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Multiple solution results for elliptic Neumann problems involving set-valued nonlinearities

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ABSTRACT

The main goal of this paper is to present multiple solution results for elliptic inclusions of Clarke's gradient type under nonlinear Neumann boundary conditions involving the p-Laplacian and set-valued nonlinearities. To be more precise, we study the inclusion

$$-\Delta_p u \in \partial F(x,u) - |u|^{p-2}u \quad \text{in } \Omega$$

with the boundary condition

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} \in a\big(u^+\big)^{p-1} - b\big(u^-\big)^{p-1} + \partial G(x,u) \quad \text{on } \partial\Omega.$$

We prove the existence of two constant-sign solutions and one sign-changing solution depending on the parameters a and b. Our approach is based on truncation techniques and comparison principles for elliptic inclusions along with variational tools like the nonsmooth Mountain-Pass Theorem, the Second Deformation Lemma for locally Lipschitz functionals as well as comparison results of local $C^1(\overline{\Omega})$-minimizers and local $W^{1,p}(\Omega)$-minimizers of nonsmooth functionals.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary and let $1 < p < \infty$. We consider the following elliptic inclusion: Find $u \in W^{1,p}(\Omega)$ and constants $a, b \in \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &\in \partial F(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &\in a\big(u^+\big)^{p-1} - b\big(u^-\big)^{p-1} + \partial G(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the negative p-Laplacian, $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative and $u^+ = \max(u,0)$ as well as $u^- = \max(-u,0)$ are the positive and negative part of u, respectively. The multivalued functions $s \mapsto \partial F(x,s)$ and $s \mapsto \partial G(x,s)$ stand for Clarke's generalized gradient of the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, respectively, which have the form

$$F(x,\eta) = \int_0^\eta f(x,s)\,ds, \quad \forall \eta \in \mathbb{R}, \qquad G(x,\xi) = \int_0^\xi g(x,s)\,ds, \quad \forall \xi \in \mathbb{R}, \tag{1.2}$$

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where we suppose the following conditions on the nonlinearities $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$.

(F1) $(x,s)\mapsto f(x,s)$ is measurable in each variable separately.
(F2) There exist $c_1>0$ and $q_0\in[p,p^*)$ such that

$$\big|f(x,s)\big|\leqslant c_1\big(1+|s|^{q_0-1}\big),$$

for a.a. $x\in\Omega$ and for all $s\in\mathbb{R}$, where p^* is given by

$$p^*=\begin{cases}\frac{Np}{N-p} & \text{if } p<N,\\ +\infty & \text{if } p\geqslant N.\end{cases}$$

(G1) $(x,s)\mapsto g(x,s)$ is measurable in each variable separately.
(G2) There exist $c_2>0$ and $q_1\in[p,p_*)$ such that

$$\big|g(x,s)\big|\leqslant c_2\big(1+|s|^{q_1-1}\big),$$

for a.a. $x\in\partial\Omega$ and for all $s\in\mathbb{R}$, where p_* is given by

$$p_*=\begin{cases}\frac{(N-1)p}{N-p} & \text{if } p<N,\\ +\infty & \text{if } p\geqslant N.\end{cases}$$

Note that for $u\in W^{1,p}(\Omega)$ defined on the boundary, we make use of the trace operator $\gamma:W^{1,p}(\Omega)\to L^{q_1}(\partial\Omega)$ with $p\leqslant q_1<p_*$ which is known to be bounded, linear and compact, where $W^{1,p}(\Omega)$ and $L^{q_1}(\partial\Omega)$ indicate the usual Sobolev and Lebesgue spaces, respectively. For the sake of simplicity we will drop the notation $\gamma(u)$ and write u for short. Concerning the assumptions above the functions $F(x,\cdot):\mathbb{R}\to\mathbb{R}$ and $G(x,\cdot):\mathbb{R}\to\mathbb{R}$ given in (1.2) are well defined and locally Lipschitz. This guarantees that their generalized gradients given in problem (1.1) exist. In order to characterize Clarke's generalized gradients $\partial F(x,\cdot)$ and $\partial G(x,\cdot)$, we set

$$\begin{aligned} f_1(x,s)&:=\lim_{\delta\to0^+}\operatorname*{ess\,inf}_{|\tau-s|<\delta} f(x,\tau), & f_2(x,s)&:=\lim_{\delta\to0^+}\operatorname*{ess\,sup}_{|\tau-s|<\delta} f(x,\tau),\\ g_1(x,t)&:=\lim_{\delta\to0^+}\operatorname*{ess\,inf}_{|\tau-t|<\delta} g(x,\tau), & g_2(x,t)&:=\lim_{\delta\to0^+}\operatorname*{ess\,sup}_{|\tau-t|<\delta} g(x,\tau),\end{aligned}\tag{1.3}$$

for all $(x,s)\in\Omega\times\mathbb{R}$ and all $(x,t)\in\partial\Omega\times\mathbb{R}$, respectively. Using Proposition 1.7 in [21] yields the representation

$$\partial F(x,\eta)=\big[f_1(x,\eta),f_2(x,\eta)\big],\qquad \partial G(x,\xi)=\big[g_1(x,\xi),g_2(x,\xi)\big].\tag{1.4}$$

Throughout the paper, we denote by q_0' and q_1' the Hölder conjugates to q_0 and q_1, respectively, meaning that $1/q_0+1/q_0'=1$ as well as $1/q_1+1/q_1'=1$.

Definition 1.1. A function $u\in W^{1,p}(\Omega)$ is said to be a solution of problem (1.1) if there exist $\eta\in L^{q_0'}(\Omega)$ and $\xi\in L^{q_1'}(\partial\Omega)$ such that

(i) $\eta(x)\in\partial F(x,u(x))$ for a.a. $x\in\Omega$,
(ii) $\xi(x)\in\partial G(x,u(x))$ for a.a. $x\in\partial\Omega$,
(iii) $\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx=\int_\Omega[\eta-|u|^{p-2}u]\varphi\,dx+\int_{\partial\Omega}[a(u^+)^{p-1}-b(u^-)^{p-1}+\xi]\varphi\,d\sigma$, $\forall\varphi\in W^{1,p}(\Omega)$.

If f and g are Carathéodory functions, problem (1.1) reduces to the single-valued elliptic Neumann boundary value problem

$$\begin{aligned}-\Delta_p u&=f(x,u)-|u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}&=a\big(u^+\big)^{p-1}-b\big(u^-\big)^{p-1}+g(x,u) && \text{on } \partial\Omega.\end{aligned}\tag{1.5}$$

With a view to the relation

$$|u|^{p-2}u=|u|^{p-2}\big(u^+-u^-\big)=\big(u^+\big)^{p-1}-\big(u^-\big)^{p-1},\tag{1.6}$$

we see that in case $a=b=\lambda$ problem (1.5) becomes

$$\begin{aligned}-\Delta_p u&=f(x,u)-|u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}&=\lambda|u|^{p-2}u+g(x,u) && \text{on } \partial\Omega.\end{aligned}\tag{1.7}$$

Multiple solution results for problems of type (1.7) were obtained by a number of authors, such as e.g. [1,12–14,19,28,31]. The main purpose of this paper is to provide a detailed multiplicity analysis of the nonsmooth elliptic problem (1.1) in dependence of the two parameters a and b. A main tool in our considerations is the method of sub- and supersolution. The idea is to construct two pairs of sub–supersolutions of problem (1.1), one with positive sign and one with negative sign, with the aid of some auxiliary problems, for example the so-called Steklov eigenvalue problem of the p-Laplacian. The existence of such pairs provides a positive and a negative solution, respectively, of the inclusion (1.1) within these pairs. Afterwards, we show the existence of extremal solutions of (1.1), meaning a smallest positive solution u_+ as well as a greatest negative solution u_-, by using the qualities of the eigenfunctions of the Steklov eigenvalue problem and the (S_+)-property of the p-Laplacian on $W^{1,p}(\Omega)$ (see Theorem 3.1). More details about the Steklov problem will be explained also in this section. In order to find a third nontrivial solution with changing sign, we use some important tools like the nonsmooth Mountain-Pass Theorem or the Second Deformation Lemma for locally Lipschitz functionals.

The main tool is the comparison of local $C^1(\overline{\Omega})$-minimizers and local $W^{1,p}(\Omega)$-minimizers of nonsmooth functionals. Let $J : W^{1,p}(\Omega) \to \mathbb{R}$ be a nonsmooth functional given in the form

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx + \int_\Omega j_1(x,u)\,dx + \int_{\partial\Omega} j_2(x,\gamma u)\,d\sigma \tag{1.8}$$

with nonsmooth potentials $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ which are measurable in the first argument and locally Lipschitz in the second one. Furthermore, growth conditions are also supposed on the elements of their Clarke's gradients similar to the assumptions (F2) and (G2). Then, every local $C^1(\overline{\Omega})$-minimizer of the functional J is also a local $W^{1,p}(\Omega)$-minimizer of J. This result was recently published by the author in [30] and is required to find a sign-changing solution of (1.1). The proof of the comparison of local minimizers is mainly based on a boundedness-result for weak solutions of nonlinear elliptic equations with nonhomogeneous Neumann boundary conditions obtained in [29] with the aid of the Moser iteration along with continuous embeddings in Besov and Lizorkin–Triebel spaces, respectively. Summarizing, we find a third nontrivial solution u_0 of our inclusion (1.1) which lies between the smallest positive solution u_+ and the greatest negative solution u_-. Hence, it must be a sign-changing solution if it is unequal to u_+ and u_-. Indeed, we prove that $u_0 \neq u_+, u_-$ which is stated in Theorem 5.1.

Problems of the form (1.1) under homogeneous Dirichlet boundary conditions and homogeneous Neumann boundary conditions, respectively, were studied in some recent papers. We refer, for example, to [2,4–6,15,24], respectively. A very related reference that contains multivalued problems with a variational treatment is the monograph of Motreanu and Rădulescu in [22]. Therein, the authors study many different topics, for example critical point theory for nonsmooth functionals, multivalued elliptic problems in variational form as well as hemivariational and variational–hemivariational inequalities. Some existence results of (variational-)hemivariational inequalities which are related to differential inclusions of the form (1.1) can be found in [8,23,26] as well.

In order to show our results, we require some additional assumptions given below.

(F3) $\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$.

(F4) $\lim_{|s|\to +\infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \Omega$.

(F5) There exists $\delta_f > 0$ such that $\frac{f(x,s)}{|s|^{p-2}s} \geqslant 0$ for all $0 < |s| \leqslant \delta_f$ and for a.a. $x \in \Omega$.

(G3) $\lim_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \partial\Omega$.

(G4) $\lim_{|s|\to +\infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \partial\Omega$.

(G5) Let $u \in W^{1,p}(\Omega)$. Then every $\xi \in \partial G(x,u)$ satisfies the condition

$$\bigl|\xi(x_1) - \xi(x_2)\bigr| \leqslant L|x_1 - x_2|^\alpha,$$

for all x_1, x_2 in $\partial\Omega$ with $\alpha \in (0,1)$.

Remark 1.2. Due to the conditions (F3) and (G3) along with the representations (1.4), we conclude that $f_1(x,0) \leqslant 0 \leqslant f_2(x,0)$ as well as $g_1(x,0) \leqslant 0 \leqslant g_2(x,0)$. This guarantees, in particular, that problem (1.1) possesses the trivial solution $u = 0$ (cf. Definition 1.1).

Remark 1.3. Note that condition (G5) is required to apply the $C^{1,\alpha}$-regularity results of Lieberman. This means that every bounded weak solution u of problem (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ if the assumption (G5) is satisfied. We refer the reader to [17] for more details.

Let us now introduce the definition of a sub- and supersolution of problem (1.1).

Definition 1.4. A function $\bar{u} \in W^{1,p}(\Omega)$ is called a supersolution of problem (1.1) if there exist $\bar{\eta} \in L^{q_0'}(\Omega)$ and $\bar{\xi} \in L^{q_1'}(\partial\Omega)$ such that

(i) $\overline{\eta}(x) \in \partial F(x, \overline{u}(x))$ for a.a. $x \in \Omega$,
(ii) $\overline{\xi}(x) \in \partial G(x, \overline{u}(x))$ for a.a. $x \in \partial\Omega$,
(iii) $\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \cdot \nabla\varphi\, dx \geqslant \int_\Omega [\overline{\eta} - |\overline{u}|^{p-2}\overline{u}]\varphi\, dx + \int_{\partial\Omega}[a(\overline{u}^+)^{p-1} - b(\overline{u}^-)^{p-1} + \overline{\xi}]\varphi\, d\sigma$, $\forall \varphi \in W^{1,p}(\Omega) \cap L^p(\Omega)_+$.

Definition 1.5. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of problem (1.1) if there exist $\underline{\eta} \in L^{q_0'}(\Omega)$ and $\underline{\xi} \in L^{q_1'}(\partial\Omega)$ such that

(i) $\underline{\eta}(x) \in \partial F(x, \underline{u}(x))$ for a.a. $x \in \Omega$,
(ii) $\underline{\xi}(x) \in \partial G(x, \underline{u}(x))$ for a.a. $x \in \partial\Omega$,
(iii) $\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \cdot \nabla\varphi\, dx \leqslant \int_\Omega [\underline{\eta} - |\underline{u}|^{p-2}\underline{u}]\varphi\, dx + \int_{\partial\Omega}[a(\underline{u}^+)^{p-1} - b(\underline{u}^-)^{p-1} + \underline{\xi}]\varphi\, d\sigma$, $\forall \varphi \in W^{1,p}(\Omega) \cap L^p(\Omega)_+$.

Next, we give a brief overview of the Fučik spectrum $\widetilde{\Sigma}_p$ for the p-Laplacian with a nonlinear boundary condition. The set $\widetilde{\Sigma}_p$ is defined by all pairs $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a\big(u^+\big)^{p-1} - b\big(u^-\big)^{p-1} && \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

has a nontrivial solution. If $a = b = \lambda$ problem (1.9) reduces to the Steklov eigenvalue problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

because of relation (1.6). It is known that (1.10) has a first eigenvalue $\lambda_1 > 0$ which is isolated and simple. Moreover, its corresponding first eigenfunction φ_1 is strictly positive in $\overline{\Omega}$ (see [18]) and belongs to $L^\infty(\Omega)$ (cf. [16, Lemma 5.6 and Theorem 4.3] or [29]). The regularity results of Lieberman in [17, Theorem 2] imply $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, and hence, $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$, where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}):\ u(x) \geqslant 0,\ \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, given by

$$\operatorname{int}\big(C^1(\overline{\Omega})_+\big) = \big\{u \in C^1(\overline{\Omega}):\ u(x) > 0,\ \forall x \in \overline{\Omega}\big\}.$$

If λ is an eigenvalue for (1.10), then the point (λ, λ) belongs to $\widetilde{\Sigma}_p$. Since the first eigenfunction of (1.10) is positive, $\widetilde{\Sigma}_p$ clearly contains the two lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$. A first nontrivial curve $\mathcal{C}$ in $\widetilde{\Sigma}_p$ through (λ_2, λ_2) was constructed and variationally characterized by a mountain-pass procedure by Martínez and Rossi [20] which implies the existence of a continuous path in $\{u \in W^{1,p}(\Omega) : I^{(a,b)}(u) < 0, \|u\|_{L^p(\partial\Omega)} = 1\}$ joining $-\varphi_1$ and φ_1 provided (a, b) is above the curve $\mathcal{C}$. The functional $I^{(a,b)}$ on $W^{1,p}(\Omega)$ is given by

$$I^{(a,b)}(u) = \int_\Omega \big(|\nabla u|^p + |u|^p\big)\, dx - \int_{\partial\Omega} \big(a\big(u^+\big)^p + b\big(u^-\big)^p\big)\, d\sigma.$$

The existence of a sign-changing solution of problem (1.1) needs an additional assumption on the constants a and b in the following way.

(H) The pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ lies above the first nontrivial curve $\mathcal{C}$ of the Fučik spectrum constructed in [20].

As demonstrated in [28], the elliptic equation

$$\begin{aligned} -\Delta_p u &= -\varsigma|u|^{p-2}u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \tag{1.11}$$

has a unique weak solution $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ where $\varsigma > 1$ is a constant. We will use the function e to construct sub- and supersolutions of problem (1.1).

Let us recall some basic facts from nonsmooth analysis. We denote by $(X, \|\cdot\|)$ a real Banach space and by X^* its dual space. By $\langle\cdot,\cdot\rangle$ we mean the duality pairing between X and X^*. Let $J : X \to \mathbb{R}$ be a locally Lipschitz functional. Clarke's generalized directional derivative of J at u in the direction $v \in X$ is defined by

$$J^{\mathrm{o}}(u; v) = \limsup_{x \to u, t \downarrow 0} \frac{J(x + tv) - J(x)}{t},$$

where $v \mapsto J^o(u; v)$ is finite, convex, positively homogeneous, subadditive on X and satisfies the estimate $|J^o(u; v)| \leqslant K\|u\|$, where $K > 0$ is the Lipschitz constant of J near the point $u \in X$ (see [10, Chapter 2]). Then, Clarke's generalized gradient of J at $u \in X$ is defined by

$$\partial J(u) = \{\xi \in X^*:\ J^o(u; v) \geqslant \langle \xi, v\rangle,\ \forall v \in X\}.$$

By means of [10], it is known that $\partial J(u)$ is a convex, weak*-compact subset of X^* with $\|\xi\|_{X^*} \leqslant K$ for all $\xi \in \partial J(u)$. Furthermore, it holds

$$J^o(u; v) = \max\{\langle \xi, v\rangle:\ \xi \in \partial J(u)\}, \quad v \in X.$$

From [10, Proposition 2.1.2] we also know that $\partial J(u)$ is nonempty. Hence, it makes sense to set

$$m_J(u) := \min\{\|\xi\|_{X^*}:\ \xi \in \partial J(u)\}.$$

We say that $u \in X$ is a critical point of J if $0 \in \partial J(u)$ which is equivalent to $J^o(u; v) \geqslant 0$ for all $v \in X$. It is clear that each local minimizer or maximizer of J is a critical point. Let us recall the nonsmooth version of the Palais–Smale condition (cf. [9]).

Definition 1.6 *(Palais–Smale condition).* Let X be real Banach space and let $J : X \to R$ be a locally Lipschitz functional. We say that J fulfills the Palais–Smale condition if any sequence (u_n) with $(J(u_n))$ is bounded and $\lim_{n\to\infty} m_J(u_n) = 0$ has a convergent subsequence.

The nonsmooth Mountain-Pass Theorem due to Chang is stated as follows (see [9, Theorem 3.4]).

Theorem 1.7 *(Mountain-Pass Theorem). Let X be a reflexive real Banach space and let $J : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying the Palais–Smale condition. If there exist $x_0, x_1 \in X$ and a constant $r > 0$ such that $\|x_1 - x_0\| > r$ and $\max\{J(x_0), J(x_1)\} < \inf_{x\in\partial B_r(x_0)} J(x)$, then J has a critical point $u_0 \in X$ such that*

$$\inf_{x\in\partial B_r(x_0)} J(x) \leqslant J(u_0) = \inf_{\pi\in\Pi} \max_{t\in[0,1]} J(\pi(t)),$$

where $\Pi = \{\pi \in C([0,1], X):\ \pi(0) = x_0,\ \pi(1) = x_1\}$ and $\partial B_r(x_0) = \{u \in X:\ \|u - x_0\| = r\}$.

Now, we want to recall some existence and comparison results involving the method of sub- and supersolutions apply on problem (1.1). We have the following results.

Theorem 1.8. *Let the hypotheses* (F1)–(F2) *and* (G1)–(G2) *be fulfilled and assume the existence of a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of problem* (1.1) *satisfying $\underline{u} \leqslant \overline{u}$. Then there exists a solution u of* (1.1) *with $\underline{u} \leqslant u \leqslant \overline{u}$.*

The proof of the theorem above was recently published in [7]. Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the ordered interval $[\underline{u}, \overline{u}]$ which is nonempty due to Theorem 1.8. A solution $u_* \in \mathcal{S}$ is said to be the smallest solution of $\mathcal{S}$ if for any element $u \in \mathcal{S}$ the inequality $u_* \leqslant u$ holds. Likewise, $u^* \in \mathcal{S}$ is called the greatest solution of $\mathcal{S}$ if $u \leqslant u^*$ holds for all $u \in \mathcal{S}$. We say $\mathcal{S}$ possesses extremal solutions if $\mathcal{S}$ has a smallest and greatest solution.

Theorem 1.9. *Let hypotheses* (F1)–(F2) *and* (G1)–(G2) *be satisfied and assume the existence of a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of* (1.1) *such that $\underline{u} \leqslant \overline{u}$. Then there exist extremal solutions of* (1.1) *within* $[\underline{u}, \overline{u}]$.

The proof of Theorem 1.9 can be done as in [3]. Note that the one-sided growth condition on Clarke's generalized gradient, which is required in [3], is not needed in the proof of the existence of extremal solutions.

2. Existence of sub- and supersolutions

In this section we prove the existence of some sub- and supersolutions of problem (1.1) according to Definition 1.4 and 1.5. Let $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ be the unique solution of the auxiliary problem (1.11). Then we have the following.

Lemma 2.1. *Let the conditions* (F1)–(F5) *and* (G1)–(G5) *be satisfied. If $a > \lambda_1$, then there exists a constant $\vartheta_a > 0$ such that for any $b \in \mathbb{R}$ the function $\vartheta_a e$ is a positive supersolution of problem* (1.1).

Proof. We put $a > \lambda_1$ and set $\overline{u} = \vartheta_a e$ with a positive constant ϑ_a to be specified. The weak formulation of the Neumann problem (1.11) reads as

$$\int_\Omega \left|\nabla(\vartheta_a e)\right|^{p-2}\nabla(\vartheta_a e)\cdot\nabla\varphi\,dx = -\varsigma\int_\Omega (\vartheta_a e)^{p-1}\varphi\,dx + \int_\Omega (\vartheta_a)^{p-1}\varphi\,dx + \int_{\partial\Omega}(\vartheta_a)^{p-1}\varphi\,d\sigma, \quad \forall\varphi\in W^{1,p}(\Omega).$$

Combining the equation above with Definition 1.4 yields a sufficient condition for $\underline{u} = \vartheta_a e$ to be a supersolution of problem (1.1). We have to show that

$$\int_\Omega \left(\vartheta_a^{p-1} + (1-\varsigma)(\vartheta_a e)^{p-1} - \overline{\eta}\right)\varphi\,dx + \int_{\partial\Omega}\left(\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} - \overline{\xi}\right)\varphi\,d\sigma \geqslant 0, \quad \forall\varphi\in W^{1,p}(\Omega)\cap L^p(\Omega)_+, \tag{2.1}$$

holds true, where $\overline{\eta}\in L^{q_0'}(\Omega)$ and $\overline{\eta}(x)\in\partial F(x,\vartheta_a e(x))$ for a.a. $x\in\Omega$ as well as $\overline{\xi}\in L^{q_1'}(\partial\Omega)$ and $\overline{\xi}(x)\in\partial G(x,\vartheta_a e(x))$ for a.a. $x\in\partial\Omega$. Note that $\varsigma > 1$. The hypothesis (F4) provides a constant $s_\varsigma > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < 1-\varsigma, \quad \text{for a.a. } x\in\Omega \text{ and all } s > s_\varsigma,$$

and by (F2) we get

$$\left|-f(x,s) + (1-\varsigma)s^{p-1}\right| \leqslant \left|f(x,s)\right| + (\varsigma+1)s^{p-1} \leqslant c_\varsigma,$$

for a.a. $x\in\Omega$ and all $s\in[0,s_\varsigma]$. This leads to

$$f(x,s) \leqslant (1-\varsigma)s^{p-1} + c_\varsigma, \quad \text{for a.a. } x\in\Omega \text{ and all } s\geqslant 0,$$

and due to the definition of f_1 we finally obtain

$$f_1(x,s) \leqslant (1-\varsigma)s^{p-1} + c_\varsigma, \quad \text{for a.a. } x\in\Omega \text{ and all } s\geqslant 0. \tag{2.2}$$

Setting $\overline{\eta}(x) = f_1(x,\vartheta_a e(x))$ as well as $\vartheta_a \geqslant c_\varsigma^{\frac{1}{p-1}}$ and applying (2.2) to the first integral in (2.1) yields

$$\begin{aligned}\int_\Omega \left(\vartheta_a^{p-1} + (1-\varsigma)(\vartheta_a e)^{p-1} - f_1\big(x,\vartheta_a e(x)\big)\right)\varphi\,dx &\geqslant \int_\Omega \left(\vartheta_a^{p-1} + (1-\varsigma)(\vartheta_a e)^{p-1} + (\varsigma-1)(\vartheta_a e)^{p-1} - c_\varsigma\right)\varphi\,dx \\ &\geqslant 0.\end{aligned} \tag{2.3}$$

Let us now study the second term in (2.1). Since $a > \lambda_1 > 0$ there exists a constant $s_a > 0$ due to condition (G4) such that

$$\frac{g(x,s)}{s^{p-1}} < -a, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s > s_a. \tag{2.4}$$

The assumption (G2) ensures the existence of a constant $c_a > 0$ such that

$$\left|-g(x,s) - as^{p-1}\right| \leqslant \left|g(x,s)\right| + as^{p-1} \leqslant c_a,$$

for a.a. $x\in\partial\Omega$ and all $s\in[0,s_a]$ which results in

$$g(x,s) \leqslant -as^{p-1} + c_a, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s\geqslant 0, \tag{2.5}$$

and hence,

$$g_1(x,s) \leqslant -as^{p-1} + c_a, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s\geqslant 0. \tag{2.6}$$

We put $\overline{\xi}(x) = g_1(x,\vartheta_a e(x))$ and $\vartheta_a \geqslant c_a^{\frac{1}{p-1}}$. One gets

$$\int_{\partial\Omega}\left(\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} - g_1(x,\vartheta_a e)\right)\varphi\,d\sigma \geqslant \int_{\partial\Omega}\left(\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} + a(\vartheta_a e)^{p-1} - c_a\right)\varphi\,d\sigma \geqslant 0. \tag{2.7}$$

If $\vartheta_a \geqslant \max(c_\varsigma^{\frac{1}{p-1}}, c_a^{\frac{1}{p-1}})$, then $\overline{u} = \vartheta_a e$ is, in fact, a positive supersolution of problem (1.1). □

The next lemma can be proven very similarly.

Lemma 2.2. *Let the assumptions* (F1)–(F5) *and* (G1)–(G5) *be satisfied. If $b > \lambda_1$, then there exists a constant $\vartheta_b > 0$ such that for any $a\in\mathbb{R}$ the function $-\vartheta_b e$ is a negative subsolution of problem* (1.1)*.*

Let $\lambda_1 > 0$ be the first eigenvalue of the Steklov eigenvalue problem and let $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ be its corresponding first eigenfunction. The next result shows that constant multipliers of φ_1 may be sub- and supersolution of problem (1.1).

Lemma 2.3. *Assume* (F1)–(F5) *and* (G1)–(G5)*. If $a > \lambda_1$, then for $\varepsilon > 0$ sufficiently small and any $b \in \mathbb{R}$ the function $\varepsilon\varphi_1$ is a positive subsolution of problem* (1.1)*. If $b > \lambda_1$, then for $\varepsilon > 0$ sufficiently small and any $a \in \mathbb{R}$ the function $-\varepsilon\varphi_1$ is a negative supersolution of problem* (1.1)*.*

Proof. Let $a > \lambda_1$ and let $\underline{u} = \varepsilon\varphi_1$. From the Steklov eigenvalue problem (1.10) we conclude

$$\int_\Omega \left|\nabla(\varepsilon\varphi_1)\right|^{p-2}\nabla(\varepsilon\varphi_1)\cdot\nabla\varphi\,dx = -\int_\Omega (\varepsilon\varphi_1)^{p-1}\varphi\,dx + \int_{\partial\Omega}\lambda_1(\varepsilon\varphi_1)^{p-1}\varphi\,d\sigma, \quad \forall\varphi \in W^{1,p}(\Omega). \tag{2.8}$$

Taking into account (2.8), a sufficient condition for $\underline{u} = \varepsilon\varphi_1$ to be a positive subsolution is

$$\int_\Omega -\underline{\eta}\varphi\,dx + \int_{\partial\Omega}\big((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - \underline{\xi}\big)\varphi\,d\sigma \leqslant 0, \tag{2.9}$$

with $\underline{\eta} \in L^{q_0'}(\Omega)$ and $\underline{\eta}(x) \in \partial F(x, \varepsilon\varphi_1(x))$ for a.a. $x \in \Omega$ as well as $\underline{\xi} \in L^{q_1'}(\partial\Omega)$ and $\underline{\xi}(x) \in \partial G(x, \varepsilon\varphi_1(x))$ for a.a. $x \in \partial\Omega$. Let us prove inequality (2.9). Concerning condition (F5) we see at once that the first integral in (2.9) is negative. Setting $\underline{\eta}(x) = f_2(x, \varepsilon\varphi_1(x))$ and $\varepsilon \in (0, \delta_f/\|\varphi_1\|_\infty]$ leads to

$$\int_\Omega -\underline{\eta}\varphi\,dx = -\int_\Omega \frac{f_2(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi\,dx \leqslant -\int_\Omega \frac{f(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi\,dx \leqslant 0, \tag{2.10}$$

where $\|\cdot\|_\infty$ denotes the usual supremum norm. In order to estimate the second integral in (2.9) we may apply the assumption (G3) which ensures the existence of a number $\delta_a > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} < a - \lambda_1, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \leqslant \delta_a.$$

Let $\varepsilon \in (0, \frac{\delta_a}{\|\varphi_1\|_\infty}]$ and let $\underline{\xi}(x) = g_2(x, \varepsilon\varphi_1(x))$ which implies that $-\underline{\xi}(x) \leqslant -g(x, \varepsilon\varphi_1)$. Then it holds

$$\begin{aligned}
\int_{\partial\Omega}\big((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - \underline{\xi}\big)\varphi\,d\sigma &\leqslant \int_{\partial\Omega}\big((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} + \left|g(x,\varepsilon\varphi_1)\right|\big)\varphi\,d\sigma \\
&= \int_{\partial\Omega}\left(\lambda_1 - a + \frac{|g(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi\,d\sigma \\
&\leqslant \int_{\partial\Omega}(\lambda_1 - a + a - \lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi\,d\sigma \\
&= 0.
\end{aligned}$$

Finally, we select $\varepsilon > 0$ such that $0 < \varepsilon \leqslant \min\{\delta_f/\|\varphi_1\|_\infty, \delta_a/\|\varphi_1\|_\infty\}$ which yields that both integrals in (2.9) are nonpositive and hence, $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution of problem (1.1). The proof of the existence of a negative supersolution $\underline{u} = -\varepsilon\varphi_1$ acts in the same way and is dropped now. □

To sum up, we proved the existence of two sub- and two supersolutions of problem (1.1). If we choose $\varepsilon > 0$ sufficiently small, we get $\underline{u}_1 = \varepsilon\varphi_1 \leqslant \vartheta_a e = \overline{u}_1$ and $\underline{u}_2 = -\vartheta_b e \leqslant -\varepsilon\varphi_1 = \overline{u}_2$ which means that we have two ordered pairs of sub- and supersolution namely $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_2, \overline{u}_2]$, respectively. The next result gives an answer about the regularity of weak solutions of problem (1.1).

Lemma 2.4. *Let the conditions* (F1)–(F5) *and* (G1)–(G5) *be satisfied and let $a, b > \lambda_1$. If $u \in [0, \vartheta_a e]$ (respectively, $u \in [-\vartheta_b e, 0]$) is a solution of problem* (1.1) *which is not identically zero in Ω, then it holds $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (respectively, $u \in -\operatorname{int}(C^1(\overline{\Omega})_+)$).*

Proof. Let $u \in [0, \vartheta_a e]$ be a solution of problem (1.1) satisfying $u \not\equiv 0$. Then we directly obtain the boundedness of u meaning $u \in L^\infty(\Omega)$. Applying the results of Lieberman in [17, Theorem 2] guarantees that $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. By the assumptions (F2) and (F3) as well as (G2) and (G3), we find constants $c_f, c_g > 0$ such that

$$\begin{aligned}
\left|f(x,s)\right| &\leqslant c_f s^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \leqslant s \leqslant \vartheta_a\|e\|_\infty, \\
\left|g(x,s)\right| &\leqslant c_g s^{p-1}, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 \leqslant s \leqslant \vartheta_a\|e\|_\infty.
\end{aligned} \tag{2.11}$$

Applying (2.11) to (1.1) implies

$$\Delta_p u \leqslant (1 + c_f)u^{p-1} \quad \text{a.e. in } \Omega.$$

Then, we set $\beta(s) = (1 + c_f)s^{p-1}$ for all $s > 0$ and note that $\int_{0+} 1/(s\beta(s))^{\frac{1}{p}}\, ds = +\infty$. Hence, the assumptions of Vázquez's strong maximum principle (cf. [25]) are satisfied and we obtain $u > 0$ in Ω. In order to prove that u is strictly positive in the closure of Ω, we suppose there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$. Applying again the maximum principle yields $\frac{\partial u}{\partial \nu}(x_0) < 0$. However, we know that $0 \in \partial G(x_0, u(x_0)) = \partial G(x_0, 0)$ which leads to a contradiction in view of problem (1.1) because in this case we have $\frac{\partial u}{\partial \nu}(x_0) = 0$. Therefore, it holds $u > 0$ in $\overline{\Omega}$ which implies $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. The case $u \in [-\vartheta_b e, 0]$ can be shown by using similar arguments. □

3. Extremal constant-sign solutions

One of our main results about the existence of constant-sign solutions of (1.1) reads as follows.

Theorem 3.1. *Let the conditions* (F1)–(F5) *and* (G1)–(G5) *be satisfied. For every $a > \lambda_1$ and $b \in \mathbb{R}$ there exists a smallest positive solution $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of* (1.1) *in the order interval $[0, \vartheta_a e]$ with the constant ϑ_a as in Lemma* 2.1*. For every $b > \lambda_1$ and $a \in \mathbb{R}$ there exists a greatest solution $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[-\vartheta_b e, 0]$ with the constant ϑ_b as in Lemma* 2.2.

Proof. Let $a > \lambda_1$. By means of Lemma 2.3 we know that $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a positive subsolution of problem (1.1) provided $\varepsilon > 0$ is sufficiently small and Lemma 2.1 ensures that $\overline{u} = \vartheta_a e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a positive supersolution of problem (1.1). Additionally, we can take $\varepsilon > 0$ such that $\varepsilon\varphi_1 \leqslant \vartheta_a e$. Due to Theorem 1.9 there exists a smallest positive solution $u_\varepsilon = u_\varepsilon(a)$ of problem (1.1) satisfying $\varepsilon\varphi_1 \leqslant u_\varepsilon \leqslant \vartheta_a e$. The regularity results in Lemma 2.4 can be applied because $u_\varepsilon \not\equiv 0$ which ensures that $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Consequently, we find for every positive integer n choosing sufficiently large a smallest positive solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) which lies in $[\frac{1}{n}\varphi_1, \vartheta_a e]$. This construction creates a sequence (u_n) of smallest solutions which is monotone decreasing. One gets

$$u_n \downarrow u_+ \quad \text{for a.a. } x \in \Omega, \tag{3.1}$$

with some function $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \leqslant u_+ \leqslant \vartheta_a e$. Note that $u_n \in [\frac{1}{n}\varphi_1, \vartheta_a e]$ and $\gamma(u_n) \in [\gamma(\frac{1}{n}\varphi_1), \gamma(\vartheta_a e)]$ imply, in particular, that u_n belongs to $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$, respectively. As $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ solves problem (1.1), we obtain by taking the test function $\varphi = u_n$ in the weak formulation of problem (1.1) along with (F2) and (G2)

$$\begin{aligned}
\|\nabla u_n\|^p_{L^p(\Omega)} &\leqslant \int_\Omega |\eta_n| u_n\, dx + \|u_n\|^p_{L^p(\Omega)} + a\|u_n\|^p_{L^p(\partial\Omega)} + \int_{\partial\Omega} |\xi_n| u_n\, d\sigma \\
&\leqslant \widetilde{c}_1 \|u_n\|_{L^p(\Omega)} + c_1 \|u_n\|^{q_0}_{L^{q_0}(\Omega)} + \|u_n\|^p_{L^p(\Omega)} + a\|u_n\|^p_{L^p(\partial\Omega)} + \widetilde{c}_2 \|u_n\|^p_{L^p(\partial\Omega)} + \widetilde{c}_3 \|u_n\|^{q_1}_{L^{q_1}(\partial\Omega)} \\
&\leqslant \widetilde{C},
\end{aligned} \tag{3.2}$$

where $\eta_n \in L^{q_0'}(\Omega)$ with $\eta_n(x) \in \partial F(x, u_n(x))$ for a.a. $x \in \Omega$ as well as $\xi_n \in L^{q_1'}(\partial\Omega)$ with $\xi_n(x) \in \partial G(x, u_n(x))$ for a.a. $x \in \partial\Omega$. Relation (3.2) yields the boundedness of ∇u_n in $L^p(\Omega)$ and thus, $\|u_n\|_{W^{1,p}(\Omega)} \leqslant C$, for all $n \in \mathbb{N}$ with some positive constant C independent of n. The reflexivity of the Sobolev space $W^{1,p}(\Omega)$ in case $1 < p < \infty$ yields the existence of a weakly convergent subsequence of u_n. The compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, the monotony of the sequence u_n and the compactness of $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ provide for the entire sequence u_n

$$\begin{aligned}
&u_n \rightharpoonup u_+ \quad \text{in } W^{1,p}(\Omega), \\
&u_n \to u_+ \quad \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega, \\
&u_n \to u_+ \quad \text{in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega.
\end{aligned} \tag{3.3}$$

The solution u_n of problem (1.1) fulfills

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla\varphi\, dx = \int_\Omega \left(\eta_n - u_n^{p-1}\right)\varphi\, dx + \int_{\partial\Omega} \left(a u_n^{p-1} + \xi_n\right)\varphi\, d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega), \tag{3.4}$$

where $\eta_n \in L^{q_0'}(\Omega)$ with $\eta_n(x) \in \partial F(x, u_n(x))$ for a.a. $x \in \Omega$ as well as $\xi_n \in L^{q_1'}(\partial\Omega)$ with $\xi_n(x) \in \partial G(x, u_n(x))$ for a.a. $x \in \partial\Omega$. Setting $\varphi = u_n - u_+ \in W^{1,p}(\Omega)$ in (3.4) it results in

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla(u_n - u_+)\, dx = \int_\Omega \left(\eta_n - u_n^{p-1}\right)(u_n - u_+)\, dx + \int_{\partial\Omega} \left(a u_n^{p-1} + \xi_n\right)(u_n - u_+)\, d\sigma. \tag{3.5}$$

The convergence properties of (u_n) along with the assumptions (F2) and (G2) as well as the uniform boundedness of the sequence (u_n) allow us to apply Lebesgue's dominated convergence theorem. We obtain

$$\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u_+)\,dx\leqslant 0,$$

which provides by the (S_+)-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ along with (3.3) the strong convergence in $W^{1,p}(\Omega)$, meaning

$$u_n\to u_+\quad\text{in } W^{1,p}(\Omega). \tag{3.6}$$

Due to (F2) and (G2) in conjunction with the uniform boundedness of (u_n), there exist constants $b_1,b_2>0$ such that

$$\begin{aligned}&\big|\eta_n(x)\big|\leqslant b_1\quad\text{for a.a. } x\in\Omega,\ \forall n\in\mathbb{N},\\&\big|\xi_n(x)\big|\leqslant b_2\quad\text{for a.a. } x\in\partial\Omega,\ \forall n\in\mathbb{N}.\end{aligned}\tag{3.7}$$

Hence, we get

$$\begin{aligned}&\eta_n\rightharpoonup\eta_+\quad\text{in } L^{q_0'}(\Omega),\\&\xi_n\rightharpoonup\xi_+\quad\text{in } L^{q_1'}(\partial\Omega),\end{aligned}\tag{3.8}$$

for some subsequences, not relabeled. From calculus of Clarke's generalized gradient one gets that $\eta_+(x)\in\partial F(x,u_+(x))$ for a.a. $x\in\Omega$ and $\xi_+(x)\in\partial G(x,u_+(x))$ for a.a. $x\in\partial\Omega$, respectively. Passing to the limit in (3.4) for some subsequences if necessary proves that u_+ is a solution of problem (1.1).

Applying Lemma 2.4 yields $u_+\in\operatorname{int}(C^1(\overline{\Omega})_+)$ provided $u_+\not\equiv 0$ in Ω. Assume $u_+\equiv 0$ in Ω. Then, by (3.1), we obtain

$$u_n\downarrow 0\quad\text{for a.a. } x\in\Omega. \tag{3.9}$$

We set

$$\widetilde{u}_n=\frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}\quad\text{for all } n.$$

The boundedness of the sequence $(\widetilde{u}_n)$ in $W^{1,p}(\Omega)$ can be proved similarly as for (u_n). Hence, we find a subsequence, not relabelled, such that

$$\begin{aligned}&\widetilde{u}_n\rightharpoonup\widetilde{u}\quad\text{in } W^{1,p}(\Omega),\\&\widetilde{u}_n\to\widetilde{u}\quad\text{in } L^p(\Omega)\text{ and for a.a. } x\in\Omega,\\&\widetilde{u}_n\to\widetilde{u}\quad\text{in } L^p(\partial\Omega)\text{ and for a.a. } x\in\partial\Omega,\end{aligned}\tag{3.10}$$

with some function $\widetilde{u}:\Omega\to\mathbb{R}$ belonging to $W^{1,p}(\Omega)$. Furthermore, there exist functions $z_1\in L^p(\Omega)_+$ and $z_2\in L^p(\partial\Omega)_+$ such that

$$\begin{aligned}&\big|\widetilde{u}_n(x)\big|\leqslant z_1(x)\quad\text{for a.a. } x\in\Omega,\\&\big|\widetilde{u}_n(x)\big|\leqslant z_2(x)\quad\text{for a.a. } x\in\partial\Omega.\end{aligned}\tag{3.11}$$

Due to the representation $u_n=\widetilde{u}_n\cdot\|u_n\|_{W^{1,p}(\Omega)}$ and because u_n solves (1.1), we get the following variational equation

$$\begin{aligned}&\int_\Omega|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\cdot\nabla\varphi\,dx\\&\quad=\int_\Omega\left(\frac{\eta_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}-\widetilde{u}_n^{p-1}\right)\varphi\,dx+\int_{\partial\Omega}a\widetilde{u}_n^{p-1}\varphi\,d\sigma+\int_{\partial\Omega}\frac{\xi_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}\varphi\,d\sigma,\quad\forall\varphi\in W^{1,p}(\Omega).\end{aligned}\tag{3.12}$$

Selecting $\varphi=\widetilde{u}_n-\widetilde{u}\in W^{1,p}(\Omega)$ in (3.12) provides

$$\begin{aligned}&\int_\Omega|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\cdot\nabla(\widetilde{u}_n-\widetilde{u})\,dx\\&\quad=\int_\Omega\left(\frac{\eta_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}-\widetilde{u}_n^{p-1}\right)(\widetilde{u}_n-\widetilde{u})\,dx+\int_{\partial\Omega}a\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})\,d\sigma+\int_{\partial\Omega}\frac{\xi_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})\,d\sigma.\end{aligned}\tag{3.13}$$

Applying (2.11) and (3.11), one obtains

$$\frac{|\eta_n(x)|}{u_n^{p-1}(x)}\widetilde{u}_n^{p-1}(x)\big|\widetilde{u}_n(x)-\widetilde{u}(x)\big|\leqslant c_f z_1(x)^{p-1}\big(z_1(x)+\big|\widetilde{u}(x)\big|\big),\tag{3.14}$$

respectively,

$$\frac{|\xi_n(x)|}{u_n^{p-1}(x)}\widetilde{u}_n^{p-1}(x)\big|\widetilde{u}_n(x)-\widetilde{u}(x)\big| \leqslant c_g z_2(x)^{p-1}\big(z_2(x)+\big|\widetilde{u}(x)\big|\big). \tag{3.15}$$

Obviously, the right-hand sides of (3.14) and (3.15) belong to $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, which allows us to apply Lebesgue's dominated convergence theorem which in conjunction with (3.10) yields

$$\lim_{n\to\infty}\int_\Omega \frac{\eta_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})\,dx=0,$$
$$\lim_{n\to\infty}\int_{\partial\Omega} \frac{\xi_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})\,d\sigma=0. \tag{3.16}$$

Taking into account (3.10) and (3.16) we get from (3.13)

$$\limsup_{n\to\infty}\int_\Omega |\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\cdot\nabla(\widetilde{u}_n-u_n)\,dx=0.$$

As before, the (S_+)-property of $-\Delta_p$ corresponding to $W^{1,p}(\Omega)$ implies

$$\widetilde{u}_n\to\widetilde{u}\quad\text{in } W^{1,p}(\Omega). \tag{3.17}$$

From the definition of $\widetilde{u}_n$ we see at once that $\|\widetilde{u}\|_{W^{1,p}(\Omega)}=1$, meaning $\widetilde{u}\not\equiv 0$. Passing to the limit in (3.12) in conjunction with (3.9), (3.17) as well as the assumptions (F3) and (G3) it results in

$$\int_\Omega |\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u}\cdot\nabla\varphi\,dx=-\int_\Omega \widetilde{u}^{p-1}\varphi\,dx+\int_{\partial\Omega} a\widetilde{u}^{p-1}\varphi\,d\sigma,\quad \forall\varphi\in W^{1,p}(\Omega).$$

The equation above is nothing less than the weak formulation of the Steklov eigenvalue problem corresponding to the eigenvalue $a>\lambda_1$ and the eigenfunction $\widetilde{u}\geqslant 0$. However, this is a contradiction because $\widetilde{u}$ must change sign on $\partial\Omega$ (see [18, Lemma 2.4]). Hence, $u_+\not\equiv 0$ which guarantees that $u_+\in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Finally, we have to prove that u_+ is the smallest solution in $[0,\vartheta_a e]$. Fix a positive solution $u\in W^{1,p}(\Omega)$ of (1.1) such that $0\leqslant u\leqslant \vartheta_a e$. Lemma 2.4 provides $u\in\operatorname{int}(C^1(\overline{\Omega})_+)$. Then, there exists an integer n sufficiently large such that $u\in[\frac{1}{n}\varphi_1,\vartheta_a e]$. However, u_n is the smallest solution in $[\frac{1}{n}\varphi_1,\vartheta_a e]$ which yields $u_n\leqslant u$ if n is large enough. Due to the monotonicity of u_n, we obtain $u_+\leqslant u$ which proves that u_+ is, indeed, the smallest positive solution of (1.1) in $[0,\vartheta_a e]$. The existence of a greatest negative solution can be done similarly and is omitted. □

4. Variational characterization of extremal solutions

In this section we give a variational characterization of the extremal solutions of (1.1) which we obtained in the last section. To this end, we introduce truncation operators $T_+, T_- : \Omega\times\mathbb{R}\to\mathbb{R}$ as well as $T_+^{\partial\Omega}, T_-^{\partial\Omega} : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ in the following way:

$$T_+(x,s)=\begin{cases}0 & \text{if } s<0,\\ s & \text{if } 0\leqslant s\leqslant u_+(x),\\ u_+(x) & \text{if } s>u_+(x),\end{cases}\qquad T_+^{\partial\Omega}(x,s)=\begin{cases}0 & \text{if } s<0,\\ s & \text{if } 0\leqslant s\leqslant u_+(x),\\ u_+(x) & \text{if } s>u_+(x),\end{cases}$$
$$T_-(x,s)=\begin{cases}u_-(x) & \text{if } s<u_-(x),\\ s & \text{if } u_-(x)\leqslant s\leqslant 0,\\ 0 & \text{if } s>0,\end{cases}\qquad T_-^{\partial\Omega}(x,s)=\begin{cases}u_-(x) & \text{if } s<u_-(x),\\ s & \text{if } u_-(x)\leqslant s\leqslant 0,\\ 0 & \text{if } s>0.\end{cases}$$

Note that the truncation operators on $\partial\Omega$ apply to the corresponding traces $\gamma(u)$, where $u\in W^{1,p}(\Omega)$. For the sake of simplicity we just write $T_+^{\partial\Omega}(x,u)$ and $T_-^{\partial\Omega}(x,u)$ without the notation γ. It is clear that the truncation operators are continuous, uniformly bounded, and Lipschitz continuous with respect to the second argument. Additionally, we introduce truncations related to the nonlinearities $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ as follows:

$$f_+(x,s)=\begin{cases}0 & \text{if } s<0,\\ f(x,s) & \text{if } 0\leqslant s\leqslant u_+(x),\\ \eta_+(x) & \text{if } s>u_+(x),\end{cases}\qquad g_+(x,s)=\begin{cases}0 & \text{if } s<0,\\ g(x,s) & \text{if } 0\leqslant s\leqslant u_+(x),\\ \xi_+(x) & \text{if } s>u_+(x),\end{cases}$$

$$f_-(x,s)=\begin{cases}\eta_-(x) & \text{if } s<u_-(x),\\ f(x,s) & \text{if } u_-(x)\leqslant s\leqslant 0,\\ 0 & \text{if } s>0,\end{cases}\qquad g_-(x,s)=\begin{cases}\xi_-(x) & \text{if } s<u_-(x),\\ g(x,s) & \text{if } u_-(x)\leqslant s\leqslant 0,\\ 0 & \text{if } s>0,\end{cases}$$

$$f_0(x,s)=\begin{cases}\eta_-(x) & \text{if } s<u_-(x),\\ f(x,s) & \text{if } u_-(x)\leqslant s\leqslant u_+(x),\\ \eta_+(x) & \text{if } s>u_+(x),\end{cases}\qquad g_0(x,s)=\begin{cases}\xi_-(x) & \text{if } s<u_-(x),\\ g(x,s) & \text{if } u_-(x)\leqslant s\leqslant u_+(x),\\ \xi_+(x) & \text{if } s>u_+(x).\end{cases}$$

Here, η_+,ξ_+ and η_-,ξ_- correspond to the extremal solutions $u_+\in\operatorname{int}(C^1(\overline{\Omega})_+)$ and $u_-\in-\operatorname{int}(C^1(\overline{\Omega})_+)$, respectively. By means of these truncations, we define the following associated functionals given by

$$E_+(u)=\frac{1}{p}\big[\|\nabla u\|^p_{L^p(\Omega)}+\|u\|^p_{L^p(\Omega)}\big]-\int_\Omega\int_0^{u(x)} f_+(x,s)\,ds\,dx-\int_{\partial\Omega}\int_0^{u(x)}\big[aT_+^{\partial\Omega}(x,s)^{p-1}+g_+(x,s)\big]\,ds\,d\sigma,$$

$$E_-(u)=\frac{1}{p}\big[\|\nabla u\|^p_{L^p(\Omega)}+\|u\|^p_{L^p(\Omega)}\big]-\int_\Omega\int_0^{u(x)} f_-(x,s)\,ds\,dx+\int_{\partial\Omega}\int_0^{u(x)}\big[b\big|T_-^{\partial\Omega}(x,s)\big|^{p-1}-g_-(x,s)\big]\,ds\,d\sigma,$$

$$E_0(u)=\frac{1}{p}\big[\|\nabla u\|^p_{L^p(\Omega)}+\|u\|^p_{L^p(\Omega)}\big]-\int_\Omega\int_0^{u(x)} f_0(x,s)\,ds\,dx$$
$$-\int_{\partial\Omega}\int_0^{u(x)}\big[aT_+^{\partial\Omega}(x,s)^{p-1}-b\big|T_-^{\partial\Omega}(x,s)\big|^{p-1}+g_0(x,s)\big]\,ds\,d\sigma.$$

With a view to (F2) and (G2), we see that the functionals $E_+,E_-,E_0:W^{1,p}(\Omega)\to\mathbb{R}$ are locally Lipschitz continuous. The truncations involved guarantee that these functionals are bounded below, coercive and weakly sequentially lower semicontinuous which implies that their global minimizers exist. A characterization of the critical points of these functionals is stated in the next lemma.

Lemma 4.1. *The extremal constant-sign solutions of* (1.1) *are denoted by u_+ and u_-. Then one has*:

(i) *A critical point $v\in W^{1,p}(\Omega)$ of E_+ is a nonnegative solution of* (1.1) *satisfying* $0\leqslant v\leqslant u_+$.
(ii) *A critical point $v\in W^{1,p}(\Omega)$ of E_- is a nonpositive solution of* (1.1) *satisfying* $u_-\leqslant v\leqslant 0$.
(iii) *A critical point $v\in W^{1,p}(\Omega)$ of E_0 is a solution of* (1.1) *satisfying* $u_-\leqslant v\leqslant u_+$.

Proof. Let us only prove the third assertion, because the other cases can be done likewise. Let v be a critical point of E_0 which means $0\in\partial E_0(v)$. By the definition of E_0 we obtain

$$\int_\Omega|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx=\int_\Omega\big[\eta-|v|^{p-2}v\big]\varphi\,dx+\int_{\partial\Omega}\big[aT_+^{\partial\Omega}(x,v)^{p-1}-b\big|T_-^{\partial\Omega}(x,v)\big|^{p-1}+\xi\big]\varphi\,d\sigma,\quad\forall\varphi\in W^{1,p}(\Omega),\tag{4.1}$$

with some $\eta\in L^{q_0'}(\Omega)$ and $\xi\in L^{q_1'}(\partial\Omega)$ such that $\eta(x)\in\partial F_0(x,v(x))$ for a.a. $x\in\Omega$ and $\xi(x)\in\partial G_0(x,v(x))$ for a.a. $x\in\partial\Omega$, where

$$F_0(x,\eta)=\int_0^\eta f_0(x,s)\,ds,\qquad G_0(x,\xi)=\int_0^\xi g_0(x,s)\,ds.$$

The function u_+ is the smallest positive solution of (1.1) meaning that it satisfies the weak formulation given in Definition 1.1 by

$$\int_\Omega|\nabla u_+|^{p-2}\nabla u_+\cdot\nabla\varphi\,dx=\int_\Omega\big[\eta_+-|u_+|^{p-2}u_+\big]\varphi\,dx+\int_{\partial\Omega}\big[a(u_+)^{p-1}+\xi_+\big]\varphi\,d\sigma,\quad\forall\varphi\in W^{1,p}(\Omega),\tag{4.2}$$

where $\eta_+\in L^{q_0'}(\Omega)$ with $\eta_+(x)\in\partial F(x,u_+(x))$ for a.a. $x\in\Omega$ and $\xi_+\in L^{q_1'}(\partial\Omega)$ with $\xi_+(x)\in\partial G(x,u_+(x))$ for a.a. $x\in\partial\Omega$. Subtracting (4.2) from (4.1) and setting $\varphi=(v-u_+)^+\in W^{1,p}(\Omega)$ provides

$$\int_\Omega \left[|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+\right]\cdot\nabla(v-u_+)^+\,dx + \int_\Omega \left[|v|^{p-2}v - u_+^{p-1}\right](v-u_+)^+\,dx$$
$$= \int_\Omega [\eta-\eta_+](v-u_+)^+\,dx + \int_{\partial\Omega}\left[aT_+^{\partial\Omega}(x,v)^{p-1} - b\left|T_-^{\partial\Omega}(x,v)\right|^{p-1} - au_+^{p-1}\right](v-u_+)^+\,d\sigma$$
$$+ \int_{\partial\Omega}[\xi-\xi_+](v-u_+)^+\,d\sigma.$$

Clearly, it holds $\eta(x)=\eta_+(x)$ for a.a. $x \in \{x\in\Omega\colon\ v(x)>u_+(x)\}$. Furthermore, we get $T_+^{\partial\Omega}(x,v)=u_+$, $T_-^{\partial\Omega}(x,v)=0$ and $\xi(x)=\xi_+(x)$ for a.a. $x\in\partial\Omega$ satisfying $v(x)>u_+(x)$. Thus, the right-hand side of the equality above vanishes. However, the left-hand side is strictly positive in case $v>u_+$ which is a contradiction and hence $v\leqslant u_+$. The proof for $v\geqslant u_-$ acts in the same way. Summarizing, v belongs to the ordered interval $[u_-,u_+]$ which provides that $T_+^{\partial\Omega}(x,v)=v^+$ and $T_-^{\partial\Omega}(x,v)=v^-$. Since $\partial F_0(x,v(x))\subset\partial F(x,v(x))$ as well as $\partial G_0(x,v(x))\subset\partial G(x,v(x))$, from (4.1) it follows that v solves our original problem (1.1) satisfying $u_-\leqslant v\leqslant u_+$. This completes the proof. □

Let us now consider some results about local and global minimizers with respect to the functionals $E_+, E_-, E_0 : W^{1,p}(\Omega)\to\mathbb{R}$.

Lemma 4.2. *Let $a>\lambda_1$ and $b>\lambda_1$. Then the extremal positive solution u_+ of (1.1) is the unique global minimizer of the functional E_+ and the extremal negative solution u_- of (1.1) is the unique global minimizer of the functional E_-. Both u_+ and u_- are local minimizers of the functional E_0. Moreover, the functional $E_0 : W^{1,p}(\Omega)\to\mathbb{R}$ has a global minimizer v_0 which is a nontrivial solution of (1.1) satisfying $u_-\leqslant v_0\leqslant u_+$.*

Proof. The functional $E_+ : W^{1,p}(\Omega)\to\mathbb{R}$ is bounded below, coercive and weakly sequentially lower semicontinuous. This ensures that its global minimizer, namely $v_+\in W^{1,p}(\Omega)$, exists. Since v_+ is a critical point of E_+, Lemma 4.1 can be applied which yields that v_+ is a nonnegative solution of (1.1) fulfilling $0\leqslant v_+\leqslant u_+$. Applying the condition (G3) guarantees the existence of a number $\delta_a>0$ such that

$$\left|g(x,s)\right|\leqslant(a-\lambda_1)s^{p-1},\quad \forall s\colon\ 0<s\leqslant\delta_a. \tag{4.3}$$

We take $\varepsilon<\min\{\frac{\delta_f}{\|\varphi_1\|_\infty},\frac{\delta_a}{\|\varphi_1\|_\infty}\}$. Then, due to (F5) and (4.3) in combination with the Steklov eigenvalue problem in (1.10), we obtain

$$E_+(\varepsilon\varphi_1) = -\int_\Omega\int_0^{\varepsilon\varphi_1(x)} f(x,s)\,ds\,dx + \frac{\lambda_1-a}{p}\varepsilon^p\|\varphi_1\|_{L^p(\partial\Omega)}^p - \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} g(x,s)\,ds\,d\sigma$$
$$< \frac{\lambda_1-a}{p}\varepsilon^p\|\varphi_1\|_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)}(a-\lambda_1)s^{p-1}\,ds\,d\sigma$$
$$=0.$$

We see that $E_+(v_+)\neq0$ which means $v_+\neq0$. Applying Lemma 2.4 yields $v_+\in\operatorname{int}(C^1(\overline{\Omega})_+)$. As u_+ is the smallest positive solution of (1.1) in the ordered interval $[0,\vartheta_a e]$ satisfying $0\leqslant v_+\leqslant u_+$, it must hold $v_+=u_+$. This proves that u_+ is the unique global minimizer of the functional $E_+ : W^{1,p}(\Omega)\to\mathbb{R}$. Likewise, u_- is the unique global minimizer of E_-. In order to show that u_+ and u_- are local minimizers of E_0, we argue as follows. As $u_+\in\operatorname{int}(C^1(\overline{\Omega})_+)$ there exists a neighborhood V_{u_+} of u_+ in the space $C^1(\overline{\Omega})$ satisfying $V_{u_+}\subset C^1(\overline{\Omega})_+$. Hence, $E_+=E_0$ on V_{u_+} meaning that u_+ is a local minimizer of E_0 on $C^1(\overline{\Omega})$. Applying the recent results of the author in [30, Theorem 3.1] ensures that u_+ is also a local minimizer of E_0 on the space $W^{1,p}(\Omega)$. The same arguments can be applied on u_- which point out that u_- is a local minimizer of E_0 as well.

In the last step we have to show the existence of a global minimizer of E_0. As already mentioned the functional $E_0 : W^{1,p}(\Omega)\to\mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Thus, a global minimizer v_0 of E_0 exists which is, in particular, a critical point of E_0. Taking into account Lemma 4.1 proves that v_0 is a solution of (1.1) satisfying $u_-\leqslant v_0\leqslant u_+$. Since $E_0(u_+)=E_+(u_+)<0$, it guarantees $v_0\neq0$ which completes the proof. □

5. Existence of sign-changing solutions

This section is devoted to the proof of the existence of a sign-changing solution of problem (1.1). The idea is to find a nontrivial solution u_0 of problem (1.1) which belongs to $[u_-,u_+]$. If $u_0\neq u_-$ and $u_0\neq u_+$, then it must be a sign-changing solution of (1.1), because Theorem 3.1 ensures that $u_+\in\operatorname{int}(C^1(\overline{\Omega})_+)$ is the smallest positive solution in $[0,\vartheta_a e]$ and $u_-\in-\operatorname{int}(C^1(\overline{\Omega})_+)$ is the greatest negative solution in $[-\vartheta_b e,0]$.

Theorem 5.1. *Let the assumptions* (F1)–(F5), (G1)–(G5) *and* (H) *be satisfied. Then problem* (1.1) *has a nontrivial sign-changing solution* $u_0 \in C^1(\overline{\Omega})$.

Proof. As regards Lemma 4.2, the function $v_0 \in W^{1,p}(\Omega) \setminus \{0\}$ is a global minimizer of E_0 lying in $[u_-, u_+]$. Obviously, in the cases $v_0 \neq u_-$ and $v_0 \neq u_+$, the function $u_0 := v_0$ must be a sign-changing solutions of (1.1) because of the extremality properties of u_- and u_+, respectively. Let us now consider the case $v_0 = u_+$, the other case $v_0 = u_-$ can be done similarly. With a view to Lemma 4.2, we know that u_- is a local minimizer of E_0 which can be assumed to be a strict local minimizer. Otherwise we would find infinitely many critical points $v \neq 0$ of E_0 having changing sign due to $u_- \leqslant v \leqslant u_+$ and the extremality of the solutions u_- and u_+, respectively. Clearly, in this case the proof of the theorem would be done. The assumptions above ensure the existence of $\rho \in (0, \|u_+ - u_-\|_{W^{1,p}(\Omega)})$ such that

$$E_0(u_+) \leqslant E_0(u_-) < \inf\bigl\{E_0(u): u \in \partial B_\rho(u_-)\bigr\}, \tag{5.1}$$

with $\partial B_\rho(u_-) = \{u \in W^{1,p}(\Omega): \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. The functional E_0 satisfies the Palais–Smale condition (see Definition 1.6) because it is bounded below, locally Lipschitz and coercive. Hence, we can apply the Mountain-Pass Theorem as stated in Theorem 1.7 to E_0. This yields the existence of a critical point $u_0 \in W^{1,p}(\Omega)$ satisfying $0 \in \partial E_0(u_0)$ with

$$\inf\bigl\{E_0(u): u \in \partial B_\rho(u_-)\bigr\} \leqslant E_0(u_0) = \inf_{\pi \in \Pi} \max_{t \in [-1,1]} E_0\bigl(\pi(t)\bigr), \tag{5.2}$$

where

$$\Pi = \bigl\{\pi \in C\bigl([-1,1], W^{1,p}(\Omega)\bigr): \pi(-1) = u_-,\ \pi(1) = u_+\bigr\}.$$

Clearly, (5.1) and (5.2) ensure that $u_0 \neq u_-$ and $u_0 \neq u_+$ which means that u_0 is a sign-changing solution provided $u_0 \neq 0$. In order to prove that $u_0 \neq 0$, we must show that $E_0(u_0) \neq 0$ which is satisfied if there exists a path $\widetilde{\pi} \in \Pi$ such that

$$E_0\bigl(\widetilde{\pi}(t)\bigr) \neq 0, \quad \forall t \in [-1,1].$$

Such a path can be constructed as it was done in [27] with slight modifications. Additionally, the use of the Second Deformation Lemma in [27] has to be replaced by the Second Deformation Lemma for locally Lipschitz functionals as it can be found in [11, Theorem 2.10]. This completes the proof. □

References

[1] C.C. Aranda, E. Lami Dozo, Multiple solutions to a singular Lane–Emden–Fowler equation with convection term, Electron. J. Differential Equations 2008 (2008), Art. ID 05, 21 pp.

[2] D. Averna, S.A. Marano, D. Motreanu, Multiple solutions for a Dirichlet problem with p-Laplacian and set-valued nonlinearity, Bull. Aust. Math. Soc. 77 (2) (2008) 285–303.

[3] S. Carl, Existence and comparison results for noncoercive and nonmonotone multivalued elliptic problems, Nonlinear Anal. 65 (8) (2006) 1532–1546.

[4] S. Carl, D. Motreanu, Multiple solutions of nonlinear elliptic hemivariational problems, Pacific J. Appl. Math. 1 (4) (2008) 39–59.

[5] S. Carl, D. Motreanu, Multiple and sign-changing solutions for the multivalued p-Laplacian equation, Math. Nachr. 283 (7) (2010) 965–981.

[6] S. Carl, D. Motreanu, Sign-changing solutions for nonlinear elliptic problems depending on parameters, Int. J. Difference Equ. 2010 (2010), Art. ID 536236, 33 pp.

[7] S. Carl, D. Motreanu, Sub–supersolution method for multi-valued elliptic and evolution problems, in: D. Gao, D. Motreanu (Eds.), Handbook on Nonconvex Analysis and Applications, International Press of Boston, Inc., Somerville, MA, 2010, in press.

[8] S. Carl, P. Winkert, General comparison principle for variational–hemivariational inequalities, J. Inequal. Appl. 2009 (2009), Art. ID 184348, 29 pp.

[9] K.C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1) (1981) 102–129.

[10] F.H. Clarke, Optimization and Nonsmooth Analysis, second edition, Classics Appl. Math., vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[11] J.-N. Corvellec, Morse theory for continuous functionals, J. Math. Anal. Appl. 196 (3) (1995) 1050–1072.

[12] J. Fernández Bonder, Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities, Abstr. Appl. Anal. 2004 (12) (2004) 1047–1055.

[13] J. Fernández Bonder, Multiple solutions for the p-Laplace equation with nonlinear boundary conditions, Electron. J. Differential Equations 2006 (2006), Art. ID 37, 7 pp. (electronic).

[14] J. Fernández Bonder, J.D. Rossi, Existence results for the p-Laplacian with nonlinear boundary conditions, J. Math. Anal. Appl. 263 (1) (2001) 195–223.

[15] A. Iannizzotto, N.S. Papageorgiou, Existence of three nontrivial solutions for nonlinear Neumann hemivariational inequalities, Nonlinear Anal. 70 (9) (2009) 3285–3297.

[16] A. Lê, Eigenvalue problems for the p-Laplacian, Nonlinear Anal. 64 (5) (2006) 1057–1099.

[17] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (11) (1988) 1203–1219.

[18] S.R. Martínez, J.D. Rossi, Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition, Abstr. Appl. Anal. 7 (5) (2002) 287–293.

[19] S.R. Martínez, J.D. Rossi, Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance, Electron. J. Differential Equations 2003 (2003), Art. ID 27, 14 pp. (electronic).

[20] S.R. Martínez, J.D. Rossi, On the Fučik spectrum and a resonance problem for the p-Laplacian with a nonlinear boundary condition, Nonlinear Anal. 59 (6) (2004) 813–848.

[21] D. Motreanu, P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Nonconvex Optim. Appl., vol. 29, Kluwer Academic Publishers, Dordrecht, 1999.

[22] D. Motreanu, V. Rădulescu, Variational and Non-Variational Methods in Nonlinear Analysis and Boundary Value Problems, Nonconvex Optim. Appl., vol. 67, Kluwer Academic Publishers, Dordrecht, 2003.
[23] D. Motreanu, P. Winkert, Variational–hemivariational inequalities with nonhomogeneous Neumann boundary condition, Matematiche (Catania), in press.
[24] N.S. Papageorgiou, G. Smyrlis, Multiple solutions for nonlinear Neumann problems with the p-Laplacian and a nonsmooth crossing potential, Nonlinearity 23 (3) (2010) 529–548.
[25] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (3) (1984) 191–202.
[26] P. Winkert, Discontinuous variational–hemivariational inequalities involving the p-Laplacian, J. Inequal. Appl. 2007 (2007), Art. ID 13579, 11 pp.
[27] P. Winkert, Sign-changing and extremal constant-sign solutions of nonlinear elliptic Neumann boundary value problems, Bound. Value Probl. 2010 (2010), Art. ID 139126, 22 pp.
[28] P. Winkert, Constant-sign and sign-changing solutions for nonlinear elliptic equations with Neumann boundary values, Adv. Differential Equations 15 (5–6) (2010) 561–599.
[29] P. Winkert, L^∞-estimates for nonlinear elliptic Neumann boundary value problems, NoDEA Nonlinear Differential Equations Appl. 17 (3) (2010) 289–302.
[30] P. Winkert, Local $C^1(\overline{\Omega})$-minimizers versus local $W^{1,p}(\Omega)$-minimizers of nonsmooth functionals, Nonlinear Anal. 72 (11) (2010) 4298–4303.
[31] J.-H. Zhao, P.-H. Zhao, Existence of infinitely many weak solutions for the p-Laplacian with nonlinear boundary conditions, Nonlinear Anal. 69 (4) (2008) 1343–1355.