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(p, q)-LAPLACIAN PROBLEMS WITH PARAMETERS IN BOUNDED AND UNBOUNDED DOMAINS

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Abstract. The purpose of this paper is to study the existence and nonexistence of nontrivial solutions for quasilinear elliptic problems driven by the nonhomogeneous (p, q)-Laplace operator $-\Delta_p - \Delta_q$ depending on two parameters in bounded and unbounded domains. First, using variational methods, we prove the existence and nonexistence of positive solutions for a class of sublinear (p, q)-Laplacian problems with two parameters. Second, using a nonstandard variational approach, we prove the existence of bounded solutions for nonlinear problems of $(p + q)$ sublinear type involving the (p, q)-Laplace operator with two parameters in unbounded domains.

1. Introduction

In this paper, we are concerned with the study of two nonlinear problems involving a differential operator of (p, q)-Laplacian type with positive parameters of the form

$$(P_1) \qquad -\Delta_p u - \Delta_q u = \lambda f(x, u) + \mu g(x, u), \quad x \in \Omega$$

and

$$(P_2) \qquad -\Delta_p u - \mu\Delta_q u = \lambda f(x, u), \quad x \in \mathbb{R}^N,$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is either a bounded domain or an unbounded domain, $1 < q < p < +\infty$, λ and μ are positive parameters and $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ stands for the classical r-Laplacian for $1 < r < \infty$.

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It should be mentioned that in recent years several studies have been carried out to investigate related problems and many papers have appeared dealing with equations driven by the (p,q)-Laplacian in both bounded and unbounded domains. We refer, for example, to the papers of Baldelli-Filippucci [2], Baldelli-Brizi-Filippucci [3, 4], Bobkov-Tanaka [5, 6, 7], Candito-Marano-Perera [8], and Motreanu-Tanaka [14], see also the references therein. Note that the (p,q)-Laplacian appears in a wide range of applications, we mention e.g.,biophysics, plasma physics, reaction-diffusion equations, and models of elementary particles.

Regarding problem (P_1) let us point out that more works have been done in this direction in the case of nonlinear problems involving the p-Laplacian. Namely, Maya-Shivaji [13] studied the semilinear case $p=2$ and similar results have been established by Perera [15] and El Manouni-Perera [10] when only the p-Laplacian operator appears with $p>1$ and $p\neq 2$ in the cases of scalar equations and systems of two second order quasilinear equations, respectively.

As for problem (P_2), we consider different type of problems with two parameters for the (p,q)-Laplacian in $\mathbb{R}^N$. In particular each parameter is on one side of the quasilinear equation. Here, we do not minimize a Rayleigh-type quotient, but we use a nonstandard variational approach and show the existence of at least one solution corresponding to the three positive parameters. Additionally, we establish some regularity results for the solution obtained in terms of global L^∞-estimates by constructing an iteration scheme to bound the maximal norm of the solution following Moser's iteration technique, see, for example, the monograph of Drábek-Kufner-Nicolosi [9].

2. Bounded case

Let $\Omega\subset\mathbb{R}^N, N\geq 2$, be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following Dirichlet problem

$$\begin{aligned} -\Delta_p u-\Delta_q u &= \lambda f(x,u)+\mu g(x,u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $1<q<p<\infty$, $\lambda>0$, $\mu>0$, $\Delta_r u=\operatorname{div}(|\nabla u|^{r-2}\nabla u)$ stands for the usual r-Laplacian with $1<r<\infty$ and $f,g\colon \Omega\times[0,\infty)\to\mathbb{R}$ are assumed to be Carathéodory functions having subcritical growth of the form

$$|f(x,t)|\leq C_1 t^{p-1}, \quad |g(x,t)|\leq C_2 t^{q-1} \quad \text{for all } (x,t), \tag{2.2}$$

with positive constants C_1, C_2. We define

$$
\begin{aligned}
\lambda_1 &= \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega (|\nabla u|^p + |\nabla u|^q)\,\mathrm{d}x}{\int_\Omega |u|^p\,\mathrm{d}x}, \\
\mu_1 &= \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega (|\nabla u|^p + |\nabla u|^q)\,\mathrm{d}x}{\int_\Omega |u|^q\,\mathrm{d}x},
\end{aligned}
\tag{2.3}
$$

where $W = W_0^{1,p}(\Omega)$ which can be equipped with the equivalent norm

$$
\|u\| := \|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for all } u \in W_0^{1,p}(\Omega).
$$

Remark 2.1. *Regarding the definitions of λ_1 and μ_1 in* (2.3)*, since $q < p$, λ_1 is just the first eigenvalue of the p-Laplacian. Indeed, it is clear that λ_1 is greater than or equal to the first eigenvalue. Taking $u = t\phi_1$, where ϕ_1 is a first eigenfunction of the p-Laplacian with $t > 0$, and letting $t \to \infty$ shows that λ_1 is less than or equal to the first eigenvalue. A similar argument shows that μ_1 is the first eigenvalue of the q-Laplacian.*

We start by a nonexistence result for positive solution of problem (2.1).

Theorem 2.2. *There exists $\alpha, \beta > 0$ such that for all λ, μ satisfying $\lambda\alpha + \mu\beta < 1$, problem (2.1) has no positive solutions.*

Proof. Suppose that (2.1) has a positive solution $u \in W$. Testing the weak formulation of (2.1) by u, using (2.2) and (2.3) gives

$$
\begin{aligned}
\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |\nabla u|^q\,\mathrm{d}x &= \lambda \int_\Omega f(x,u)u\,\mathrm{d}x + \mu \int_\Omega g(x,u)u\,\mathrm{d}x \\
&\le \lambda C_1 \int_\Omega |u|^p\,\mathrm{d}x + \mu C_2 \int_\Omega |u|^q\,\mathrm{d}x \\
&\le \left(\lambda \frac{C_1}{\lambda_1} + \mu \frac{C_2}{\mu_1} \right) \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |\nabla u|^q\,\mathrm{d}x \right).
\end{aligned}
$$

The assertion of the theorem follows. $\square$

The next theorem shows the existence of positive solutions for problem (2.1).

Theorem 2.3. *Assume the following conditions:*

(F1) *there exists $\delta > 0$ such that*

$$
F(x,t) := \int_0^t f(x,\tau)\,\mathrm{d}\tau \le 0, \quad G(x,t) := \int_0^t g(x,\tau)\,\mathrm{d}\tau \le 0,
$$

when $t^p + t^q \le \delta$;

(F2) *there exists* $t_0 > 0$ *such that* $F(x,t_0) > 0$ *and* $G(x,t_0) > 0$ *for a.a.* $x \in \Omega$*;*
(F3) *it holds*

$$\limsup_{t\to\infty} \frac{F(x,t)}{t^p} \leq 0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{G(x,t)}{t^q} \leq 0 \quad \textit{uniformly for a.a. } x \in \Omega.$$

Then exists a number $\omega > 0$ *such that* (2.1) *has at least two positive solutions for* λ *or* $\mu \geq \omega$.

Proof. We set $f(x,t) = g(x,t) = 0$ if $t < 0$ for a.a. $x \in \Omega$ and consider the C^1-functional $\Phi_{\lambda,\mu}\colon W \to \mathbb{R}$ given by

$$\Phi_{\lambda,\mu}(u) = \int_\Omega \frac{1}{p}|\nabla u|^p \,\mathrm{d}x + \int_\Omega \frac{1}{q}|\nabla u|^q \,\mathrm{d}x - \int_\Omega \lambda F(x,u)\,\mathrm{d}x - \int_\Omega \mu G(x,u)\,\mathrm{d}x$$

for all $u \in W$. If u is a critical point of $\Phi_{\lambda,\mu}$, denoting by u^- the negative part of u, we have

$$\begin{aligned} 0 = \langle \Phi'_{\lambda,\mu}(u)), u^- \rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u^- \,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla u^- \,\mathrm{d}x \\ &\quad - \int_\Omega \lambda f(x,u)u^- \,\mathrm{d}x - \int_\Omega \mu g(x,u)u^- \,\mathrm{d}x \\ &= \|\nabla u^-\|_p^p + \|\nabla u^-\|_q^q. \end{aligned}$$

This shows $u \geq 0$. Furthermore, $u \in L^\infty(\Omega) \cap C^1(\Omega)$ (see, for example, Ho-Kim-Winkert-Zhang [11, Theorem 3.1] and Lieberman [12, Theorem 1.7]), so the positivity of u now follows from the weak Harnack type inequality proved by Trudinger [18, Theorem 1.1], that is, either $u > 0$ or $u \equiv 0$. Thus, nontrivial critical points of $\phi_{\lambda,\mu}$ are positive solutions of (2.1).

By the growth condition (2.2) we get

$$\text{(2.4)} \qquad |F(x,t)| \leq C_3|t|^p, \quad |G(x,t)| \leq C_4|t|^q \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

for some positive constants C_3 and C_4. By (F3) there is $B_{\lambda,\mu} > 0$ such that

$$\text{(2.5)} \qquad F(x,t) \leq \frac{\lambda_1}{3p\lambda}|t|^p \quad \text{and} \quad G(x,t) \leq \frac{\mu_1}{3p\mu}|t|^q \quad \text{for all } |t| \geq B_{\lambda,\mu}.$$

By using (2.4) and (2.5), there is a constant $C_{\lambda,\mu} > 0$ such that

$$\text{(2.6)} \qquad \lambda F(x,t) + \mu G(x,t) \leq \frac{\lambda_1}{3p}|t|^p + \frac{\mu_1}{3p}|t|^q + C_{\lambda,\mu} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Hence, since $q < p$, by applying (2.6) and (2.3), it follows that

$$\begin{aligned}\Phi_{\lambda,\mu}(u) &\geq \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla u|^q - \frac{\lambda_1}{3p}|u|^p - \frac{\mu_1}{3p}|u|^q - C_{\lambda,\mu}\right) \mathrm{d}x,\\ &\geq \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{1}{p}|\nabla u|^q - \frac{2}{3p}(|\nabla u|^p + |\nabla u|^q) - C_{\lambda,\mu}\right) \mathrm{d}x\\ &\geq \gamma\|u\|^p - C_{\lambda,\mu}|\Omega|_N,\end{aligned}$$

where $\gamma = \dfrac{1}{3p}$ and $|\cdot|_N$ is the Lebesgue measure in $\mathbb{R}^N$. Hence, $\Phi_{\lambda,\mu}$ is bounded from below and coercive. In addition, $\Phi_{\lambda,\mu}$ is sequentially weakly lower semicontinuous which implies the existence of a global minimizer $u_1 \in W^{1,p}(\Omega)$ of $\Phi_{\lambda,\mu}(u)$.

Claim 1: There exists $\omega > 0$ such that $\inf \Phi_{\lambda,\mu} < 0$ for $\lambda \geq \omega$ or $\mu \geq \omega$.

In order to prove this, we take a sufficiently large compact subset Ω' of Ω and $u_0 \in W_0^{1,p}(\Omega)$ such that $u_0 = t_0$ on Ω' and $0 \leq u_0 \leq t_0$ on $\Omega \setminus \Omega'$, where t_0 is as in (F2). Then we have

$$\begin{aligned}\int_\Omega F(x, u_0)\,\mathrm{d}x &\geq \int_{\Omega'} F(x, t_0)\,\mathrm{d}x - C_1|t_0|^p|\Omega \setminus \Omega'|_N > 0\\ \int_\Omega G(x, u_0)\,\mathrm{d}x &\geq \int_{\Omega'} G(x, t_0)\,\mathrm{d}x - C_2|t_0|^q|\Omega \setminus \Omega'|_N > 0,\end{aligned}$$

for $|\Omega \setminus \Omega'|_N$ sufficiently small. This yields

$$\begin{aligned}&\Phi_{\lambda,\mu}(u_0)\\ &\leq \int_\Omega \left(\frac{1}{p}\,|\nabla u_0|^p + \frac{1}{q}\,|\nabla u_0|^q\right) \mathrm{d}x - \lambda\int_\Omega F(x, u_0)\,\mathrm{d}x - \mu\int_\Omega G(x, u_0)\,\mathrm{d}x < 0\end{aligned}$$

for λ or μ large enough. This proves Claim 1.

From Claim 1, choosing λ or $\mu \geq \omega$, we get that $\Phi_{\lambda,\mu}(u_1) < 0 = \Phi_{\lambda,\mu}(0)$ and so $u_1 \neq 0$. Now, let us fix λ, μ with λ or $\mu \geq \omega$.

Claim 2: The origin is a strict local minimizer of $\Phi_{\lambda,\mu}$.

Let $u \in W_0^{1,p}(\Omega)$. We set $\Omega_u = \{x \in \Omega : |u(x)|^p + |u(x)|^q > \delta\}$, where $\delta > 0$ is given in (F1). By hypothesis (F1), $F(x, u) \leq 0$ and $G(x, u) \leq 0$ on $\Omega \setminus \Omega_u$. Then we have

$$
\begin{aligned}
\Phi_{\lambda,\mu}(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_p^q - \lambda\int_{\Omega_u} F(x,u)\,\mathrm{d}x - \lambda\int_{\Omega\setminus\Omega_u} F(x,u)\,\mathrm{d}x \\
&\quad - \mu\int_{\Omega_u} G(x,u)\,\mathrm{d}x - \mu\int_{\Omega\setminus\Omega_u} G(x,u)\,\mathrm{d}x \\
&\geq \frac{1}{p}\|u\|^p - \lambda\int_{\Omega_u} F(x,u)\mathrm{d}x - \mu\int_{\Omega_u} G(x,u)\,\mathrm{d}x.
\end{aligned}
\tag{2.7}
$$

Applying (2.2), Hölder's inequality and the Sobolev embedding theorem, it follows that

$$
\begin{aligned}
&\lambda\int_{\Omega_u} F(x,u)\,\mathrm{d}x + \mu\int_{\Omega_u} G(x,u)\,\mathrm{d}x \\
&\leq \lambda C_1\int_{\Omega_u} |u|^p\,\mathrm{d}x + \mu C_2\int_{\Omega_u} |u|^q\,\mathrm{d}x \\
&\leq \lambda C_1'|\Omega_u|_N^{1-\frac{p}{r}}\|\nabla u\|_p^p + \mu C_2'|\Omega_u|_N^{1-\frac{q}{s}}\|\nabla u\|_q^q,
\end{aligned}
\tag{2.8}
$$

for some positive constants C_1' and C_1' and where $r = \dfrac{Np}{N-p}$ if $p < N, r > p$ if $p \geq N$ and $s = \dfrac{Nq}{N-q}$ if $q < N$, $s > q$ if $q \geq N$. Since $q < p$ and

$$1 < \frac{|u|^p + |u|^q}{\delta} \quad \text{on } \Omega_u,$$

we obtain

$$|\Omega_u|_N \leq \frac{1}{\delta}\int_{\Omega_u} (|u|^p + |u|^q)\;\mathrm{d}x \leq C'\|u\|^p,$$

for some positive constant C'. Hence $|\Omega_u|_N \to 0$ as $\|u\| \to 0$ and Claim 2 follows from (2.7) and (2.8).

Since $\Phi_{\lambda,\mu}$ is coercive, every Palais-Smale sequence is bounded and hence contains a convergent subsequence. So the mountain pass lemma now gives a critical point u_2 of $\Phi_{\lambda,\mu}$ at the level

$$c := \inf_{\gamma\in\Gamma}\max_{u\in\gamma([0,1])} \Phi_{\lambda,\mu}(u) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1\}$ is the class of paths joining the origin to u_1. Therefore we found two positive solutions u_1, u_2 such that $\Phi_{\lambda,\mu}(u_1) < 0 = \Phi_{\lambda,\mu}(0) < \Phi_{\lambda,\mu}(u_2)$.
This completes the proof of the theorem. $\square$

3. Unbounded case

In this part we consider the following nonlinear problem in $\mathbb{R}^N$

$$-\Delta_p u - \mu\Delta_q u = \lambda f(x, u) \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $1 < q < p < N, N < p + q, \lambda, \mu > 0$ are two real parameters and $f\colon \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^+$ is a Carathéodory function that satisfies the following conditions:

(H1) There exists $\delta \in (0, 1)$ such that $0 \leq f(x, t) \leq m(x)t^{p+q-1}$ for a.a. $x \in \mathbb{R}^N$ and for all $t \geq 0$, where m is a positive function such that $m \in L^r\left(\mathbb{R}^N\right) \cap L^{r+\delta}\left(\mathbb{R}^N\right)$ with $r = \dfrac{1}{1 - \frac{p}{p^*} - \frac{q}{q^*}}$;

(H2) There exists a nonempty open subset Ω of $\mathbb{R}^N$ and constants $\delta_0, \delta_1 > 0$ such that $f(x, t) > 0$ for all $(x, t) \in \Omega \times (\delta_0, \delta_1)$.

We denote by X the reflexive Banach space $D^{1,p}\left(\mathbb{R}^N\right) \cap D^{1,q}\left(\mathbb{R}^N\right)$, where for $1 < s < N$

$$D^{1,s}\left(\mathbb{R}^N\right) = \left\{u \in L^{s^*}\left(\mathbb{R}^N\right) : \nabla u \in L^s\left(\mathbb{R}^N\right)\right\},$$

is equipped with the norm

$$\|u\| := \|u\|_X = \|u\|_{D^{1,p}(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)}.$$

By a weak solution of problem (3.1) we mean any $u \in X$ such that

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x + \mu \int_{\mathbb{R}^n} |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \lambda \int_{\mathbb{R}^N} f(x, u)\varphi \,\mathrm{d}x \tag{3.2}$$

is satisfied for all $\varphi \in X$.

Note that $r = \frac{N}{p+q-N}$ and the assumptions $N < p+q$ as well as (H1) guarantee that r and both integrals in (3.2) are well-defined, respectively.

Our first result is the following one.

Theorem 3.1. *Let $1 < q < p < N$ with $N < p + q$ and suppose that* (H1)–(H2) *are satisfied. Then there exist $\lambda, \mu > 0$ such that problem* (3.1) *has a positive solution $u \in X$.*

Proof. Let $1 < \alpha < q$ be fixed and set $F(x, t) = \int_0^t f(x, \tau) \,\mathrm{d}\tau$. We consider the functional $J\colon X \setminus \{0\} \to \mathbb{R}$ defined by

$$J(u) = \frac{\displaystyle\int_{\mathbb{R}^N} F(x, u) \,\mathrm{d}x}{(\|\nabla u\|_p^\alpha + \|\nabla u\|_p^{p^*})(\|\nabla u\|_q^\alpha + \|\nabla u\|_q^{q^*})},$$

which is well defined and bounded. Indeed, in view of (H1), the Sobolev embedding theorem and since $\alpha < p < p^*$ and $\alpha < q < q^*$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x &\leq \|m\|_r \|u\|_{p^*}^p \|u\|_{q^*}^q\\ &\leq c_1 \|\nabla u\|_p^p \|\nabla u\|_q^q\\ &\leq c_1(\|\nabla u\|_p^\alpha + \|\nabla u\|_p^{p^*})(\|\nabla u\|_q^\alpha + \|\nabla u\|_q^{q^*}),\end{aligned}$$

for some $c_1 > 0$. Hence, we have

$$J(u) \leq c_1 \quad \text{for all } u \in X \setminus \{0\}.$$

Let $\mathcal{S} := \sup_{0 \neq u \in X} J(u)$ and choose $\varphi_0 \in \mathcal{C}_0^\infty\left(\mathbb{R}^N\right)$ such that

$$\operatorname{supp} \varphi_0 \subset\subset \Omega \quad \text{and} \quad \sup_{x \in \mathbb{R}^N} \varphi_0(x) > \delta_0,$$

where Ω and δ_0 are given in (H2). Thanks to (H2), it holds $F(x,t) > 0$ and $G(x,t) > 0$ on $\Omega \times (\delta_0, \infty)$. It follows that $\mathcal{S} > \mathcal{S}_0$ with $\mathcal{S}_0 = \frac{1}{2} J(\varphi_0) > 0$. Let $\{u_k\}_{k \geq 1} \subset X$, with $u_k \neq 0$ for all $k \in \mathbb{N}$, be a sequence such that

$$J(u_k) \to \mathcal{S} \quad \text{as } k \to +\infty.$$

Since $\mathcal{S} > \mathcal{S}_0$ and $J({u_k}^+) \geq J(u_k)$, one can choose

$$u_k \geq 0 \quad \text{and} \quad J(u_k) \geq \mathcal{S}_0 \quad \text{for all } k \in \mathbb{N}.$$

Then we obtain

$$\begin{aligned}\mathcal{S}_0(\|\nabla u_k\|_p^\alpha + \|\nabla u_k\|_p^{p^*})(\|\nabla u_k\|_q^\alpha + \|\nabla u_k\|_q^{q^*}) &\leq \int_{\mathbb{R}^N} F(x,u_k)\,\mathrm{d}x\\ &\leq c_1 \|\nabla u_k\|_p^p \|\nabla u_k\|_q^q,\end{aligned}$$

for all $k \in \mathbb{N}$. It follows that $\|\nabla u_k\|_p$ and $\|\nabla u_k\|_q$ are bounded. Hence there exist constants $0 < A_1 < A_2 < \infty$ and $0 < A_3 < A_4 < \infty$ such that

$$A_1 \leq \|\nabla u_k\|_p \leq A_2 \quad \text{and} \quad A_3 \leq \|\nabla u_k\|_q \leq A_4 \quad \text{for all } k \in \mathbb{N}.$$

Then, due to the reflexivity of X, one can find a subsequence (still denoted by $\{u_k\}_{k \geq 1}$) and an element $u \in X$ such that

$$u_k \rightharpoonup u \quad \text{in } X \text{ and a.e. in } \mathbb{R}^N.$$

Then $u_k \rightharpoonup u$ in $D^{1,p}\left(\mathbb{R}^N\right)$ and $u_k \rightharpoonup u$ in $D^{1,q}\left(\mathbb{R}^N\right)$.

Let now $R > 0$. Using the Rellich–Kondrachov theorem (see Adams [1, p. 144], p. 144), the embedding $D^{1,p}(B_R) \hookrightarrow L^p(B_R)$ is compact, where B_R denotes the ball in $\mathbb{R}^N$ with center zero and radius $R > 0$. This implies that

$$u_k \to u \quad \text{in } L^p(B_R).$$

Since $\mathbb{R}^N = \bigcup_{R>0} B_R$, we deduce that $u \geq 0$ a.e. in $\mathbb{R}^N$. By using Hölder's inequality, (H1) and (H2), we have for all $R > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}
&\left| \int_{|x| \geq R} F(x, u_k)\, \mathrm{d}x \right| \\
&\leq \left(\int_{|x| \geq R} m(x)^r\, \mathrm{d}x \right)^{\frac{1}{r}} \left(\int_{|x| \geq R} |u_k|^{p^*}\, \mathrm{d}x \right)^{\frac{N-p}{N}} \left(\int_{|x| \geq R} |u_k|^{q^*}\, \mathrm{d}x \right)^{\frac{N-q}{N}} \\
&\leq c_2 \left(\int_{|x| \geq R} m(x)^r\, \mathrm{d}x \right)^{\frac{1}{r}},
\end{aligned}$$

where c_2 is a positive constant independent of R and k. The same holds also for u, that is

$$\left| \int_{|x| \geq R} F(x, u)\, \mathrm{d}x \right| \leq c_3 \left(\int_{|x| \geq R} m(x)^r\, \mathrm{d}x \right)^{\frac{1}{r}},$$

for some constant $c_3 > 0$ independent of R. Since $m \in L^r(\mathbb{R}^N)$, we have

$$\lim_{R \to \infty} \int_{|x| \geq R} m(x)^r\, \mathrm{d}x = 0,$$

which implies that for each $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\left| \int_{|x| \geq R_\varepsilon} F(x, u_k)\, \mathrm{d}x \right| \leq \varepsilon, \quad \left| \int_{|x| \geq R_\varepsilon} |F(x, u)\, \mathrm{d}x \right| \quad \leq \varepsilon \tag{3.3}$$

for all $k \in \mathbb{N}$. On the other hand, applying Young's inequality, for any $x \in \mathbb{R}$ and $t \in (0, \infty)$, we have the following estimate

$$F(x, t) \leq \frac{m(x)^{r+\delta}}{r+\delta} + \frac{t^{(p+q)(r+\delta)'}}{(r+\delta)'},$$

for a.a. $x \in B_\varepsilon = \{x \in \mathbb{R}^N : |x| < R_\varepsilon\}$ and for all $t \in \mathbb{R}$. Note that $(p + q)(r + \delta)' < p^*$, since $(r + \delta)' < r' < \frac{p^*}{p+q}$. Hence the continuity of the Nemytskij operator implies

$$\int_{|x| < R_\varepsilon} F(x, u_k)\, \mathrm{d}x \to \int_{|x| < R_\varepsilon} F(x, u)\, \mathrm{d}x, \tag{3.4}$$

as $k \to \infty$ and any fixed $\varepsilon > 0$. Combining (3.3) and (3.4) we conclude that

$$\int_{\mathbb{R}^N} F(x, u_k)\,\mathrm{d}x \to \int_{\mathbb{R}^N} F(x, u)\,\mathrm{d}x,$$

as $k \to \infty$. Hence, since $J(u_k) \geq \mathcal{S}_0$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} F(x, u)\,\mathrm{d}x &\geq \mathcal{S}_0(\|\nabla u_k\|_p^\alpha + \|\nabla u_k\|_p^{p^*})(\|\nabla u_k\|_q^\alpha + \alpha\|\nabla u_k\|_q^{q^*})\\ &\geq \mathcal{S}_0(A_1^\alpha + A_1^{p^*})(A_3^\alpha + A_3^{p^*})\\ &> 0.\end{aligned}$$

Therefore $u \not\equiv 0$ in $\mathbb{R}^N$.

From the weak lower semicontinuity of the norm in $D^{1,p}(\mathbb{R}^N)$ and in $D^{1,q}(\mathbb{R}^N)$, we obtain

$$\mathcal{S} = \limsup_{k\to\infty} J(u_k) \leq \frac{\displaystyle\int_{\mathbb{R}^N} F(x, u)\,\mathrm{d}x}{\liminf\limits_{k\to\infty}(\|\nabla u_k\|_p^\alpha + \|\nabla u_k\|_p^{p^*})(\|\nabla u_k\|_q^\alpha + \|\nabla u_k\|_q^{q^*})}.$$

Thus

$$\mathcal{S} \leq \frac{\displaystyle\int_{\mathbb{R}^N} F(x, u)\,\mathrm{d}x}{(\|\nabla u\|_p^\alpha + \|\nabla u\|_p^{p^*})(\|\nabla u\|_q^\alpha + \|\nabla u\|_q^{q^*})} = J(u).$$

Consequently, we get $J(u) = \mathcal{S}$.

Let φ be a fixed element in X and consider

$$\xi_p(\varepsilon) = \|u + \varepsilon\varphi\|_p, \quad \xi_q(\varepsilon) = \|u + \varepsilon\varphi\|_q.$$

Due to the fact that $u \neq 0$, and the continuity of ξ_p and ξ_q, we can find $\varepsilon_0 > 0$ such that

$$\|u + \varepsilon\varphi\|_p > 0 \quad \text{and} \quad \|u + \varepsilon\varphi\|_q > 0 \quad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we define the function

$$\begin{aligned}\eta(\varepsilon) &= J(u + \varepsilon\varphi)\\ &= \frac{\displaystyle\int_{\mathbb{R}^N} F(x, u + \varepsilon\varphi)\,\mathrm{d}x}{(\|\nabla u + \varepsilon\nabla\varphi\|_p^\alpha + \|\nabla u + \varepsilon\nabla\varphi\|_p^{p^*})(\|\nabla u + \varepsilon\nabla\varphi\|_q^\alpha + \|\nabla u + \varepsilon\nabla\varphi\|_q^{q^*})}.\end{aligned}$$

The functions

$$\begin{aligned}
&\int_{\mathbb{R}^N} F(x, u+\varepsilon\varphi)\,\mathrm{d}x, \\
&K_1(\varepsilon) = \|\nabla u + \varepsilon\nabla\varphi\|_p^{\alpha}, \quad K_2(\varepsilon) = \|\nabla u + \varepsilon\nabla\varphi\|_p^{p^*}, \\
&K_3(\varepsilon) = \|\nabla u + \varepsilon\nabla\varphi\|_q^{\alpha}, \quad K_4(\varepsilon) = \|\nabla u + \varepsilon\nabla\varphi\|_q^{q^*}
\end{aligned}$$

are differentiable with respect to ε with derivatives

$$\begin{aligned}
K_1'(\varepsilon) &= \alpha\|\nabla u + \varepsilon\nabla\varphi\|_p^{\alpha-p}\int_{\mathbb{R}^N} |\nabla u + \varepsilon\nabla\varphi|^{p-2}(\nabla u + \varepsilon\nabla\varphi)\cdot\nabla\varphi\,\mathrm{d}x, \\
K_2'(\varepsilon) &= p^*\|\nabla u + \varepsilon\nabla\varphi\|_p^{p^*-p}\int_{\mathbb{R}^N} |\nabla u + \varepsilon\nabla\varphi|^{p-2}(\nabla u + \varepsilon\nabla\varphi)\cdot\nabla\varphi\,\mathrm{d}x, \\
K_3'(\varepsilon) &= \alpha\|\nabla u + \varepsilon\nabla\varphi\|_q^{\alpha-q}\int_{\mathbb{R}^N} |\nabla u + \varepsilon\nabla\varphi|^{q-2}(\nabla u + \varepsilon\nabla\varphi)\cdot\nabla\varphi\,\mathrm{d}x, \\
K_4'(\varepsilon) &= q^*\|\nabla u + \varepsilon\nabla\varphi\|_q^{q^*-q}\int_{\mathbb{R}^N} |\nabla u + \varepsilon\nabla\varphi|^{q-2}(\nabla u + \varepsilon\nabla\varphi)\cdot\nabla\varphi\,\mathrm{d}x.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\eta'(\varepsilon) = {}& \frac{(K_1(\varepsilon)+K_2(\varepsilon))(K_3(\varepsilon)+K_4(\varepsilon))\int_{\mathbb{R}^N} f(x,u+\varepsilon\varphi)\varphi\,\mathrm{d}x}{(K_1(\varepsilon)+K_2(\varepsilon))^2(K_3(\varepsilon)+K_4(\varepsilon))^2} \\
&- \frac{(K_1'(\varepsilon)+K_2'(\varepsilon))(K_3(\varepsilon)+K_4(\varepsilon))\int_{\mathbb{R}^N} F(x,u+\varepsilon\varphi)\,\mathrm{d}x}{(K_1(\varepsilon)+K_2(\varepsilon))^2(K_3(\varepsilon)+K_4(\varepsilon))^2} \\
&- \frac{(K_3'(\varepsilon)+K_4'(\varepsilon))(K_1(\varepsilon)+K_2(\varepsilon))\int_{\mathbb{R}^N} F(x,u+\varepsilon\varphi)\,\mathrm{d}x}{(K_1(\varepsilon)+K_2(\varepsilon))^2(K_3(\varepsilon)+K_4(\varepsilon))^2}.
\end{aligned}$$

Since zero is a global maximum of the function η, one has $\eta'(0) = 0$. This implies that

$$\begin{aligned}
&(K_1(0)+K_2(0))(K_3(0)+K_4(0))\int_{\mathbb{R}^N} f(x,u)\varphi\,\mathrm{d}x \\
&= (K_1'(0)+K_2'(0))(K_3(0)+K_4(0))\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\
&\quad + (K_3'(0)+K_4'(0))(K_1(0)+K_2(0))\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x.
\end{aligned}$$

Thus,

$$\begin{aligned}&(K_1'(0)+K_2'(0))(K_3(0)+K_4(0))+(K_3'(0)+K_4'(0))(K_1(0)+K_2(0))\\&=\frac{(K_1(0)+K_2(0))(K_3(0)+K_4(0))}{\displaystyle\int_{\mathbb{R}^N}F(x,u)\,\mathrm{d}x}\int_{\mathbb{R}^N}f(x,u)\varphi\,\mathrm{d}x.\end{aligned}$$

Finally, we get

$$\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x+\mu\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x=\lambda\int_{\mathbb{R}^N}f(x,u)\varphi\,\mathrm{d}x,$$

for all $\varphi\in X$, where

$$\begin{aligned}\lambda&=\frac{(\|\nabla u\|_p^\alpha+\|\nabla u\|_p^{p^*})(\|\nabla u\|_q^\alpha+\|\nabla u\|_q^{q^*})}{(\alpha\|\nabla u\|_p^{\alpha-p}+p^*\|\nabla u\|_p^{p^*-p})(\|\nabla u\|_q^\alpha+\|\nabla u\|_q^{q^*})\displaystyle\int_{\mathbb{R}^N}F(x,u)\,\mathrm{d}x},\\\mu&=\frac{(\alpha\|\nabla u\|_q^{\alpha-q}+q^*\|\nabla u\|_q^{q^*-q})(\|\nabla u\|_p^\alpha+\|\nabla u\|_p^{p^*})}{(\alpha\|\nabla u\|_p^{\alpha-p}+p^*\|\nabla u\|_p^{p^*-p})(\|\nabla u\|_q^\alpha+\|\nabla u\|_q^{q^*})}.\end{aligned}$$

This completes the proof of Theorem 3.1. $\square$

The following theorem gives the boundedness, positivity and decay of weak solutions to (3.1).

Theorem 3.2. *Under the same assumptions of Theorem 3.1, we have $u\in L^\sigma(\mathbb{R}^N)$ for any $\sigma\in[q^*,\infty]$. Moreover, $u>0$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty}u(x)=0$.*

Proof. Let $M>0$ and define the cut-off function $u_M(x)=\inf\{u(x),M\}$. Let $\kappa>0$ and $\varphi=u_M^{\kappa p+1}$ be a test function in (3.2). Note that $\varphi\in X\cap L^\infty(\mathbb{R}^N)$. This yields

$$(\kappa p+1)\left(\int_{\mathbb{R}^N}u_M^{\kappa p}|\nabla u_M|^p\,\mathrm{d}x+\mu\int_{\mathbb{R}^N}u_M^{\kappa p}|\nabla u_M|^q\,\mathrm{d}x\right)\le\lambda\int_{\mathbb{R}^N}mu^{p+q-1}u_M^{\kappa p+1}\,\mathrm{d}x.$$

On one hand, we have

$$\frac{1}{c_4^p}\frac{\kappa p+1}{(\kappa+1)^p}\left(\int_{\mathbb{R}^N}u_M^{(\kappa+1)p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}\le\frac{\kappa p+1}{(\kappa+1)^p}\int_{\mathbb{R}^N}|\nabla u_M^{\kappa+1}|^p\,\mathrm{d}x,$$

for some constant $c_4 > 0$. We also have

$$\begin{aligned}
&\frac{\kappa p+1}{(\kappa+1)^p}\int_{\mathbb{R}^N}|\nabla u_M^{\kappa+1}|^p\,\mathrm{d}x\\
&=(\kappa p+1)\int_{\mathbb{R}^N}u_M^{\kappa p}|\nabla u_M|^p\,\mathrm{d}x\\
&\le(\kappa p+1)\left(\int_{\mathbb{R}^N}u_M^{\kappa p}|\nabla u_M|^p\,\mathrm{d}x+\mu\int_{\mathbb{R}^N}u_M^{\kappa p}|\nabla u_M|^q\,\mathrm{d}x\right).
\end{aligned}$$

Hence

$$\frac{1}{c_4^p}\frac{\kappa p+1}{(\kappa+1)^p}\left(\int_{\mathbb{R}^N}u_M^{(\kappa+1)p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}\le\lambda\int_{\mathbb{R}^N}mu^{p+q-1}u_M^{\kappa p+1}\,\mathrm{d}x. \tag{3.5}$$

On the other hand, for $0<\delta<1$ small enough, let $t=\dfrac{p^*pr}{pr+\frac{\delta p^*}{r+\delta}}$. Remark that $p<t<p^*$ and $\dfrac{1}{r+\delta}+\dfrac{q}{q^*}+\dfrac{p}{t}=1$. Regarding the second term in the right-hand side of (3.5), using (H1) and Hölder's inequality, we derive

$$\begin{aligned}
&\int_{\mathbb{R}^N}mu^{p+q-1}u_M^{\kappa p+1}\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^N}mu^qu^{(\kappa+1)p}\,\mathrm{d}x\\
&\le\left(\int(m(x))^{r+\delta}\,\mathrm{d}x\right)^{\frac{1}{r+\delta}}\left(\int u^{q^*}\,\mathrm{d}x\right)^{\frac{q}{q^*}}\left(\int u^{(k+1)t}\,\mathrm{d}x\right)^{\frac{p}{t}}.
\end{aligned}\tag{3.6}$$

Combining (3.5) and (3.6), there exists a constant $c_5>0$ independent of $M>0$ and $\kappa>0$ such that

$$\left(\int_{\mathbb{R}^N}u_M^{(\kappa+1)p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}\le c_5\frac{(\kappa+1)^p}{\kappa p+1}\left(\int u^{(k+1)t}\,\mathrm{d}x\right)^{\frac{p}{t}},$$

that is

$$\|u_M\|_{(\kappa+1)p^*}\le c_6^{\frac{1}{\kappa+1}}\left[\frac{\kappa+1}{(\kappa p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\kappa+1}}\left(\int u^{(k+1)t}\,\mathrm{d}x\right)^{\frac{1}{t(\kappa+1)}}, \tag{3.7}$$

with $c_6=c_5^{\frac{1}{p}}$. Since $u\in D^p(\mathbb{R}^N)$, and hence $u\in L^{p^*}(\mathbb{R}^N)$, one can choose κ_1 in (3.7) such that $(\kappa_1+1)t=p^*$, that is $\kappa_1=\frac{p^*}{t}-1$. Then we have

$$\|u_M\|_{(\kappa_1+1)p^*}\le c_6^{\frac{1}{\kappa_1+1}}\left[\frac{\kappa_1+1}{(\kappa_1p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\kappa_1+1}}\|u\|_{(\kappa_1+1)t},$$

for all $M > 0$. Hence since $\lim_{M\to\infty} u_M(x) = u(x)$, Fatou's lemma implies

$$\|u\|_{(\kappa_1+1)p^*} \leq c_6^{\frac{1}{\kappa_1+1}} \left[\frac{\kappa_1+1}{(\kappa_1 p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\kappa_1+1}} \|u\|_{p^*}.$$

This gives $u \in L^{(\kappa_1+1)p^*}\left(\mathbb{R}^N\right)$. By the same argument, one can choose κ_2 in (3.7) such that $(\kappa_2+1)t = (\kappa_1+1)\,p^*$, that is $\kappa_2 = \left(\frac{p^*}{t}\right)^2 - 1$ to have

$$\|u\|_{(\kappa_2+1)p^*} \leq c_6^{\frac{1}{\kappa_2+1}} \left[\frac{\kappa_2+1}{(\kappa_2 p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\kappa_2+1}} \|u\|_{(\kappa_1+1)p^*}.$$

By iteration, we obtain $\kappa_n = (\frac{p^*}{t})^n - 1$ such that

$$\|u\|_{(\kappa_n+1)p^*} \leq c_6^{\frac{1}{\kappa_n+1}} \left[\frac{\kappa_n+1}{(\kappa_n p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\kappa_n+1}} \|u\|_{(\kappa_{n-1}+1)p^*} \quad \text{for all } n \in \mathbb{N}.$$

It follows

$$\|u\|_{(\kappa_n+1)p^*} \leq c_6^{\sum_{i=1}^n \frac{1}{\kappa_i+1}} \prod_{i=1}^n \left[\frac{\kappa_i+1}{(\kappa_i p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\kappa_i+1}} \|u\|_{p^*},$$

or equivalently

$$\|u\|_{(\kappa_n+1)p^*} \leq c_6^{\sum_{i=1}^n \frac{1}{\kappa_i+1}} \prod_{i=1}^n \left[\left[\frac{\kappa_i+1}{(\kappa_i p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{\kappa_i+1}}}\right]^{\frac{1}{\sqrt{\kappa_i+1}}} \|u\|_{p^*}.$$

Since

$$\left[\frac{z+1}{(zp+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{z+1}}} > 1 \quad \text{for all } z > 0 \quad \text{and} \quad \lim_{z\to\infty} \left[\frac{z+1}{(zp+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{z+1}}} = 1,$$

there exists a constant $c_7 > 0$ independent of $n \in \mathbb{N}$ such that

$$\|u\|_{(\kappa_n+1)p^*} \leq c_6^{\sum_{i=1}^n \frac{1}{\kappa_i+1}} c_7^{\sum_{i=1}^n \frac{1}{\sqrt{\kappa_i+1}}} \|u\|_{p^*},$$

where

$$\frac{1}{\kappa_i+1} = \left(\frac{t}{p^*}\right)^i, \quad \frac{1}{\sqrt{\kappa_i+1}} = \left(\sqrt{\frac{t}{p^*}}\right)^i, \quad \frac{t}{p^*} < \sqrt{\frac{t}{p^*}} < 1.$$

Hence, there exists a constant $c_8 > 0$ independent of $n \in \mathbb{N}$ such that

$$\|u\|_{(\kappa_n+1)p^*} \leq c_8 \|u\|_{p^*} \tag{3.8}$$

for all $n \in \mathbb{N}$. Passing to the limit as n goes to infinity we get

$$\|u\|_\infty \le c_8 \|u\|_{p^*}. \tag{3.9}$$

Therefore, due to (3.8) and (3.9), we deduce that

$$u \in L^\sigma\left(\mathbb{R}^N\right) \quad \text{for all} \quad p^* \le \sigma \le \infty.$$

Finally, since $q^* < p^*$ we have $u \in L^\sigma\left(\mathbb{R}^N\right)$ for all $q^* \le \sigma \le \infty$.

The positivity of u follows from the weak Harnack type inequality proved in Trudinger [18]. Indeed, let us remark for the reader's convenience that since $u \in L^\infty(\mathbb{R}^N)$ and $u \ge 0$ in a cube $K = K(3\rho) \subset \mathbb{R}^N$, then using Theorem 1.1 of [18], there exists a constant $C = C(\rho)$ such that

$$\max_{K(\rho)} u(x) \le C \min_{K(\rho)} u(x),$$

where $K(\rho)$ is the cube of side ρ centered at 0 whose sides are parallel to the coordinate axes. Suppose that there is a subset E of $\mathbb{R}^N$ such that $u \equiv 0$ a.e. in E with $|E|_N \ne 0$, where $|\cdot|_N$ is the Lebesgue measure in $\mathbb{R}^N$. Then, since $E = \bigcup_{\rho \in \mathbb{Q}} (E \cap K(\rho))$ and $|E|_N \ne 0$, there exists ρ' with $|E \cap K(\rho')|_N \ne 0$. Hence

$$0 \le \max_{K(\rho')} u(x) \le C \min_{K(\rho')} u(x) \le C \min_{E \cap K(\rho')} u(x) = 0.$$

This implies that $u \equiv 0$ a.e. in $K(\rho')$. A similar argument shows that $u \equiv 0$ a.e. in $K(\rho)$ for all $\rho > \rho'$. Thus $u \equiv 0$ a.e. in $\mathbb{R}^N$, which leads to a contradiction. Therefore $u > 0$ in $\mathbb{R}^N$.

Finally, since $hu^q \in L^{\frac{N}{p-\varepsilon}}(\mathbb{R}^N), 0 < \varepsilon < 1$, then the decay of u follows directly from Theorem 1 of Serrin [16].

This finished the proof of Theorem 3.2. □

Remark 3.3. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then $u \in C^{1,\alpha}(B_R(0))$ for any $R > 0$ with some $\alpha = \alpha(R) \in (0, 1)$. The proof follows immediately from the regularity results of Tolksdorf* [17].

Remark 3.4. *One can prove similar results as in Theorems 3.1 and 3.2 for problem* (3.1) *under the more general assumptions*

$$0 \le f(x, t) \le \sum_{i=1}^{I} m_i(x) t^{\alpha_i + \beta_i + 1},$$

for a.a. $x \in \mathbb{R}^N$ and for all $t \ge 0$, where m_i are positive functions such that $m_i(x) \in L^{s_i}(\mathbb{R}^N)$, $s_i \in [r_i, r_i + \delta]$ with $r_i = \frac{1}{1 - (\frac{\alpha_i+1}{p^*} + \frac{\beta_i+1}{q^*})}$, $\frac{\alpha_i+1}{p^*} + \frac{\beta_i+1}{q^*} < 1$, $\alpha_i, \beta_i > 0$, $i = 1, .., I$ and some $\delta \in (0, 1)$.

The decay and the positivity of u can be obtained by the above processes if we suppose moreover that $\alpha_i \geq p-1$ and $\beta_i \geq q-1, i=1,..,I$.

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