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Variational–hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions

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ABSTRACT

In this paper variational–hemivariational inequalities with nonhomogeneous Neumann boundary conditions are investigated. Under an appropriate oscillating behavior of the nonlinear term, the existence of infinitely many solutions to this type of problems, even under small perturbations of nonhomogeneous Neumann boundary conditions, is established.

1. Introduction

The aim of this paper is to investigate variational–hemivariational inequalities with a nonhomogeneous Neumann boundary condition. Precisely, let Ω be a non-empty, bounded, open subset of the Euclidean space $\mathbb{R}^N$, $N \geqslant 1$, with C^1-boundary $\partial\Omega$, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $q \geqslant 0, q \not\equiv 0$. Our purpose is to study the following problem: Find $u \in K$ such that, for all $v \in K$,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\big(v(x)-u(x)\big)\,dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)\big(v(x)-u(x)\big)\,dx + \int_\Omega \lambda\alpha(x)F^\circ\big(u(x);v(x)-u(x)\big)\,dx + \int_{\partial\Omega}\mu\beta(x)G^\circ\big(\gamma u(x);\gamma v(x)-\gamma u(x)\big)\,d\sigma \geqslant 0, \tag{P}$$

where K is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, and $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial\Omega)$, with $\alpha(x) \geqslant 0$ for a.a. $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geqslant 0$ for a.a. $x \in \partial\Omega$, and λ, μ are real parameters, with $\lambda > 0$ and $\mu \geqslant 0$. Here, F° and G° stand for Clarke's generalized directional derivatives of locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)\,dt$, $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator. If $\Omega =]a,b[\subseteq \mathbb{R}$ and $h : \{a,b\} \to \mathbb{R}$, then $\int_{\partial\Omega} h(x)\,d\sigma$ reads $h(b)+h(a)$, so problem (P) makes sense even for $N = 1$.

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A prototype of (P) for $K = W^{1,p}(\Omega)$ is the following boundary value problem with non-smooth potential and nonhomogeneous, non-smooth Neumann boundary condition

$$\begin{cases} \Delta_p u - q(x)|u|^{p-2}u \in \lambda\alpha(x)\partial F(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} \in -\mu\beta(x)\partial G(\gamma u) & \text{on } \partial\Omega. \end{cases} \tag{N}$$

The main result of this paper is Theorem 3.1, which establishes, under an appropriate oscillating behavior of F and a suitable growth of G at infinity, the existence of a precise interval for the real parameter λ such that, provided μ is small enough, problem (P) admits infinitely many solutions. Some consequences and applications are also pointed out (see Theorems 3.6, 3.8 and Section 4). Just as an example, we illustrate the applicability of our approach by stating the following consequence of our results in the special case of ordinary differential equations.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, locally essentially bounded function and set $F(\xi) = \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for each non-negative, continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g_\infty := \lim_{t\to+\infty} \frac{g(t)}{t} < +\infty,$$

and for every $\mu \in \left]0, \frac{1}{8g_\infty}\right[$, there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0,1[)$ such that for all $n \in \mathbb{N}$ one has

$$\begin{cases} -u_n''(x) + u_n(x) \in \left[f^-\big(u_n(x)\big), f^+\big(u_n(x)\big)\right] & \textit{for a.a. } x \in]0,1[, \\ u_n'(0) = \mu g\big(u_n(0)\big), \\ u_n'(1) = -\mu g\big(u_n(1)\big), \end{cases} \tag{ON}$$

where $f^-(t) = \lim_{\delta\to0^+} \operatorname{ess\,inf}_{|t-z|<\delta} f(z)$ and $f^+(t) = \lim_{\delta\to0^+} \operatorname{ess\,sup}_{|t-z|<\delta} f(z)$ for all $t \in \mathbb{R}$.

Clearly, if f is a continuous function, then Theorem 1.1 ensures the existence of infinitely many classical solutions to the Neumann boundary value problem (ON). It is worth noticing that our results are completely novel, even for continuous nonlinearities f and g, because of the presence of nonhomogeneous Neumann boundary condition (see (N)). We refer to [2] and [3], and the references therein, for smooth Neumann problems in the homogeneous case. Moreover, we also observe that our results and those of [11] are different since in [11] the Neumann boundary condition is homogeneous and the type of oscillating behavior at infinity required for f implies that f cannot be of constant sign, which is not necessary the case here. Regarding the existence of infinitely many solutions for non-smooth Neumann-type problems we also mention the paper of Candito [5] and the work of Kristály and Motreanu (see [9]) where in the second paper the authors don't require that $W^{1,p}(\Omega)$ is continuously embedded into $C^0(\overline{\Omega})$. Recently, Kristály and Moroşanu have described a new competition phenomena between oscillatory and pure power terms (cf. [8]) while existence results for variational–hemivariational inequalities of type (P) were established in [15] applying an abstract non-smooth critical point result given in [11].

2. Preliminaries

In this section we give a brief overview on some prerequisites on non-smooth analysis which are needed in the sequel. Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $h : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geqslant 0$ such that

$$\big|h(z) - h(w)\big| \leqslant L_x\|z - w\|, \quad \forall z, w \in V_x.$$

If $x, z \in X$, we write $h^\circ(x; z)$ for the generalized directional derivative of h at the point x along the direction z, i.e.,

$$h^\circ(x; z) := \limsup_{w\to x, t\to0^+} \frac{h(w + tz) - h(w)}{t}$$

(see [7, Chapter 2]). If $h_1, h_2 : X \to \mathbb{R}$ are locally Lipschitz functions, we have

$$(h_1 + h_2)^\circ(x, z) \leqslant h_1^\circ(x, z) + h_2^\circ(x, z), \quad \forall x, z \in X. \tag{2.1}$$

The generalized gradient of the function h at x, denoted by $\partial h(x)$, is the set

$$\partial h(x) := \{x^* \in X^*: \langle x^*, z\rangle \leqslant h^\circ(x; z),\ \forall z \in X\}.$$

We say that $x \in X$ is a (generalized) critical point of h when

$$h^\circ(x; z) \geqslant 0, \quad \forall z \in X,$$

that clearly means $0 \in \partial h(x)$ (see [6]).

When a non-smooth function $I : X \to]-\infty, +\infty]$ is expressed as a sum of a locally Lipschitz function, $h : X \to \mathbb{R}$, and a convex, proper and lower semicontinuous function, $j : X \to]-\infty, +\infty]$, that is $I := h + j$, a (generalized) critical point of I is every $u \in X$ such that

$$h^\circ(u; v - u) + j(v) - j(u) \geqslant 0,$$

for all $v \in X$ (see [13, Chapter 3] and [14]).

From now on, assume that X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D(j)$ is the effective domain of j. Write

$$\Psi := \Upsilon - j \quad \text{and} \quad J_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j.$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset \tag{2.2}$$

for all $r > \inf_X \Phi$. Moreover, by (2.2) and provided $r > \inf_X \Phi$, we can define

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

and

$$\varphi^+ := \liminf_{r \to +\infty} \varphi(r), \qquad \varphi^- := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Assuming also that Φ and Υ are locally Lipschitz continuous functionals, in [4] it is proved the following result, which is a version of [11, Theorem 1.1].

Theorem 2.1. *Under the above assumptions on X, Φ and Ψ, one gets*:

(a) *If $\varphi^+ < +\infty$ then, for each $\lambda \in]0, \frac{1}{\varphi^+}[$, the following alternative holds*:
either
(a_1) *J_λ possesses a global minimum,*
or
(a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of J_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\varphi^- < +\infty$ then, for each $\lambda \in]0, \frac{1}{\varphi^-}[$, the following alternative holds*:
either
(b_1) *there is a global minimum of Φ which is also a local minimum of J_λ,*
or
(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of J_λ, with $\lim_{n\to\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

We recall that the previous theorem is a non-smooth version of Ricceri's variational principle (see [16]).

On the space $W^{1,p}(\Omega)$ we consider the norm

$$\|u\| := \left(\int_\Omega \left(|\nabla u(x)|^p + q(x)|u(x)|^p\right) dx\right)^{\frac{1}{p}},$$

which is equivalent to the usual one (see for instance [12, Section 1.1.15]). Set

$$c := \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|}, \tag{2.3}$$

where $\|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|$. From (2.3) we infer that $c^p \|q\|_1 \geqslant 1$. If Ω is convex, an explicit upper bound for the constant c in (2.3) is

$$c \leqslant 2^{\frac{p-1}{p}} \max\left\{ \frac{1}{\|q\|_1^{1/p}}, \frac{d}{N^{1/p}} \left(\frac{p-1}{p-N} |\Omega| \right)^{\frac{p-1}{p}} \frac{\|q\|_\infty}{\|q\|_1} \right\}, \tag{2.4}$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω and $d := \operatorname{diam}(\Omega)$ (see, e.g., [1, Remark 1]). Finally, we set

$$A = \liminf_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi}(-F(t))}{\xi^p}, \qquad B = \limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p},$$

and

$$\lambda_1 = \frac{\|q\|_1}{\|\alpha\|_1 p B}, \qquad \lambda_2 = \frac{1}{\|\alpha\|_1 p c^p A}. \tag{2.5}$$

3. Main results

Our main result is the following.

Theorem 3.1. *Let $\alpha \in L^1(\Omega)$ and $\beta \in L^1(\partial\Omega)$ be non-negative and non-zero functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function and set $F(\xi) = \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbb{R}$. Assume that*

$$A < \frac{1}{c^p \|q\|_1} B. \tag{3.1}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (2.5), for each locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$ satisfies

$$G_\infty := \limsup_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi}(-G(t))}{\xi^p} < +\infty, \tag{3.2}$$

$$\liminf_{\xi \to +\infty} \bigl(-G(\xi)\bigr) > -\infty, \tag{3.3}$$

and for every $\mu \in [0, \delta[$, where

$$\delta = \delta_{g,\lambda} := \frac{1}{\beta^* p c^p G_\infty} \left(1 - \frac{\lambda}{\lambda_2} \right) \quad (\delta = +\infty \text{ if } G_\infty = 0),$$

with $\beta^* = \int_{\partial\Omega} \beta(x)\,d\sigma$, problem (P) admits a sequence of weak solutions that is unbounded in $W^{1,p}(\Omega)$.

Proof. Our aim is to apply Theorem 2.1. To this end, fix $\bar\lambda \in]\lambda_1, \lambda_2[$ and let g be a locally essentially bounded function satisfying our assumptions. Since $\bar\lambda < \lambda_2$, one has $\delta := \delta_{g,\bar\lambda} > 0$, so we can consider $0 \leqslant \bar\mu < \delta$. It follows that $\bar\lambda \|\alpha\|_1 p c^p A + \bar\mu \beta^* p c^p G_\infty < 1$, which implies

$$\bar\lambda < \frac{1}{\|\alpha\|_1 p c^p A + \frac{\bar\mu}{\bar\lambda} \beta^* p c^p G_\infty}. \tag{3.4}$$

Let X be the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm $\|\cdot\|$. For any $u \in X$, set

$$\Phi(u) := \frac{1}{p}\|u\|^p, \qquad \Upsilon(u) := \int_\Omega \alpha(x)\bigl[-F\bigl(u(x)\bigr)\bigr]\,dx + \frac{\bar\mu}{\bar\lambda} \int_{\partial\Omega} \beta(x)\bigl[-G(\gamma u(x)\bigr]\,d\sigma,$$

$$j(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases} \qquad \Psi(u) := \Upsilon(u) - j(u), \qquad J_\lambda(u) := \Phi(u) - \lambda\Psi(u).$$

Therefore,

$$J_{\bar\lambda}(u) = \frac{1}{p}\|u\|^p + \bar\lambda \int_\Omega \alpha(x) F\bigl(u(x)\bigr)\,dx + \bar\mu \int_{\partial\Omega} \beta(x) G\bigl(\gamma u(x)\bigr)\,d\sigma + \bar\lambda j(u) \quad \text{for all } u \in X. \tag{3.5}$$

Now, we claim that $\varphi^+ < +\infty$. Let $\{\rho_n\}$ be a real sequence such that $\lim_{n\to+\infty} \rho_n = +\infty$ and

$$\lim_{n \to +\infty} \frac{\max_{|t| \leqslant \rho_n}(-F(t))}{\rho_n^p} = A.$$

Denote $r_n = \frac{1}{p}(\frac{\rho_n}{c})^p$ and let $v \in \Phi^{-1}(]-\infty, r_n[)$. Then, taking into account that $\|v\|^p < pr_n$ and $\|v\|_\infty \leqslant c\|v\|$, one has $|v(x)| \leqslant \rho_n$ for every $x \in \Omega$. Therefore, it follows

$$\varphi(r_n) \leqslant \frac{\sup_{\|w\|^p < pr_n}(\int_\Omega \alpha(x)[-F(w(x))]\,dx + \frac{\bar{\mu}}{\bar{\lambda}}\int_{\partial\Omega}\beta(x)[-G(\gamma w(x))]\,d\sigma - j(w))}{r_n}$$
$$\leqslant \frac{\|\alpha\|_1 \max_{|t|\leqslant\rho_n}(-F(t)) + \frac{\bar{\mu}}{\bar{\lambda}}\beta^* \max_{|t|\leqslant\rho_n}(-G(t))}{r_n}$$
$$= pc^p\|\alpha\|_1 \frac{\max_{|t|\leqslant\rho_n}(-F(t))}{\rho_n^p} + pc^p\frac{\bar{\mu}}{\bar{\lambda}}\beta^* \frac{\max_{|t|\leqslant\rho_n}(-G(t))}{\rho_n^p}.$$

Hence, $\varphi^+ \leqslant \limsup_{n\to+\infty}\varphi(r_n) \leqslant pc^p\|\alpha\|_1 A + pc^p\frac{\bar{\mu}}{\bar{\lambda}}\beta^* G_\infty$. From (3.1) and (3.2) we obtain

$$\varphi^+ < +\infty,$$

and our claim is proved. Moreover, taking into account (3.4), we get

$$\bar{\lambda} < \frac{1}{\varphi^+}.$$

Next, we show that the function $J_{\bar{\lambda}}$ in (3.5) is unbounded from below. Let $\{d_n\}$ be a real sequence such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty}\frac{(-F(d_n))}{d_n^p} = B. \tag{3.6}$$

Set $w_n(x) = d_n$ for all $x \in \Omega$ and $n \in \mathbb{N}$. Clearly, $w_n \in K \subset W^{1,p}(\Omega)$ for each $n \in \mathbb{N}$. We see that

$$\|w_n\|^p = d_n^p\|q\|_1$$

and

$$\Phi(w_n) - \bar{\lambda}\Psi(w_n) = \frac{\|w_n\|^p}{p} - \bar{\lambda}\left(\int_\Omega \alpha(x)\big[-F\big(w_n(x)\big)\big]\,dx + \frac{\bar{\mu}}{\bar{\lambda}}\int_{\partial\Omega}\beta(x)\big[-G\big(\gamma w_n(x)\big)\big]\,d\sigma\right) + \bar{\lambda} j(w_n)$$
$$= \frac{d_n^p\|q\|_1}{p} - \bar{\lambda}\left(\|\alpha\|_1\big(-F(d_n)\big) + \frac{\bar{\mu}}{\bar{\lambda}}\beta^*\big(-G(d_n)\big)\right),$$

thus

$$J_{\bar{\lambda}}(w_n) = \frac{d_n^p\|q\|_1}{p} - \bar{\lambda}\left(\|\alpha\|_1\big(-F(d_n)\big) + \frac{\bar{\mu}}{\bar{\lambda}}\beta^*\big(-G(d_n)\big)\right). \tag{3.7}$$

If $B < +\infty$, by (2.5) and since $\bar{\lambda} > \lambda_1$, we can take $\epsilon \in]0, B - \frac{\|q\|_1}{p\|\alpha\|_1\bar{\lambda}}[$. From (3.6) there exists ν_ϵ such that

$$-F(d_n) > (B-\epsilon)d_n^p, \quad \forall n > \nu_\epsilon.$$

Combining with (3.7), one has

$$J_{\bar{\lambda}}(w_n) < d_n^p\left(\frac{\|q\|_1}{p} - \bar{\lambda}\|\alpha\|_1(B-\epsilon)\right) - \bar{\mu}\beta^*\big(-G(d_n)\big).$$

Since $\frac{\|q\|_1}{p} - \bar{\lambda}\|\alpha\|_1(B-\epsilon) < 0$ and, as known from (3.3), $\{(-G(d_n))\}$ is bounded from below, it follows that $\lim_{n\to+\infty} J_{\bar{\lambda}}(w_n) = -\infty$. If $B = +\infty$, fix $M > \frac{\|q\|_1}{p\|\alpha\|_1\bar{\lambda}}$. Then from (3.6) there exists ν_M such that

$$\big(-F(d_n)\big) > Md_n^p, \quad \forall n > \nu_M.$$

Arguing as before, one obtains

$$J_{\bar{\lambda}}(w_n) < d_n^p\left(\frac{\|q\|_1}{p} - \bar{\lambda}\|\alpha\|_1 M\right) - \bar{\mu}\beta^*\big(-G(d_n)\big).$$

By the choice of M, we have $\lim_{n\to+\infty} J_{\bar{\lambda}}(w_n) = -\infty$, which completes the proof that $J_{\bar{\lambda}}$ is unbounded from below. Then, from part (a) of Theorem 2.1, we know that the function $J_{\bar{\lambda}}$ admits a sequence of critical points $\{\bar{u}_n\} \subset X$ such that $\|\bar{u}_n\| \to \infty$ as $n \to \infty$. The fact that $\bar{u}_n \in X$ is a critical point of $J_{\bar{\lambda}}$ reads as

$$(\Phi - \bar{\lambda}\Upsilon)^\circ(\bar{u}_n; v - \bar{u}_n) + \bar{\lambda} j(v) - \bar{\lambda} j(\bar{u}_n) \geqslant 0 \quad \text{for all } v \in X. \tag{3.8}$$

It remains to prove that $\bar{u}_n$ solves problem (P). From (3.8) it follows that $\bar{u}_n \in K$ and

$$(\Phi - \bar{\lambda}\Upsilon)^\circ(\bar{u}_n; v - \bar{u}_n) \geqslant 0 \quad \text{for all } v \in K. \tag{3.9}$$

By (3.9) and (2.1) we infer that

$$\Phi'(\bar{u}_n; v - \bar{u}_n) + \bar{\lambda}(-\Upsilon)^\circ(\bar{u}_n; v - \bar{u}_n) \geqslant 0 \quad \text{for all } v \in K$$

or, equivalently,

$$\int_\Omega \left|\nabla \bar{u}_n(x)\right|^{p-2} \nabla \bar{u}_n(x) \cdot \nabla\big(v(x) - \bar{u}_n(x)\big)\, dx + \int_\Omega q(x)\left|\bar{u}_n(x)\right|^{p-2} \bar{u}_n(x)\big(v(x) - \bar{u}_n(x)\big)\, dx + \bar{\lambda}(-\Upsilon)^\circ(\bar{u}_n, v - \bar{u}_n) \geqslant 0, \quad \forall v \in K. \tag{3.10}$$

By using (2.1) and formula (2) on p. 77 in [7], we have

$$\bar{\lambda}(-\Upsilon)^\circ(\bar{u}_n; v - \bar{u}_n) \leqslant \bar{\lambda} \int_\Omega \alpha(x) F^\circ\big(\bar{u}_n(x); v(x) - \bar{u}_n(x)\big)\, dx + \bar{\mu} \int_{\partial\Omega} \beta(x) G^\circ\big(\gamma \bar{u}_n(x); \gamma v(x) - \gamma \bar{u}_n(x)\big)\, d\sigma.$$

Inserting this into (3.10) leads to

$$\int_\Omega \left|\nabla \bar{u}_n(x)\right|^{p-2} \nabla \bar{u}_n(x) \cdot \nabla\big(v(x) - \bar{u}_n(x)\big)\, dx + \int_\Omega q(x)\left|\bar{u}_n(x)\right|^{p-2} \bar{u}_n(x)\big(v(x) - \bar{u}_n(x)\big)\, dx + \bar{\lambda}\left[\int_\Omega \alpha(x) F^\circ\big(\bar{u}_n(x); v(x) - \bar{u}_n(x)\big)\, dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \beta(x) G^\circ\big(\gamma \bar{u}_n(x); \gamma v(x) - \gamma \bar{u}_n(x)\big)\, d\sigma\right] \geqslant 0$$

for every $v \in K$, which completes the proof. □

The solutions obtained in Theorem 3.1 for problem (P) corresponding to the parameters $\bar{\lambda}$ and $\bar{\mu}$ are local minima of the functional $J_{\bar{\lambda}}$ in (3.5) which is associated to (P). The following corollary demonstrates that, if the functional $J_{\bar{\lambda}}$ satisfies the Palais–Smale condition, there are solutions to (P) which are not local minima of $J_{\bar{\lambda}}$.

Corollary 3.2. *Under the hypotheses of Theorem* 3.1, *assume in addition that*

$$\min\left\{F(t) - \frac{1}{s}(-F)^\circ(t; t), G(t) - \frac{1}{s}(-G)^\circ(t; t)\right\} \geqslant -c_1|t|^\theta - d_1 \quad \textit{for all } t \in \mathbb{R}, \tag{3.11}$$

with constants $1 \leqslant \theta < p < s$ *and* $c_1, d_1 \geqslant 0$. *Let* $u \in W^{1,p}(\Omega)$ *be a solution provided by Theorem* 3.1 *for* (P) *corresponding to* $\bar{\lambda} \in]\lambda_1, \lambda_2[$ *and* $0 \leqslant \bar{\mu} < \delta$, *so* u *is a local minimum of the functional* $J_{\bar{\lambda}}$ *in* (3.5). *If* u *is isolated, there exists another solution* $w \in W^{1,p}(\Omega)$ *for* (P) *corresponding to* $\bar{\lambda}$ *and* $\bar{\mu}$ *which is not a local minimum of* $J_{\bar{\lambda}}$.

Proof. Let us check that the functional $J_{\bar{\lambda}} : W^{1,p}(\Omega) \to]-\infty, +\infty]$ given in (3.5) satisfies the Palais–Smale condition in the sense of [13, p. 64]. This amounts to saying that whenever a sequence $\{u_n\} \subset K$ is such that $J(u_n)$ is bounded and

$$J^\circ_{\bar{\lambda}}(u_n; v - u_n) \geqslant -\varepsilon_n \|v - u_n\| \quad \text{for all } v \in K, \tag{3.12}$$

with $\varepsilon_n \to 0^+$, contains a convergent subsequence. Setting $v = 0$ in (3.12) and combining with the inequality $J_{\bar{\lambda}}(u_n) \leqslant M$, for a constant $M > 0$, yield

$$\left(\frac{1}{p} - \frac{1}{s}\right)\|u_n\|^p + \bar{\lambda} \int_\Omega \alpha(x)\left[F\big(u_n(x)\big) - \frac{1}{s}(-F)^\circ(u_n; u_n)\right] dx + \bar{\mu} \int_{\partial\Omega} \beta(x)\left[G\big(\gamma u_n(x)\big) - \frac{1}{s}(-G)^\circ(\gamma u_n; \gamma u_n)\right] d\sigma \leqslant M + \frac{\varepsilon_n}{s}\|u_n\|.$$

Using hypothesis (3.11), it is straightforward to prove that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Furthermore, since $-\Delta_p$ on $W^{1,p}(\Omega)$ fulfills the (S_+) property, by handling (3.12) we find that $\{u_n\}$ contains a convergent subsequence, so the Palais–Smale condition for the functional $J_{\bar{\lambda}}$ is satisfied. We are thus in a position to apply [10, Theorem 4.2] to the functional $J_{\bar{\lambda}}$ from which we achieve the desired conclusion. □

Remark 3.3. Relation (2.4) is useful to verify inequality (3.1) and to estimate the numbers λ_2 and δ in Theorem 3.1 provided the bounded domain Ω is convex.

Remark 3.4. Actually, Theorem 3.1 ensures that the sequence $\{\bar{u}_n\}$ of solutions of problem (P) satisfies the following sharper inequality:

$$\int_\Omega \left|\nabla \bar{u}_n(x)\right|^{p-2} \nabla \bar{u}_n(x) \cdot \nabla\big(v(x) - \bar{u}_n(x)\big)\,dx + \int_\Omega q(x)\left|\bar{u}_n(x)\right|^{p-2} \bar{u}_n(x)\big(v(x) - \bar{u}_n(x)\big)\,dx$$
$$+ \bar{\lambda} U^\circ(\bar{u}_n; v - \bar{u}_n) + \bar{\mu} V^\circ(\bar{u}_n; v - \bar{u}_n) \geqslant 0, \quad \forall v \in K,$$

where $U(u) = \int_\Omega \alpha(x) F(u(x))\,dx$ and $V(u) = \int_{\partial\Omega} \beta(x) G(\gamma u(x))\,d\sigma$ for all $u \in W^{1,p}(\Omega)$.

Remark 3.5. In Theorem 3.1 the function g may not have an oscillating behavior at infinity (see, for instance, Example 4.3 where $g(u) = \sqrt{|u|}$). On the other hand, g must satisfy (3.2), namely it must have a suitable growth at infinity. It is worth noticing that when (3.2) fails, that is,

$$\limsup_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi}(-G(t))}{\xi^p} = +\infty, \tag{3.13}$$

the existence of infinitely many solutions to (P) can be again guaranteed, provided that

$$G_\infty^- := \liminf_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi}(-G(t))}{\xi^p} < +\infty \tag{3.14}$$

(for which g is then oscillating at infinity) and assuming that f, possibly even not oscillating at infinity, satisfies the following conditions:

$$B_+ := \limsup_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi}(-F(t))}{\xi^p} < +\infty, \qquad \liminf_{\xi \to +\infty}\big(-F(t)\big) > -\infty. \tag{3.15}$$

To be precise, the following result holds: Let α, β, f be as in the statement of Theorem 3.1 and assume that f satisfies (3.15). Then, for each $\lambda \in]0, \frac{1}{\|\alpha\|_1 pc^p B_+}[$, for each locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$ satisfying (3.13) and (3.14), and for each $\mu \in]0, \delta[$, where

$$\delta := \delta_{g,\lambda} := \frac{1}{\beta^* pc^p G_\infty^-}\big(1 - \lambda \|\alpha\|_1 pc^p B_+\big) \quad \big(\delta = +\infty \text{ if } G_\infty^- = 0\big),$$

the problem (P) admits a sequence of weak solutions which is unbounded in $W^{1,p}(\Omega)$.

Now we point out two significant special cases of Theorem 3.1.

Theorem 3.6. *Let $\alpha \in L^1(\Omega)$ and $\beta \in L^1(\partial\Omega)$ be non-negative and non-zero. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-positive, locally essentially bounded function, and set $F(\xi) = \int_0^\xi f(t)\,dt$ for every $\xi \in \mathbb{R}$. Assume that*

$$\liminf_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p} < \frac{1}{c^p \|q\|_1} \limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p}. \tag{3.16}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (2.5), *for each non-positive, locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, satisfies*

$$G_\infty := \limsup_{\xi \to +\infty} \frac{-G(\xi)}{\xi^p} < +\infty, \tag{3.17}$$

and for every $\mu \in [0, \delta[$, where

$$\delta = \delta_{g,\lambda} := \frac{1}{(\int_{\partial\Omega} \beta(x)\,d\sigma) pc^p G_\infty}\left(1 - \lambda \|\alpha\|_1 pc^p \liminf_{\xi \to +\infty} \frac{(-F(\xi))}{\xi^p}\right),$$

problem (P) *admits a sequence of weak solutions that is unbounded in $W^{1,p}(\Omega)$.*

Remark 3.7. In Theorem 3.6 the assumption (3.16) can be written

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}<c^p\|q\|_1\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p},$$

as well as

$$]\lambda_1,\lambda_2[=\left]-\frac{\|q\|_1}{\|\alpha\|_1 p\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}},-\frac{1}{\|\alpha\|_1 pc^p\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}}\right[$$

and

$$\delta=-\frac{1}{(\int_{\partial\Omega}\beta(x)\,d\sigma)pc^p(\liminf_{\xi\to+\infty}\frac{G(\xi)}{\xi^p})}\left(1+\lambda\|\alpha\|_1 pc^p\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}\right).$$

Theorem 3.8. *Let $\alpha\in L^1(\Omega)$ and $\beta\in L^1(\partial\Omega)$ be non-negative and non-zero. Let $f:\mathbb{R}\to\mathbb{R}$ be a non-positive, locally essentially bounded function and set $F(\xi)=\int_0^\xi f(t)\,dt$ for every $\xi\in\mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty}\frac{-F(\xi)}{\xi^p}=0\quad\text{and}\quad\limsup_{\xi\to+\infty}\frac{-F(\xi)}{\xi^p}=+\infty.$$

Then, for each non-positive, locally essentially bounded function $g:\mathbb{R}\to\mathbb{R}$ such that

$$g_\infty:=\lim_{\xi\to+\infty}\frac{-g(\xi)}{\xi^{p-1}}<+\infty,$$

and for every $\mu\in[0,\delta[$, where

$$\delta=\delta_g:=\frac{1}{(\int_{\partial\Omega}\beta(x)\,d\sigma)c^p g_\infty},$$

there is an unbounded sequence $\{u_n\}\subset W^{1,p}(\Omega)$ such that

$$\int_\Omega\left|\nabla u_n(x)\right|^{p-2}\nabla u_n(x)\cdot\nabla\big(v(x)-u_n(x)\big)\,dx+\int_\Omega q(x)\left|u_n(x)\right|^{p-2}u_n(x)\big(v(x)-u_n(x)\big)\,dx$$
$$+\int_\Omega\alpha(x)F^\circ\big(u_n(x);\big(v(x)-u_n(x)\big)\big)\,dx+\mu\int_{\partial\Omega}\beta(x)G^\circ\big(\gamma u_n(x);\big(\gamma v(x)-\gamma u_n(x)\big)\big)\,d\sigma\geqslant 0$$

for all $v\in K$.

Remark 3.9. We explicitly observe that in Theorem 3.1 no symmetry assumption on the nonlinear term is done. On the other hand, the case $N\leqslant p$ cannot be studied by this method since the embedding of $W^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$ fails and φ^+ cannot be upper estimated, without further assumptions on the nonlinear term.

Remark 3.10. An example of application of previous results is given in the next section (see Example 4.3).

If F oscillates at zero, we can give an analogous result as in Theorem 3.1. To this end, let

$$A=\liminf_{\xi\to0^+}\frac{\max_{|t|\leqslant\xi}(-F(t))}{\xi^p},\qquad B=\limsup_{\xi\to0^+}\frac{-F(\xi)}{\xi^p},$$

and

$$\lambda_1=\frac{\|q\|_1}{\|\alpha\|_1 pB},\qquad\lambda_2=\frac{1}{\|\alpha\|_1 pc^p A}.\tag{3.18}$$

Then, our result reads as follows.

Theorem 3.11. *Let $\alpha\in L^1(\Omega)$ and $\beta\in L^1(\partial\Omega)$ be non-negative and non-zero functions. Let $f:\mathbb{R}\to\mathbb{R}$ be a locally essentially bounded function and set $F(\xi)=\int_0^\xi f(t)\,dt$ for all $\xi\in\mathbb{R}$. Assume that*

$$A<\frac{1}{c^p\|q\|_1}B.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (3.18), for each locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$ satisfies

$$G_0 := \limsup_{\xi \to 0^+} \frac{\max_{|t| \leqslant \xi}(-G(t))}{\xi^p} < +\infty,$$
$$\liminf_{\xi \to 0^+} \big(-G(\xi)\big) \geqslant 0,$$

and for every $\mu \in [0, \delta[$, where

$$\delta = \delta_{g,\lambda} := \frac{1}{\beta^* p c^p G_0}\left(1 - \frac{\lambda}{\lambda_2}\right) \quad (\delta = +\infty \text{ if } G_0 = 0),$$

with $\beta^* = \int_{\partial\Omega} \beta(x)\,d\sigma$, problem (P) admits a sequence of distinct weak solutions converging uniformly to zero.

Proof. The proof can be done similarly as the proof of Theorem 3.1 by applying part (b) of Theorem 2.1 instead of part (a), thus obtaining the assertion. □

4. Applications and examples

Here we present an application of Theorem 3.1 to an ordinary differential problem with discontinuous nonlinearities.

Theorem 4.1. *Let $\alpha \in L^1(]0,1[)$ be a non-negative and non-zero function and let β_1, β_0 be non-negative constants such that at least one of them is positive. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-positive, locally essentially bounded function and set $F(\xi) = \int_0^\xi f(t)\,dt$ for every $\xi \in \mathbb{R}$. Assume that*

$$\liminf_{\xi \to +\infty} \frac{-F(\xi)}{\xi^2} < \frac{1}{2} \limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^2}. \tag{4.1}$$

Then, for each $\lambda \in \left]\frac{1}{\|\alpha\|_1 2 \limsup_{\xi\to+\infty} \frac{-F(\xi)}{\xi^2}}, \frac{1}{\|\alpha\|_1 4 \liminf_{\xi\to+\infty} \frac{-F(\xi)}{\xi^2}}\right[$, for each non-positive, continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, satisfies

$$G_\infty := \limsup_{\xi \to +\infty} \frac{-G(\xi)}{\xi^2} < +\infty,$$

and for every $\mu \in [0, \delta[$, where

$$\delta = \delta_{g,\lambda} := \frac{1}{(\beta_1 + \beta_0) 4 G_\infty}\left(1 - \lambda \|\alpha\|_1 4 \liminf_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p}\right) \quad (\delta = +\infty \text{ if } G_\infty = 0),$$

there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0,1[)$ such that for all $n \in \mathbb{N}$ one has

$$\begin{cases} u_n''(x) - u_n(x) \in \big[\lambda\alpha(x) f^-\big(u_n(x)\big), \lambda\alpha(x) f^+\big(u_n(x)\big)\big] & \text{for a.a. } x \in]0,1[, \\ u_n'(0) = \mu\beta_0 g\big(u_n(0)\big), \\ u_n'(1) = -\mu\beta_1 g\big(u_n(1)\big). \end{cases} \tag{N1}$$

Proof. The result is a consequence of Theorem 3.6. For the sake of clarity, we first point out three facts specific for the ordinary differential case that enable us to adapt the proof of Theorem 3.1 to this situation. The first one is the inequality

$$\int_{\partial\Omega} \beta(x)\big[-G\big(\gamma u(x)\big)\big]\,d\sigma = \beta(1)\big[-G\big(u(1)\big)\big] + \beta(0)\big[-G\big(u(0)\big)\big]$$
$$\leqslant \beta(1) \max_{|\xi| \leqslant \rho_n}\big[-G(\xi)\big] + \beta(0) \max_{|\xi| \leqslant \rho_n}\big[-G(\xi)\big] = \big(\beta(1) + \beta(0)\big) \max_{|\xi| \leqslant \rho_n}\big[-G(\xi)\big]$$

for all $\|u\|^p < p r_n$, from which we derive

$$\varphi^+ \leqslant \limsup_{n \to +\infty} \varphi(r_n) \leqslant p c^p \|\alpha\|_1 A + p c^p \frac{\bar{\mu}}{\bar{\lambda}}\big(\beta(1) + \beta(0)\big) G_\infty.$$

The second one is the estimate

$$\int_{\partial\Omega} \beta(x)\big[-G\big(\gamma w_n(x)\big)\big]\,d\sigma = \big(\beta(1)+\beta(0)\big)\big[-G(d_n)\big] \geqslant \big(\beta(1)+\beta(0)\big)\liminf_{\xi\to+\infty}\big(-G(\xi)\big) \geqslant 0,$$

from which we deduce $\lim_{n\to+\infty} J_{\bar\lambda}(w_n) = -\infty$. The last one is

$$\left[\int_{\partial\Omega} \beta(x)G\big(\gamma\bar u_n(x);\big(\gamma v(x)-\gamma\bar u_n(x)\big)\big)\,d\sigma\right]^\circ = \big[\beta(1)G\big(\bar u_n(1);v(1)-\bar u_n(1)\big)+\beta(0)G\big(\bar u_n(0);v(0)-\bar u_n(0)\big)\big]^\circ$$
$$\leqslant \beta(1)G^\circ\big(\bar u_n(1);v(1)-\bar u_n(1)\big)+\beta(0)G^\circ\big(\bar u_n(0);v(0)-\bar u_n(0)\big),$$

from which it turns out

$$\int_0^1 \big|\overline{u'}_n(x)\big|^{p-2}\overline{u'}_n(x)\cdot\big(v'(x)-\overline{u'}_n(x)\big)\,dx + \int_0^1 q(x)\big|\bar u_n(x)\big|^{p-2}\bar u_n(x)\big(v(x)-\bar u_n(x)\big)\,dx$$
$$+\bar\lambda\int_0^1 \alpha(x)F^\circ\big(\bar u_n(x);v(x)-\bar u_n(x)\big)\,dx + \bar\mu\big[\beta(1)G^\circ\big(\bar u_n(1);v(1)-\bar u_n(1)\big)+\beta(0)G^\circ\big(\bar u_n(0);v(0)-\bar u_n(0)\big)\big] \geqslant 0,$$
$$\forall v\in K.$$

The proof of Theorem 4.1 is carried out as follows. Fix $\bar\lambda$ and $\bar\mu$ as in the conclusion of Theorem 4.1. We may apply Theorem 3.6 (see also Remark 3.4) by choosing $\Omega =]0,1[$, $p=2$, $q\equiv 1$, $K = W^{1,2}(]0,1[)$, and noticing that hypothesis (4.1) in conjunction with (2.4) implies that (3.16) holds true. Then there exists an unbounded sequence $\{\bar u_n\}\subset W^{1,2}(]0,1[)$ such that

$$\int_0^1 \overline{u'}_n(x)v'(x)\,dx + \int_0^1 \bar u_n(x)v(x)\,dx + \bar\lambda U^\circ(\bar u_n;v) + \bar\mu\big[\beta_1 G^\circ\big(\bar u_n(1);v(1)\big)+\beta_0 G^\circ\big(\bar u_n(0);v(0)\big)\big] \geqslant 0,$$
$$\forall v\in W^{1,2}\big(]0,1[\big),$$

where $\beta_0=\beta(0)$ and $\beta_1=\beta(1)$, while the function U was introduced in Remark 3.4. Setting

$$T_n(v) = -\left[\int_0^1 \overline{u'}_n(x)v'(x)\,dx + \int_0^1 \bar u_n(x)v(x)\,dx\right] - \bar\mu\big[\beta_1 g\big(\bar u_n(1)\big)v(1)+\beta_0 g\big(\bar u_n(0)\big)v(0)\big]$$

for all $v\in W^{1,2}(]0,1[)$, we see that T_n is linear and continuous on $W^{1,2}(]0,1[)$, and $T_n\in\bar\lambda\partial U(\bar u_n)$. Taking into account that $W^{1,2}(]0,1[)$ is continuously and densely embedded in $L^2(]0,1[)$, from [6, Theorem 2.2] we know that there is $h_n\in L^2(]0,1[)$ such that

$$-\left[\int_0^1 \overline{u'}_n(x)v'(x)\,dx + \int_0^1 \bar u_n(x)v(x)\,dx\right] - \bar\mu\big[\beta_1 g\big(\bar u_n(1)\big)v(1)+\beta_0 g\big(\bar u_n(0)\big)v(0)\big] = \int_0^1 h_n(x)v(x)\,dx$$

for all $v\in W^{1,2}(]0,1[)$. This ensures that $\bar u_n$ is the unique solution of the problem

$$\begin{cases} u''-u = h_n(x) \quad \text{in }]0,1[,\\ u'(0) = -\bar\mu\beta_0 g\big(u(0)\big),\\ u'(1) = \bar\mu\beta_1 g\big(u(1)\big)\end{cases}$$

and, in addition, $\bar u_n\in W^{2,2}(]0,1[)$. Moreover, since $T_n\in\bar\lambda\partial U(\bar u_n)$, we deduce through [6, Corollary, p. 111] that

$$h_n(x)\in\big[\bar\lambda\alpha(x)f^-\big(\bar u_n(x)\big),\bar\lambda\alpha(x)f^+\big(\bar u_n(x)\big)\big] \quad \text{for a.a. } x\in]0,1[.$$

Hence the conclusion regarding problem (N1) is obtained with $u_n=\bar u_n$. □

Remark 4.2. Theorem 1.1 in the Introduction is a direct consequence of Theorem 4.1.

Example 4.3. Set

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n \in \mathbb{N}$ and define the non-negative (discontinuous) function $f_p : \mathbb{R} \to \mathbb{R}$ by

$$f_p(t) = \begin{cases} 2(n+1)![n^{p-1}(n+1)!^p - (n-1)^{p-1}n!^p] & \text{if } t \in \bigcup_{n\in\mathbb{N}}]a_n, b_n[, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting $F_p(\xi) = \int_0^\xi f_p(t)\,dt$ for every $\xi \in \mathbb{R}$, a simple computation shows that $\liminf_{\xi\to+\infty} \frac{F_p(\xi)}{\xi^p} = 0$ and $\limsup_{\xi\to+\infty} \frac{F_p(\xi)}{\xi^p} = +\infty$. Owing to Theorem 3.8, there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{1,p}(\Omega)$ such that

$$\int_\Omega \left|\nabla u_n(x)\right|^{p-2} \nabla u_n(x) \cdot \nabla\big(v(x) - u_n(x)\big)\,dx + \int_\Omega \left|u_n(x)\right|^{p-2} u_n(x)\big(v(x) - u_n(x)\big)\,dx$$
$$+ \int_\Omega (-F)^\circ\big(u_n(x); v(x) - u_n(x)\big)\,dx + \int_{\partial\Omega} \Big[-\left|\gamma u_n(x)\right|^{\frac{p-1}{2}}\Big]\big(\gamma v(x) - \gamma u_n(x)\big)\,d\sigma \geqslant 0, \quad \forall v \in K.$$

In particular, Theorem 4.1 ensures that there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0,1[)$ such that for all $n \in \mathbb{N}$ there holds

$$\begin{cases} -u_n''(x) + u_n(x) \in \big[f_2^-\big(u(x)\big), f_2^+\big(u(x)\big)\big] & \text{for a.a. } x \in]0,1[, \\ u_n'(0) = -\sqrt{\left|u_n(0)\right|}, \\ u_n'(1) = \sqrt{\left|u_n(1)\right|}. \end{cases}$$

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