into account that $\{u_n\} \subseteq W^{1,\mathcal{H}}(\Omega)$ is bounded, we may suppose that

$$u_n \rightharpoonup u_* \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in} \quad L^{s(\tau-1)}(\Omega) \text{ and in } L^{s(\tau-1)}(\partial\Omega).$$

From (3.5), (3.6) and hypothesis (H2)(i) it follows that

$$|k(z,x)| \le b_3|x|^{\tau-1} \tag{3.15}$$

for some $b_3 > 0$, for a.a. $z \in \Omega$ and for all $x \in \mathbb{R}$. Then, from (3.12) and (3.15) along with a Moser-iteration type argument as it was explained in Colasuonno-Squassina [12, Section 3.2], we obtain, as $s > \frac{N-1}{p}$, that

$$\|u_n\|_\infty \le b_4\|u_n\|_{s(\tau-1)}^{\frac{\tau-1}{p-1}} \quad \text{for some } b_4 > 0 \text{ and for all } n \in \mathbb{N}.$$

Now, we show that $u_* \neq 0$. We argue indirectly and suppose $u_* = 0$ which implies $\|u_n\|_\infty \to 0$ as $n \to +\infty$. Hence, we deduce

$$0 < u_n(z) \le \delta \quad \text{for a.a. } z \in \Omega \text{ and for all } n \ge n_0,$$

where $\delta \in \left(0, \min\{\frac{\eta_0}{2}, 1\}\right)$. We further have

$$k(z,u_n(z)) = f(z,u_n(z)) + u_n(z)^{p^*-1} \quad \text{for a.a. } z \in \Omega \text{ and for all } n \ge n_0.$$

We put $y_n = \frac{u_n}{\|u_n\|_{1,\mathcal{H}}}$ for all $n \in \mathbb{N}$. Thus, $\|y_n\|_{1,\mathcal{H}} = 1$ and $y_n \ge 0$ for all $n \in \mathbb{N}$. Moreover, we can suppose

$$y_n \rightharpoonup y \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{q(\cdot)}(\Omega) \quad \text{with} \quad y \ge 0.$$

Now, we remark that (3.12) gives

$$\begin{aligned}
&\int_\Omega \left(\|u_n\|_{1,\mathcal{H}}^{p-1}|\nabla y_n|^{p-2}\nabla y_n + \|u_n\|_{1,\mathcal{H}}^{q(z)-1}\mu(z)|\nabla y_n|^{q(z)-2}\nabla y_n\right)\cdot\nabla h \,\mathrm{d}z\\
&+ \int_{\partial\Omega} \beta(z)\|u_n\|_{1,\mathcal{H}}^{p_*-1}|y_n|^{p_*-2}y_n h \,\mathrm{d}\sigma\\
&= \int_\Omega \|u_n\|_{1,\mathcal{H}}^{p-1}\left[\frac{f(z,u_n)}{u_n^{p-1}} + u_n^{p^*-p}\right] y_n^{p-1} h \,\mathrm{d}z
\end{aligned}$$

for all $h \in W^{1,\mathcal{H}}(\Omega)$ and for all $n \in \mathbb{N}$. Hence, since $0 < u_n(z) \leq \delta$, we deduce that

$$\begin{aligned}
&\int_\Omega |\nabla y_n|^{p-2}\nabla y_n \cdot \nabla h \,\mathrm{d}z \\
&+ \int_\Omega \|u_n\|_{1,\mathcal{H}}^{q(z)-p}\mu(z)|\nabla y_n|^{q(z)-2}\nabla y_n \cdot \nabla h \,\mathrm{d}z \\
&+ \int_{\partial\Omega} \beta(z)\|u_n\|_{1,\mathcal{H}}^{p_*-p}|y_n|^{p_*-2}y_n \cdot h \,\mathrm{d}\sigma \\
&= \int_\Omega \left[\frac{f(z,u_n)}{u_n^{p-1}} + u_n^{p^*-p}\right] y_n^{p-1} h \,\mathrm{d}z
\end{aligned} \tag{3.16}$$

for all $h \in W^{1,\mathcal{H}}(\Omega)$. Note that the left-hand side in (3.16) is bounded for all $h \in W^{1,\mathcal{H}}(\Omega)$. From this, using hypothesis (H2)(iii), we infer

$$y = 0 \quad \text{and} \quad \frac{f(z,u_n)}{u_n^{p-1}} y_n^{p-1} \to 0 \quad \text{for a.a.}\, z \in \Omega.$$

Furthermore, if we choose $h = y_n$ in (3.16) and we pass to limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \int_\Omega |\nabla y_n|^p = 0.$$

This guarantees, at least for a subsequence, that $\nabla y_n(z) \to 0$ for a.a. $z \in \Omega$. Hence, we infer that $\mathcal{H}(z, \nabla y_n) \to 0$ for a.a. $z \in \Omega$. Taking into account that $\{\mathcal{H}(\cdot, \nabla y_n)\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ is uniformly integrable, by Vitali's convergence theorem, we get that

$$\rho_\mathcal{H}(\nabla y_n) \to 0 \quad \text{in} \quad W^{1,\mathcal{H}}(\Omega)$$

as $n \to +\infty$, which by Proposition 2.1(iv) implies that

$$\|\nabla y_n\|_\mathcal{H} \to 0 \quad \text{as } n \to +\infty.$$

Finally, we recall that $\|y_n\|_{1,\mathcal{H}} = \|\nabla y_n\|_\mathcal{H} + \|y_n\|_\mathcal{H} = 1$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\lim_{n\to+\infty} \|y_n\|_{1,\mathcal{H}} &= \lim_{n\to+\infty} (\|\nabla y_n\|_\mathcal{H} + \|y_n\|_\mathcal{H}) \\
&= \lim_{n\to+\infty} \|\nabla y_n\|_\mathcal{H} + \lim_{n\to+\infty} \|y_n\|_\mathcal{H} \\
&= \lim_{n\to+\infty} \|y_n\|_\mathcal{H} = 1.
\end{aligned}$$

Also, as the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^\mathcal{H}(\Omega)$ is continuous (see Proposition 2.1(v)), we know that $\|y_n\|_\mathcal{H} \leq e\|y_n\|_{q(\cdot)}$ for some $e > 0$. Next, taking into account that $y_n \to 0$ in $L^{q(\cdot)}(\Omega)$, we have that $\|y_n\|_{q(\cdot)} \to 0$. Then, we get a contradiction. This allows to conclude that $u_* \neq 0$. Therefore, $u_* \in \mathcal{S}_+$ and u_* is the smallest positive solution in $\mathcal{S}_+$. □

4. **Proof of Theorem 1.2.** In this section we prove our main result, namely Theorem 1.2. To be more precise, we prove the existence of a sequence of nodal (that is, sign changing) solutions for problem (1.1). Furthermore, we are going to show that such a sequence converges to 0 in $W^{1,\mathcal{H}}(\Omega)$ and in $L^\infty(\Omega)$.

For this purpose, we start by the extremal constant sign solutions u_* and v_* determinated in Proposition 3.3. Our aim is to focus on the order interval

$$[v_*, u_*] := \left\{u \in W^{1,\mathcal{H}}(\Omega) \,:\, v_*(z) \leq u(z) \leq u_*(z) \quad \text{for a.a.}\, z \in \Omega\right\}.$$

Therefore, we use truncations of $k(z,\cdot)$ at $v_*(z)$ and $u_*(z)$ in order to introduce a new C^1-functional Ψ_*. Thus, let $k_*\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be the truncation function given by

$$k_*(z,x) := \begin{cases} k(z,v_*(z)) & \text{if } x < v_*(z),\\ k(z,x) & \text{if } v_*(z)\le x\le u_*(z),\\ k(z,u_*(z)) & \text{if } u_*(z) < x. \end{cases} \tag{4.1}$$

Then, we consider the C^1-functional $\Psi_*\colon W^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$ defined by

$$\begin{aligned}\Psi_*(u) = &\int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{1}{q(z)}\mu(z)|\nabla u|^{q(z)}\right]\mathrm{d}z\\ &+\int_{\partial\Omega}\frac{1}{p_*}\beta(z)|u|^{p_*}\,\mathrm{d}\sigma - \int_\Omega K_*(z,u)\,\mathrm{d}z,\end{aligned}$$

for all $u\in W^{1,\mathcal{H}}(\Omega)$, where $K_*(z,x)=\int_0^x k_*(z,s)\,\mathrm{d}s$.

As a first step, we observe that $K_{\Psi_*}=\{u\in W^{1,\mathcal{H}}(\Omega):(\Psi_*)'(u)=0\}$ is contained in the order interval $[v_*,u_*]$. So, let $u\in K_{\Psi_*}\setminus\{u_*,v_*\}$, then we know that

$$\begin{aligned}&\int_\Omega \left(|\nabla u|^{p-2}\nabla u+\mu(z)|\nabla u|^{q(z)-2}\nabla u\right)\cdot\nabla h\,\mathrm{d}z\\ &+\int_{\partial\Omega}\beta(z)|u|^{p_*-2}uh\,\mathrm{d}\sigma = \int_\Omega k_*(z,u)h\,\mathrm{d}z\end{aligned} \tag{4.2}$$

for all $h\in W^{1,\mathcal{H}}(\Omega)$. If we take $h=(u-u_*)_+$ in (4.2) we have

$$\begin{aligned}&\int_\Omega \left(|\nabla u|^{p-2}\nabla u+\mu(z)|\nabla u|^{q(z)-2}\nabla u\right)\cdot\nabla(u-u_*)_+\,\mathrm{d}z\\ &+\int_{\partial\Omega}\beta(z)|u|^{p_*-2}u\cdot(u-u_*)_+\,\mathrm{d}\sigma\\ &=\int_\Omega k_*(z,u)(u-u_*)_+\,\mathrm{d}z\\ &=\int_\Omega k(z,u_*)(u-u_*)_+\,\mathrm{d}z\\ &=\int_\Omega \left(|\nabla u_*|^{p-2}\nabla u_*+\mu(z)|\nabla u_*|^{q(z)-2}\nabla u_*\right)\cdot\nabla(u-u_*)_+\,\mathrm{d}z\\ &\quad+\int_{\partial\Omega}\beta(z)|u_*|^{p_*-2}u_*\cdot(u-u_*)_+\,\mathrm{d}\sigma\end{aligned}$$

since $u_*\in\mathcal{S}_+$. It follows that

$$\begin{aligned}&\int_\Omega \left(|\nabla u|^{p-2}\nabla u-|\nabla u_*|^{p-2}\nabla u_*\right)\cdot\nabla(u-u_*)_+\,\mathrm{d}z\\ &+\int_\Omega\mu(z)\left(|\nabla u|^{q(z)-2}\nabla u-|\nabla u_*|^{q(z)-2}\nabla u_*\right)\cdot\nabla(u-u_*)_+\,\mathrm{d}z\\ &+\int_{\partial\Omega}\beta(z)\left(|u|^{p_*-2}u-|u_*|^{p_*-2}u_*\right)\cdot(u-u_*)_+\,\mathrm{d}\sigma=0.\end{aligned}$$

Now, taking into account that $\beta(z)\ge 0$ for a.a. $z\in\partial\Omega$, we may conclude that $u\le u_*$. Arguing in a similar way but choosing $h=(v_*-u)_+$ in (4.2), we get that $v_*\le u$.

Let $V\subseteq W^{1,\mathcal{H}}(\Omega)\cap L^\infty(\Omega)$ be a finite dimensional subspace. For $v\in V$ we put

$$\{v<v_*\}:=\{z\in\Omega\,:\,v(z)<v_*(z)\},$$

$$\begin{aligned}\{v_* \le v \le u_*\} &:= \{z \in \Omega \,:\, v_*(z) \le v(z) \le u_*(z)\},\\ \{u_* < v\} &:= \{z \in \Omega \,:\, u_*(z) < v(z)\}.\end{aligned}$$

Next, we establish the following result.

Proposition 4.1. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then, we can find* $r_V > 0$ *such that*

$$\sup[\Psi_*(v) \,:\, v \in V,\, \|v\|_{1,\mathcal{H}} = r_V] < 0.$$

Proof. Recall that V is a finite dimensional space and thus we know that all the norms on V are equivalent, see for example Papageorgiou-Winkert [36, Proposition 3.1.17, p.183]. This guarantees that we can find $r_V > 0$ such that

$$v \in V \text{ and } \|v\|_{1,\mathcal{H}} \le r_V \text{ imply } |v(z)| \le \delta \text{ for a. a. } z \in \Omega$$

with $\delta \in \left(0, \min\{\frac{\eta_0}{2}, 1\}\right)$. Since $\delta \le \frac{\eta_0}{2}$, by (3.6) we know that $\theta(v(z)) = 1$ for a.a. $z \in \Omega$. Consequently, for $v \in V$ with $\|v\|_{1,\mathcal{H}} \le r_V$, we have

$$k_*(z, v(z)) = \begin{cases} f(z, v_*(z)) + |v_*(z)|^{p^*-2} v_*(z) & \text{if } v(z) < v_*(z),\\ f(z, v(z)) + |v(z)|^{p^*-2} v(z) & \text{if } v_*(z) \le v(z) \le u_*(z),\\ f(z, u_*(z)) + |u_*(z)|^{p^*-2} u_*(z) & \text{if } u_*(z) < v(z).\end{cases}$$

Now, we consider the function $f_* \colon \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$f_*(z, v(z)) = \begin{cases} f(z, v_*(z)) & \text{if } v(z) < v_*(z),\\ f(z, v(z)) & \text{if } v_*(z) \le v(z) \le u_*(z),\\ f(z, u_*(z)) & \text{if } u_*(z) < v(z)\end{cases}$$

and we put $F_*(z, v) := \int_0^v f_*(z, s)\,\mathrm{d}s$. For $v < v_*$ we know

$$\begin{aligned} F_*(z, v) &= \int_0^{v_*} f_*(z, s)\,\mathrm{d}s + \int_{v_*}^{v} f_*(z, s)\,\mathrm{d}s\\ &= \int_0^{v_*} f(z, s)\,\mathrm{d}s + \int_{v_*}^{v} f(z, v_*)\,\mathrm{d}s\\ &= F(z, v_*) + f(z, v_*)(v - v_*).\end{aligned}$$

Note that $f(z, v_*)$ is negative, see Remark 1.1, then $f(z, v_*)(v - v_*) > 0$. From this, we deduce that

$$\begin{aligned} F(z, v) - F_*(z, v) &= [F(z, v) - F(z, v_*)] + f(z, v_*)(v_* - v)\\ &\le [F(z, v) - F(z, v_*)],\end{aligned}$$

where $F(z, v) := \int_0^v f(z, s)\,\mathrm{d}s$. Likewise, for $u_* < v$ we have

$$F_*(z, v) = F(z, u_*) + f(z, u_*)(v - u_*),$$

which gives

$$\begin{aligned} F(z, v) - F_*(z, v) &= [F(z, v) - F(z, u_*)] + f(z, u_*)(u_* - v)\\ &\le [F(z, v) - F(z, u_*)],\end{aligned}$$

since $f(z, u_*)(u_* - v) < 0$, see Remark 1.1.

Consequently, we infer that

$$\Psi_*(v) = \int_\Omega \left[\frac{1}{p}|\nabla v|^p + \frac{1}{q(z)}\mu(z)|\nabla v|^{q(z)}\right]\mathrm{d}z$$

$$
\begin{aligned}
&+\int_{\partial\Omega}\frac{1}{p_*}\beta(z)|v|^{p_*}\,\mathrm{d}\sigma-\int_\Omega K_*(z,v)\,\mathrm{d}z\\
&\le \frac{1}{p}\int_\Omega|\nabla v|^p\,\mathrm{d}z+\frac{1}{q^-}\int_\Omega\mu(z)|\nabla v|^{q(z)}\,\mathrm{d}z+\frac{1}{p_*}\int_{\partial\Omega}\beta(z)|v|^{p_*}\,\mathrm{d}\sigma\\
&-\int_{\{v<v_*\}}\left[F_*(z,v)+\frac{1}{p^*}|v_*|^{p^*}\right]\mathrm{d}z-\int_{\{v_*\le v\le u_*\}}\left[F(z,v)+\frac{1}{p^*}|v|^{p^*}\right]\mathrm{d}z\\
&-\int_{\{u_*<v\}}\left[F_*(z,v)+\frac{1}{p^*}|u_*|^{p^*}\right]\mathrm{d}z\\
&\le \frac{1}{p}\int_\Omega|\nabla v|^p\,\mathrm{d}z+\frac{1}{q^-}\int_\Omega\mu(z)|\nabla v|^{q(z)}\,\mathrm{d}z+\frac{1}{p_*}\int_{\partial\Omega}\beta(z)|v|^{p_*}\,\mathrm{d}\sigma\\
&-\int_{\{v<v_*\}}F_*(z,v)\,\mathrm{d}z-\int_{\{v_*\le v\le u_*\}}F(z,v)\,\mathrm{d}z-\int_{\{u_*<v\}}F_*(z,v)\,\mathrm{d}z
\end{aligned}
$$

where we used that

$$
\frac{1}{p^*}|v_*|^{p^*},\ \ \frac{1}{p^*}|v|^{p^*}\quad\text{and}\quad\frac{1}{p^*}|u_*|^{p^*}
$$

are positive. We can further write

$$
\begin{aligned}
\Psi_*(v)&\le \frac{1}{p}\int_\Omega|\nabla v|^p\,\mathrm{d}z+\frac{1}{q^-}\int_\Omega\mu(z)|\nabla v|^{q(z)}\,\mathrm{d}z+\frac{1}{p_*}\int_{\partial\Omega}\beta(z)|v|^{p_*}\,\mathrm{d}\sigma\\
&-\int_\Omega F(z,v)\,\mathrm{d}z+\int_{\{v<v_*\}}[F(z,v)-F_*(z,v)]\,\mathrm{d}z\\
&+\int_{\{u_*<v\}}[F(z,v)-F_*(z,v)]\,\mathrm{d}z\\
&\le \frac{1}{p}\int_\Omega|\nabla v|^p\,\mathrm{d}z+\frac{1}{q^-}\int_\Omega\mu(z)|\nabla v|^{q(z)}\,\mathrm{d}z+\frac{1}{p_*}\int_{\partial\Omega}\beta(z)|v|^{p_*}\,\mathrm{d}\sigma\\
&-\int_\Omega F(z,v)\,\mathrm{d}z+\int_{\{v<v_*\}}[F(z,v)-F(z,v_*)]\,\mathrm{d}z\\
&+\int_{\{u_*<v\}}[F(z,v)-F(z,u_*)]\,\mathrm{d}z.
\end{aligned}
$$

Next, we recall that f is odd and thus by hypothesis (H2)(iii) we know that for each $\eta>0$ it is possible to find $\delta\in\left(0,\min\{\frac{\eta_0}{2},1\}\right)$ such that

$$
F(z,x)\ge\frac{\eta}{p}|x|^p\quad\text{for all } |x|\le\delta.
$$

Then, if we take r_V small so that

$$
\int_{\{v<v_*\}}[F(z,v)-F(z,v_*)]\,\mathrm{d}z+\int_{\{u_*<v\}}[F(z,v)-F(z,u_*)]\,\mathrm{d}z<\delta^p,
$$

we obtain

$$
\begin{aligned}
\Psi_*(v)\le{}&\frac{1}{p}\int_\Omega|\nabla v|^p\mathrm{d}\,z+\frac{1}{q^-}\int_\Omega\mu(z)|\nabla v|^{q(z)}\,\mathrm{d}z\\
&+\frac{1}{p_*}\int_{\partial\Omega}\beta(z)|v|^{p_*}\,\mathrm{d}\sigma-\frac{\eta}{p}\int_\Omega|v|^p\,\mathrm{d}z+\delta^p.
\end{aligned}
$$

Now, taking into account that $\|v\|_{1,\mathcal{H}} = \|\nabla v\|_{\mathcal{H}} + \|v\|_{\mathcal{H}}$, by Proposition 2.1(ii), (iii) we know that

$$\begin{aligned}\int_\Omega \mu(z)|\nabla v|^{q(z)}\,\mathrm{d}z &\le \rho_{\mathcal{H}}(\nabla v) \le \max\{\|\nabla v\|_{\mathcal{H}}^p,\ \|\nabla v\|_{\mathcal{H}}^{q^+}\}\\ &\le \max\{\|v\|_{1,\mathcal{H}}^p,\ \|v\|_{1,\mathcal{H}}^{q^+}\}.\end{aligned}$$

In addition, we recall that $L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous, see Proposition 2.2(i). Then, for some $\hat{e} > 0$ we have $\|\nabla v\|_p \le \hat{e}\,\|\nabla v\|_{\mathcal{H}}$ which gives $\|\nabla v\|_p \le \hat{e}\,\|v\|_{1,\mathcal{H}}$.

Finally, we recall that V is finite dimensional and thus all the norms on V are equivalent. On account of this, we find positive constants c_1, c_2, c_3, c_4, independent of δ, such that

$$\Psi_*(v) \le c_1\|v\|_\infty^p + c_2 \max\left\{\|v\|_\infty^p,\ \|v\|_\infty^{q^+}\right\} + c_3\|v\|_\infty^{p_*} - \eta\, c_4\|v\|_\infty^p + \delta^p.$$

Thanks to the equivalence of the norms on V, for $v \in V$ with $\|v\|_{1,\mathcal{H}} = r_V$ we deduce that

$$\begin{aligned}\Psi_*(v) &\le c_1\delta^p + c_2\max\{\delta^p, \delta^{q^+}\} + c_3\delta^{p_*} - \eta\, c_4\delta^p + \delta^p\\ &\le (c_1 + c_2 + c_3 - \eta\, c_4 + 1)\,\delta^p\end{aligned}$$

as $\delta < 1$ and $p < p_*$. Next, since η is arbitrary, we can choose $\eta > \frac{c_1+c_2+c_3+1}{c_4}$ in order to get

$$\Psi_*(v) < 0 \quad \text{for all } v \in V \text{ with } \|v\|_{1,\mathcal{H}} = r_V,$$

which gives the claim. □

Now we can give the proof of Theorem 1.2. In order to prove it, we use a generalized version of the symmetric mountain pass theorem due to Kajikiya, see [24, Theorem 1].

Proof of Theorem 1.2. With view to the definition of the truncation function $k_*\colon \Omega\times\mathbb{R} \to \mathbb{R}$ given in (4.1), we easily deduce that $\Psi_*\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ is even and coercive. This guarantees that $\Psi_*\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ is bounded from below. Moreover, we know that $\Psi_*\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale condition, see Papageorgiou-Radulescu-Repovs [32, Proposition 5.1.15]. According to this and thanks to Proposition 4.1, we can apply Theorem 1 of Kajikiya [24] which gives the existence of a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}}(\Omega)\cap L^\infty(\Omega)$ satisfying the following properties

$$w_n \in K_{\Psi_*} \subseteq [v_*, u_*], \quad w_n \neq 0, \quad \Psi_*(w_n) \le 0 \quad \text{for all } n \in \mathbb{N}$$

and

$$\|w_n\|_{1,\mathcal{H}} \to 0 \quad \text{as } n \to +\infty.$$

Recall that v_* and u_* are the extremal constant sign solutions for problem (3.8). Therefore, as $w_n \in K_{\Psi_*} \subseteq [v_*, u_*]$ and $w_n \neq 0$ for all $n \in \mathbb{N}$, we infer that w_n is a nodal solution of problem (3.8) for all $n \in \mathbb{N}$. Also, arguing as in the proof of Proposition 3.3 we obtain

$$\|w_n\|_\infty \le d\,\|w_n\|_{s(\tau-1)}^{\frac{\tau-1}{p-1}}$$

for some $d > 0$ and for all $n \in \mathbb{N}$ with $s > \frac{N-1}{p}$ as well as $s(\tau - 1) < p_*$. Hence, due to $\|w_n\|_{1,\mathcal{H}} \to 0$, we derive that $\|w_n\|_\infty \to 0$ as $n \to +\infty$. Then, we can find $n_0 \in \mathbb{N}$ such that $|w_n(z)| \leq \frac{\eta_0}{2}$ for a.a. $z \in \Omega$ and for all $n \geq n_0$. Hence, we have

$$\theta(w_n(z)) = 1 \quad \text{for a.a. } z \in \Omega \text{ and for all } n \geq n_0.$$

Therefore, with view to (4.1) and (3.5), we conclude that w_n is a sign changing solution of problem (1.1) for all $n > n_0$. $\square$

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Received February 2023; revised April 2023; early access May 2023.