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Partial Differential Equations. – *A multiplicity theorem for anisotropic Robin equations*, by Nikolaos S. Papageorgiou and Patrick Winkert, communicated on 2 July 2021.

Abstract. – In this paper, we consider an anisotropic Robin problem driven by the $p(x)$-Laplacian and a superlinear reaction. Applying variational tools along with truncation and comparison techniques as well as critical groups, we prove that the problem has at least five nontrivial smooth solutions to be ordered and with sign information: two positive, two negative, and the fifth nodal.

Keywords. – Anisotropic maximum principle, anisotropic regularity theory, comparison and truncation techniques, constant sign and nodal solutions, critical groups, variable exponent spaces.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the anisotropic Robin problem

$$\begin{aligned} -\Delta_{p(\cdot)}u + \xi(x)|u|^{p(x)-2}u &= f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\nabla u \cdot \nu + \beta(x)|u|^{p(x)-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\nu(x)$ denotes the outer unit normal at $x \in \partial\Omega$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$, $\beta \geq 0$, and for $p \in C^{0,1}(\overline{\Omega})$ with $1 < \min_{x\in\overline{\Omega}} p(x)$ we denote by $\Delta_{p(\cdot)}$ the $p(x)$-Laplacian which is given by

$$\Delta_{p(\cdot)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

In the left-hand side of (1.1) there is also a potential term $\xi(x)|u|^{p(x)-2}u$ with $\xi \in L^\infty(\Omega)$ and $\xi \geq 0$. In the right-hand side of (1.1) there is a Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$, that is, $x \to f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \to f(x,s)$ is continuous for a.a. $x \in \Omega$. We suppose that $f(x,\cdot)$ is $(p_+ - 1)$-superlinear as $s \to \pm\infty$ but without assuming the usual Ambrosetti–Rabinowitz condition, where $p_+ = \max_{x\in\overline{\Omega}} p(x)$. Near zero $f(x,\cdot)$ exhibits an oscillatory behavior.

Using variational tools from the critical point theory along with appropriate truncation and comparison techniques, we prove the existence of at least five nontrivial smooth solutions, all with sign information and ordered.

Elliptic equations driven by the anisotropic Dirichlet p-Laplacian have been studied extensively in the last decade. The books of Diening–Harjulehto–Hästö–Růžička [6] and Rădulescu–Repovš [19] contain a rich bibliography on the subject. In contrast, the study of anisotropic Robin problems is lagging behind. Deng [2] studied the Robin problem

$$
\begin{aligned}
-\Delta_{p(\cdot)} u &= \lambda f(x,u) && \text{in } \Omega,\\
|\nabla u|^{p(x)-2}\nabla u\cdot\nu + \beta(x)|u|^{p(x)-2}u &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.2}
$$

and proved the existence of two positive solutions of problem (1.2) when $p \in C^1(\overline{\Omega})$ and under the Ambrosetti–Rabinowitz condition. A similar problem under the same assumptions as in [2] was treated by Fan–Deng [10], namely

$$
\begin{aligned}
-\Delta_{p(\cdot)} u + \lambda|u|^{p(x)-2}u &= f(x,u) && \text{in } \Omega,\\
|\nabla u|^{p(x)-2}\nabla u\cdot\nu &= \varphi && \text{on } \partial\Omega.
\end{aligned}
\tag{1.3}
$$

Only positive solutions for (1.3) is shown but no sign-changing solution is obtained. In 2010, Deng–Wang [3] considered existence and nonexistence of a nonhomogeneous Neumann problem given by

$$
\begin{aligned}
-\Delta_{p(\cdot)} u + \lambda|u|^{p(x)-2}u &= f(x,u) && \text{in } \Omega,\\
|\nabla u|^{p(x)-2}\nabla u\cdot\nu &= g(x,u) && \text{on } \partial\Omega.
\end{aligned}
\tag{1.4}
$$

It is proved that there exists a parameter $\lambda^* > 0$ such that problem (1.4) has at least two positive solutions for all $\lambda > \lambda^*$. We also mention the works of Gasiński–Papageorgiou [11], Papageorgiou–Rădulescu–Tang [18], and Wang–Fan–Ge [21]. Except for [11], the above-mentioned works consider parametric equations and focus on the existence and multiplicity of positive solutions. Gasiński–Papageorgiou [11] considered the Neumann problem

$$
\begin{aligned}
-\Delta_{p(\cdot)} u &= f(x,u) && \text{in } \Omega,\\
|\nabla u|^{p(x)-2}\nabla u\cdot\nu &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

and proved the existence of three nontrivial smooth solutions but they did not produce nodal solutions. The novelties in our work in contrast to the above-mentioned papers can be summarized in the following.

- We only need p to be Lipschitz continuous.

- We do not need to assume the Ambrosetti–Rabinowitz condition. We can weaken the assumptions; see hypotheses (H_1)(ii), (iii) in Section 2 and Remark 2.4.
- We obtain not only constant sign solutions, but also a sign-changing solution.
- All the solutions we obtain are ordered with concrete sign information.

Finally, we mention the works of Deng [1] and Deng–Wang–Cheng [4] concerning the Steklov and Robin eigenvalue problems of the anisotropic p-Laplacian, respectively.

The paper is organized as follows. In Section 2, we recall the basic properties of the variable exponent Sobolev spaces and the anisotropic p-Laplacian, mention some tools/definitions we need later (Cerami-condition, critical groups), and state the main hypotheses on the data of our problem. Section 3 deals with the existence of constant sign solutions. The first pair of positive and negative solutions is obtained in Proposition 3.1 by using the direct method of calculus of variations and the existence of the second pair of positive and negative solutions, stated in Proposition 3.2, is proved via the mountain pass theorem. The rest of the section is devoted to the existence of extremal constant sign solutions, see Proposition 3.5, which are needed later in order to find a sign-changing solution. Finally, Section 4 is concerned with the existence of a nodal solution to problem (1.1) which lies between the extremal constant sign solutions. This result is stated in Proposition 4.1 and the proof relies on the combination of the mountain pass theorem and critical groups. The full multiplicity result is given at the end in Theorem 4.2.

2. Preliminaries and Hypotheses

The study of problem (1.1) uses function spaces with variable exponents. A comprehensive introduction on the subject can be found in the book of Diening–Harjulehto–Hästö–Růžička [6].

In what follows we denote by $M(\Omega)$ the vector space of functions $u\colon \Omega \to \mathbb{R}$ which are measurable. As usual, we identify two such functions when they differ only on a Lebesgue-null set. Given $r \in C(\overline{\Omega})$ we define

$$r_- = \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_+ = \max_{x\in\overline{\Omega}} r(x)$$

and introduce the set

$$E_1 = \left\{ r \in C(\overline{\Omega}) : 1 < r_- \right\}.$$

Then, for $r \in E_1$, we introduce the variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ defined by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{r(x)}\,\mathrm{d}x < \infty \right\}.$$

We equip this space with the Luxemburg norm defined by

$$\|u\|_{r(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda}\right)^{r(x)} \mathrm{d}x \le 1\right\}.$$

Then $L^{r(\cdot)}(\Omega)$ is a separable, reflexive Banach space.

Moreover, we denote by $r'(x) = \frac{r(x)}{r(x)-1}$ the conjugate variable exponent to $r \in E_1$, that is,

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

It is clear that $r' \in E_1$. We know that $L^{r(\cdot)}(\Omega)^* = L^{r'(\cdot)}(\Omega)$ and the version of Hölder's inequality

$$\int_\Omega |uv| \,\mathrm{d}x \le \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(\cdot)} \|v\|_{r'(\cdot)}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$.

On the boundary $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Using this measure we can define the boundary variable exponent Lebesgue spaces $L^{r(\cdot)}(\partial\Omega)$ for $r \in E_1$.

The corresponding variable exponent Sobolev spaces can be defined in a natural way using the variable exponent Lebesgue spaces. So, given $r \in E_1$, we define

$$W^{1,r(\cdot)}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) : |\nabla u| \in L^{r(\cdot)}(\Omega)\right\}$$

with ∇u being the gradient of $u\colon \Omega \to \mathbb{R}$. This space is equipped with the norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W^{1,r(\cdot)}(\Omega)$$

with $\|\nabla u\|_{r(\cdot)} = \|\,|\nabla u|\,\|_{r(\cdot)}$. The space $W^{1,r(\cdot)}(\Omega)$ is a separable and reflexive Banach space.

For $r \in E_1$ we introduce the critical Sobolev variable exponents r^* and r_* defined by

$$r^*(x) = \begin{cases} \frac{Nr(x)}{N-r(x)} & \text{if } r(x) < N, \\ \ell_1(x) & \text{if } N \le r(x), \end{cases} \qquad \text{for all } x \in \overline{\Omega},$$

$$r_*(x) = \begin{cases} \frac{(N-1)r(x)}{N-r(x)} & \text{if } r(x) < N, \\ \ell_2(x) & \text{if } N \le r(x), \end{cases} \qquad \text{for all } x \in \partial\Omega,$$

where $\ell_1 \in C(\overline{\Omega})$ and $\ell_2 \in C(\partial\Omega)$ are arbitrarily chosen such that $r(x) < \ell_1(x)$ for all $x \in \overline{\Omega}$ and $r(x) < \ell_2(x)$ for all $x \in \partial\Omega$.

Suppose that $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $q \in C(\overline{\Omega})$ with $1 \leq q_-$. Then we have the anisotropic Sobolev embeddings

$$\begin{aligned} W^{1,r(\cdot)}(\Omega) &\hookrightarrow L^{q(\cdot)}(\Omega) \quad \text{continuously if } q(x) \leq r^*(x) \text{ for all } x \in \overline{\Omega},\\ W^{1,r(\cdot)}(\Omega) &\hookrightarrow L^{q(\cdot)}(\Omega) \quad \text{compactly if } q(x) < r^*(x) \quad \text{for all } x \in \overline{\Omega}. \end{aligned}$$

Similarly, if $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $q \in C(\partial\Omega)$ with $1 \leq q_-$, then we have the anisotropic trace embeddings

$$\begin{aligned} W^{1,r(\cdot)}(\Omega) &\hookrightarrow L^{q(\cdot)}(\partial\Omega) \quad \text{continuously if } q(x) \leq r_*(x) \text{ for all } x \in \overline{\Omega},\\ W^{1,r(\cdot)}(\Omega) &\hookrightarrow L^{q(\cdot)}(\partial\Omega) \quad \text{compactly if } q(x) < r_*(x) \quad \text{for all } x \in \overline{\Omega}. \end{aligned}$$

We refer to Diening–Harjulehto–Hästö–Růžička [6] and Fan [9].

In the study of these variable exponent spaces, the following modular function is useful:

$$\varrho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}\,\mathrm{d}x \quad \text{for all } u \in L^{r(\cdot)}(\Omega).$$

For $u \in W^{1,r(\cdot)}(\Omega)$ we write $\varrho_{r(\cdot)}(\nabla u) = \varrho_{r(\cdot)}(|\nabla u|)$.

The following proposition illustrates the relation between this modular and the Luxemburg norm.

Proposition 2.1. *Let $r \in E_1$, let $u \in L^{r(\cdot)}(\Omega)$, and let $\{u_n\}_{n\in\mathbb{N}} \subseteq L^{r(\cdot)}(\Omega)$. The following assertions hold:*

(i) $\|u\|_{r(\cdot)} = \eta \iff \varrho_{r(\cdot)}(\frac{u}{\eta}) = 1$;

(ii) $\|u\|_{r(\cdot)} < 1$ (*resp.* $= 1, > 1$) $\iff \varrho_{r(\cdot)}(u) < 1$ (*resp.* $= 1, > 1$);

(iii) $\|u\|_{r(\cdot)} \leq 1 \implies \|u\|_{r(\cdot)}^{r_+} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_-}$;
$\|u\|_{r(\cdot)} \geq 1 \implies \|u\|_{r(\cdot)}^{r_-} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_+}$;

(iv) $\|u_n\|_{r(\cdot)} \to 0 \iff \varrho_{r(\cdot)}(u_n) \to 0$;

(v) $\|u_n\|_{r(\cdot)} \to \infty \iff \varrho_{r(\cdot)}(u_n) \to \infty$.

Let $A_{r(\cdot)}\colon W^{1,r(\cdot)}(\Omega) \to W^{1,r(\cdot)}(\Omega)^*$ be the nonlinear operator defined by

$$\left\langle A_{r(\cdot)}(u), h\right\rangle = \int_\Omega |\nabla u|^{r(x)-2}\nabla u \cdot \nabla h\,\mathrm{d}x \quad \text{for all } u, h \in W^{1,r(\cdot)}(\Omega).$$

This operator has the following properties; see, for example Gasiński–Papageorgiou [11] and Rădulescu–Repovš [19, p. 40].

PROPOSITION 2.2. *The operator $A_{r(\cdot)}\colon W^{1,r(\cdot)}(\Omega) \to W^{1,r(\cdot)}(\Omega)^*$ is bounded (so it maps bounded sets to bounded sets), continuous, strictly monotone (which implies that it is also maximal monotone) and of type* (S_+)*; that is,*

$$u_n \overset{\mathrm{w}}{\to} u \text{ in } W^{1,r(\cdot)}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \left\langle A_{r(\cdot)}(u_n), u_n - u \right\rangle \leq 0$$

imply that $u_n \to u$ in $W^{1,r(\cdot)}(\Omega)$.

In the anisotropic regularity theory we need the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive order cone

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$ defined by

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \Omega \text{ and } \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega\cap u^{-1}(0)} < 0\right\},$$

where $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

Given $u \in W^{1,r(\cdot)}(\Omega)$, we set $u^\pm = \max\{\pm u, 0\}$ to be the positive and negative part of u, respectively. We know that $u = u^+ - u^-$, $|u| = u^+ + u^-$, and $u^\pm \in W^{1,r(\cdot)}(\Omega)$. If $u, v\colon \Omega \to \mathbb{R}$ are measurable functions and $u(x) \leq v(x)$ for a.a. $x \in \Omega$, then we introduce the following order interval in $W^{1,r(\cdot)}(\Omega)$:

$$[u, v] = \left\{h \in W^{1,r(\cdot)}(\Omega) : u(x) \leq h(x) \leq v(x) \text{ for a.a. } x \in \Omega\right\}.$$

Moreover, we denote by $\operatorname{int}_{C^1(\overline{\Omega})}[u, v]$ the interior of $[u, v] \cap C^1(\overline{\Omega})$ in $C^1(\overline{\Omega})$. Furthermore, we define

$$[u) = \left\{h \in W^{1,r(\cdot)}(\Omega) : u(x) \leq h(x) \text{ for a.a. } x \in \Omega\right\}.$$

Suppose that X is a Banach space and $\varphi \in C^1(X)$. We introduce the sets

$$\begin{aligned} K_\varphi &= \left\{u \in X : \varphi'(u) = 0\right\}, \\ \varphi^c &= \left\{u \in X : \varphi(u) \leq c\right\} \quad \text{with } c \in \mathbb{R}. \end{aligned}$$

We say that φ satisfies the "Cerami condition," C-condition for short, if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\left(1 + \|u_n\|_X\right)\varphi'(u_n) \to 0 \text{ in } X^* \quad \text{as } n \to \infty,$$

admits a strongly convergent subsequence.

If $Y_2 \subseteq Y_1 \subseteq X$, then we denote by $H_k(Y_1, Y_2)$, with $k \in \mathbb{N}_0$, the k-th relative singular homology group with integer coefficients. If $u \in K_\varphi$ is isolated, then the k-th critical group of φ at u is defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}\big) \quad \text{for all } k \in \mathbb{N}_0$$

with $c = \varphi(u)$ and a neighborhood U of u such that $\varphi^c \cap K_\varphi \cap U = \{u\}$. The excision property of singular homology implies that this definition of critical groups is independent of the isolating neighborhood U.

Now we introduce our hypotheses on the exponent $p(\cdot)$, the potential $\xi(\cdot)$, and the boundary coefficient $\beta(\cdot)$.

(H$_0$) $p \in C^{0,1}(\overline{\Omega})$, $1 < p_-(x) = \min_{x \in \overline{\Omega}} p(x) < N$, $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$, $\xi(x) \geq 0$ for a.a. $x \in \Omega$, $\beta(x) \geq 0$ for all $x \in \partial\Omega$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Note that the case $\beta = 0$ is also included and corresponds to the Neumann problem.

We introduce the C^1-functional $\gamma_{p(\cdot)} \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_{p(\cdot)}(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, \mathrm{d}x + \int_\Omega \frac{\xi(x)}{p(x)} |u|^{p(x)} \, \mathrm{d}x + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} \, \mathrm{d}\sigma$$

for all $u \in W^{1,p(\cdot)}(\Omega)$. We have

$$\left\langle \gamma'_{p(\cdot)}(u), h \right\rangle = \left\langle A_{p(\cdot)}(u), h \right\rangle + \int_\Omega \xi(x) |u|^{p(x)-2} u h \, \mathrm{d}x + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} u h \, \mathrm{d}\sigma$$

for all $u, h \in W^{1,p(\cdot)}(\Omega)$. Moreover, let $\varrho_0 \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ be the modular function defined by

$$\varrho_0(u) = \varrho_{p(\cdot)}(\nabla u) + \int_\Omega \xi(x) |u|^{p(x)} \, \mathrm{d}x + \int_{\partial\Omega} \beta(x) |u|^{p(x)} \, \mathrm{d}\sigma$$

for all $u \in W^{1,p(\cdot)}(\Omega)$.

In the sequel, we denote by $\|\cdot\|$ the norm of the Sobolev space $W^{1,p(\cdot)}(\Omega)$ defined by

$$\|u\| = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \quad \text{for } u \in W^{1,p(\cdot)}(\Omega).$$

The following estimates for $\gamma_{p(\cdot)}(\cdot)$ will be useful in what follows. The result can be found in the recent work of Papageorgiou–Rădulescu–Tang [18].

Proposition 2.3. *If hypothesis* (H$_0$) *holds, then there exist* $\hat{c}_0, \hat{c} > 0$ *such that*

$$\hat{c}\|u\|^{p_+} \leq \frac{1}{p_+}\varrho_0(u) \leq \gamma_{p(\cdot)}(u) \leq \frac{1}{p_-}\varrho_0(u) \leq \hat{c}_0\|u\|^{p_-} \quad \textit{if } \|u\| \leq 1,$$

$$\hat{c}\|u\|^{p_-} \leq \frac{1}{p_+}\varrho_0(u) \leq \gamma_{p(\cdot)}(u) \leq \frac{1}{p_-}\varrho_0(u) \leq \hat{c}_0\|u\|^{p_+} \quad \textit{if } \|u\| \geq 1.$$

Now we are ready to state our hypotheses on the nonlinearity $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$.

(H$_1$) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0)=0$ for a.a. $x \in \Omega$ and

(i) there exists $a \in L^\infty(\Omega)$ such that

$$\left|f(x,s)\right| \leq a(x)\left[1+|s|^{r(x)-1}\right]$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ with $r \in L^\infty(\Omega)$ such that $p_+ < r(x) < p^*(x)$ for a.a. $x \in \Omega$;

(ii) if $F(x,s)=\int_0^s f(x,t)\,\mathrm{d}t$, then

$$\lim_{s\to\pm\infty} \frac{F(x,s)}{|s|^{p_+}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii) there exists a function $q \in C(\overline{\Omega})$ such that

$$q(x) \in \left((r_+ - p_-)\frac{N}{p_-}, p^*(x)\right) \quad \text{for all } x \in \overline{\Omega},$$
$$0 < \eta \leq \liminf_{s\to+\infty} \frac{f(x,s)s - p_+F(x,s)}{|s|^{q(x)}}$$

uniformly for a.a. $x \in \Omega$;

(iv) there exist $\eta_- < 0 < \eta_+$, $\tau \in C(\overline{\Omega})$, and $\delta > 0$ such that

$$f(x,\eta_+) \leq -c_0 < 0 < c_1 \leq f(x,\eta_-)$$

for a.a. $x \in \Omega$, for some positive constants c_0, c_1, $\tau_+ < p_-$ and

$$f(x,s)s \geq c_2|s|^{\tau(x)}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for some $c_2 > 0$;

(v) there exists $\hat{\xi} > 0$ such that

$$s \to f(x,s) + \hat{\xi}|s|^{p(x)-2}s$$

is nondecreasing on $[\eta_-, \eta_+]$ for a.a. $x \in \Omega$.

Remark 2.4. Hypotheses (H$_1$)(ii), (iii) imply that $f(x,\cdot)$ is (p_+-1)-superlinear for a.a. $x \in \Omega$. However, we do not use the Ambrosetti–Rabinowitz condition as it was done in most previous works on the subject; see Deng [2], Deng–Wang [3], and Fan–Deng [10], for example. Hypothesis (H$_1$)(iv) dictates an oscillatory behavior near zero. In contrast, in [2,3,10] the reaction $f(x,\cdot)$ is required to be positive. Moreover, in [10],

$f(x,\cdot)$ is nondecreasing. So we see that our hypotheses provide a broader framework for the analysis of problem (1.1). The following function satisfies hypothesis (H_1), but fails to fulfill the Ambrosetti–Rabinowitz condition:

$$f(x,s)=\begin{cases} |s|^{\tau(x)-2}s-2|s|^{\mu(x)-2}s & \text{if } |s|\leq 1,\\ |s|^{p_+-2}s\ln(|s|)-|s|^{q(x)-2}s & \text{if } 1<|s|,\end{cases}$$

with $\tau\in E_1$, $\mu,q\in L^\infty(\Omega)$ and $q(x)\leq p_+$ for a.a. $x\in\Omega$. Note that f fails to satisfy the requirements in [2, 3, 10].

3. Constant sign solutions

We start by producing two localized constant sign solutions. To do this, we do not need the complete set of hypothesis (H_1). More precisely, we do not need the asymptotic conditions (H_1)(ii), (iii) as $s\to\pm\infty$.

Proposition 3.1. *If hypotheses* (H_0), (H_1)(i), (iv), (v) *hold, then problem* (1.1) *has two constant sign solutions*

$$u_0\in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)\quad \textit{and}\quad v_0\in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$$

such that

$$\eta_-<v_0(x)<0<u_0(x)<\eta_+\quad \textit{for all } x\in\overline{\Omega}.$$

Proof. First we show the existence of the positive solution. To this end, we introduce the Carathéodory function $\hat{f}_+\colon\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$\hat{f}_+(x,s)=\begin{cases} f(x,s^+) & \text{if } s\leq\eta_+,\\ f(x,\eta_+) & \text{if } \eta_+<s.\end{cases}\tag{3.1}$$

We set $\widehat{F}_+(x,s)=\int_0^s \hat{f}_+(x,t)\,\mathrm{d}t$ and consider the C^1-functional $\widehat{\psi}_+\colon W^{1,p(\cdot)}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\psi}_+(u)=\gamma_{p(\cdot)}(u)-\int_\Omega \widehat{F}_+(x,u)\,\mathrm{d}x\quad \text{for all } u\in W^{1,p(\cdot)}(\Omega).$$

From the truncation in (3.1) and Proposition 2.3 it is clear that $\widehat{\psi}_+\colon W^{1,p(\cdot)}(\Omega)\to\mathbb{R}$ is coercive. Moreover, the anisotropic Sobolev embedding theorem and the compactness of the anisotropic trace map imply that $\widehat{\psi}_+\colon W^{1,p(\cdot)}(\Omega)\to\mathbb{R}$ is also sequentially weakly lower semicontinuous. So, by the Weierstraß–Tonelli theorem we can find $u_0 \in W^{1,p(\cdot)}(\Omega)$ such that

$$\hat{\psi}_+(u_0) = \min\left[\hat{\psi}_+(u) : u \in W^{1,p(\cdot)}(\Omega)\right]. \tag{3.2}$$

Let $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and choose $t \in (0,1)$ small enough such that

$$0 < tu(x) \le \min\{\eta_+, \delta\} \quad \text{for all } x \in \overline{\Omega},$$

see hypothesis (H$_1$)(iv). Applying hypothesis (H$_1$)(iv) and recalling that $t \in (0,1)$, we have

$$\hat{\psi}_+(tu) \le \frac{t^{p_-}}{p_-}\varrho_0(u) - \frac{t^{\tau_+}}{\tau_+}c_0\varrho_{\tau(\cdot)}(u).$$

Since $\tau_+ < p_-$, we can choose $t \in (0,1)$ sufficiently small such that $\hat{\psi}_+(tu) < 0$. Hence, since u_0 is the global minimizer of $\hat{\psi}_+$, see (3.2), we know that

$$\hat{\psi}_+(u_0) < 0 = \hat{\psi}_+(0).$$

Thus, $u_0 \neq 0$.

From (3.2) we have $(\hat{\psi}_+)'(u_0) = 0$ which is equivalent to

$$\left\langle \gamma'_{p(\cdot)}(u_0), h\right\rangle = \int_\Omega \hat{f}_+(x,u_0)h\,\mathrm{d}x \quad \text{for all } h \in W^{1,p(\cdot)}(\Omega). \tag{3.3}$$

Choosing $h = -u_0^- \in W^{1,p(\cdot)}(\Omega)$ in (3.3) and using (3.1) gives

$$\varrho_0(u_0^-) = 0.$$

Hence, from Proposition 2.3, we get $u_0 \ge 0$ with $u_0 \neq 0$.

Next, we choose $h = (u_0 - \eta_+)^+ \in W^{1,p(\cdot)}(\Omega)$ in (3.3). Applying the definition of the truncation in (3.1) and hypothesis (H$_1$)(iv), we obtain

$$\begin{aligned}\left\langle \gamma'_{p(\cdot)}(u_0), (u_0-\eta_+)^+\right\rangle &= \int_\Omega f(x,\eta_+)(u_0-\eta_+)^+\,\mathrm{d}x \\ &\le 0 = \left\langle \gamma'_{p(\cdot)}(\eta_+), (u_0-\eta_+)^+\right\rangle.\end{aligned}$$

So, $u_0 \le \eta_+$; see hypothesis (H$_0$).

We have proved that

$$u_0 \in [0,\eta_+], \quad u_0 \neq 0. \tag{3.4}$$

Then (3.4), (3.1), and (3.3) imply that u_0 is a positive solution of problem (1.1). From the anisotropic regularity theory, see Fan [7, Theorem 1.3], we have $u_0 \in$

$C^1(\overline{\Omega})_+ \setminus \{0\}$. Finally, the anisotropic maximum principle of Zhang [23, Theorem 1.2] implies that $u_0 \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Let $\widehat{\xi} > 0$ be as given in hypothesis (H$_1$)(v). Then by using (3.4) and hypothesis (H$_1$)(v) one gets

$$\begin{aligned}-\Delta_{p(\cdot)} u_0 + \big(\xi(x) + \widehat{\xi}\,\big) u_0^{p(x)-1} &= f(x, u_0) + \widehat{\xi} u_0^{p(x)-1} \\ &\leq f(x, \eta_+) + \widehat{\xi} \eta_+^{p(x)-1} \\ &\leq -c_0 + \widehat{\xi} \eta_+^{p(x)-1} \\ &\leq -\Delta_{p(\cdot)} \eta_+ + \big(\xi(x) + \widehat{\xi}\,\big) \eta_+^{p(x)-1} \quad \text{in } \Omega.\end{aligned}$$

From Proposition 2.5 of Papageorgiou–Rădulescu–Repovš [17] we then conclude that $\eta_+ - u_0 \in D_+$.

Similarly, using the Carathéodory function $\hat{f}_-: \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}_-(x, s) = \begin{cases} f(x, \eta_-) & \text{if } s < \eta_-, \\ f(x, s) & \text{if } \eta_- \leq s \end{cases}$$

and reasoning as above, we produce a negative solution

$$v_0 \in -\operatorname{int}\big(C^1(\overline{\Omega})_+\big) \quad \text{and} \quad v_0 - \eta_- \in D_+. \qquad \blacksquare$$

From Proposition 3.1 it follows that

$$u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[0, \eta_+] \quad \text{and} \quad v_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[\eta_-, 0]. \tag{3.5}$$

Now, using these localized constant sign solutions, we are going to show the existence of two more such solutions, one is larger than u_0 and the other one is smaller than v_0. So, we will have four smooth constant sign solutions which are ordered. For this we will use the asymptotic conditions (H$_1$)(ii), (iii) as $s \to \pm\infty$.

Proposition 3.2. *If hypotheses* (H$_0$), (H$_1$) *hold, then problem* (1.1) *has two more constant sign solutions*

$$\hat{u} \in \operatorname{int}\big(C^1(\overline{\Omega})_+\big) \quad \textit{and} \quad \hat{v} \in -\operatorname{int}\big(C^1(\overline{\Omega})_+\big)$$

such that

$$\hat{u} \neq u_0,\ u_0 \leq \hat{u} \quad \textit{and} \quad \hat{v} \neq v_0,\ \hat{v} \leq v_0.$$

Proof. We start with the existence of a second positive solution. To this end, we introduce the Carathéodory function $g_+: \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g_+(x, s) = \begin{cases} f\big(x, u_0(x)\big) & \text{if } s \leq u_0(x), \\ f(x, s) & \text{if } u_0(x) < s. \end{cases} \tag{3.6}$$

Moreover, we will use the truncation of $g_+(x,\cdot)$ at η_+ and recall that $u_0(x) < \eta_+$ for all $x \in \Omega$. So we introduce the Carathéodory function $\hat{g}_+ \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{g}_+(x,s) = \begin{cases} g(x,s) & \text{if } s \leq \eta_+, \\ g(x,\eta_+) & \text{if } \eta_+ < s. \end{cases} \tag{3.7}$$

We set $G_+(x,s) = \int_0^s g_+(x,t)\,\mathrm{d}t$, $\widehat{G}_+(x,s) = \int_0^s \hat{g}_+(x,t)\,\mathrm{d}t$ and consider the C^1-functionals $\sigma_+, \widehat{\sigma}_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \gamma_{p(\cdot)}(u) - \int_\Omega G_+(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega),$$
$$\widehat{\sigma}_+(u) = \gamma_{p(\cdot)}(u) - \int_\Omega \widehat{G}_+(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

Using (3.6), (3.7) and the anisotropic regularity theory, see Winkert–Zacher [22] (see also Ho–Kim–Winkert–Zhang [12]) and Fan [7], we have

$$K_{\sigma_+} \subseteq [u_0) \cap \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and} \quad K_{\widehat{\sigma}_+} \subseteq [u_0,\eta_+] \cap \operatorname{int}\left(C^1(\overline{\Omega})_+\right). \tag{3.8}$$

Moreover, it is clear that from (3.6) and (3.7) we know that

$$\sigma_+|_{[0,\eta_+]} = \widehat{\sigma}_+|_{[0,\eta_+]}. \tag{3.9}$$

From (3.8) we see that we can always assume that

$$K_{\widehat{\sigma}_+} = \{u_0\}. \tag{3.10}$$

Otherwise, we would infer from (3.8) and (3.7) that we already have a second positive smooth solution of (1.1) larger than u_0 and so we are done.

From (3.7) and Proposition 2.3 it is clear that $\widehat{\sigma}_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is coercive. Also it is sequentially weakly lower semicontinuous. Hence, its global minimizer exists; that is, we find $\tilde{u}_0 \in W^{1,p(\cdot)}(\Omega)$ such that

$$\widehat{\sigma}_+(\tilde{u}_0) = \min\left[\widehat{\sigma}_+(u) : u \in W^{1,p(\cdot)}(\Omega)\right].$$

Because of (3.10) we conclude that $\tilde{u}_0 = u_0$.

From (3.9) and (3.5) it follows that u_0 is a local $C^1(\overline{\Omega})$-minimizer of σ_+. Then we know that

$$u_0 \text{ is a local } W^{1,p(\cdot)}(\Omega)\text{-minimizer of } \sigma_+; \tag{3.11}$$

see Fan [8] and Gasiński–Papageorgiou [11].

Note that from (3.8) and (3.6) we see that we may assume that

$$K_{\sigma_+} \text{ is finite.} \tag{3.12}$$

Otherwise we already have an infinity of positive smooth solutions of (1.1) all larger than u_0 and so we are done.

From (3.11), (3.12), and Theorem 5.7.6 of Papageorgiou–Rădulescu–Repovš [16, p. 449] we can find $\rho \in (0,1)$ small enough such that

$$\sigma_+(u_0) < \inf \left[\sigma_+(u) : \|u - u_0\| = \rho\right] = m_+. \tag{3.13}$$

On account of hypothesis (H_1)(ii), if $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$, we have

$$\sigma_+(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.14}$$

Moreover, hypotheses (H_1)(ii), (iii) and Proposition 4.1 of Gasiński–Papageorgiou [11] imply that

$$\sigma_+ \text{ satisfies the } C\text{-condition.} \tag{3.15}$$

From (3.13), (3.14), and (3.15) we see that we can use the mountain pass theorem and find $\hat{u} \in W^{1,p(\cdot)}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_+} \subseteq [u_0) \cap \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and} \quad \sigma_+(u_0) < m_+ \le \sigma_+(\hat{u}), \tag{3.16}$$

see (3.8) and (3.13). From (3.16) and (3.6) it follows that $\hat{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is the second positive smooth solution of problem (1.1) with $u_0 \le \hat{u}$ and $\hat{u} \ne u_0$.

In a similar way, starting with the Carathéodory function

$$g_-(x,s) = \begin{cases} f(x,s) & \text{if } s < v_0(x), \\ f\big(x, v_0(x)\big) & \text{if } v_0(x) \le s \end{cases}$$

and continuing as above, we can produce a second negative smooth solution $\hat{v} \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ with $\hat{v} \le v_0$ and $\hat{v} \ne v_0$. ∎

In fact, we will show that problem (1.1) admits extremal constant sign solutions; that is, there is a smallest positive solution $u_* \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and a greatest negative solution $v_* \in -\operatorname{int}(C^1(\overline{\Omega})_+)$. In Section 4, we will use these extremal constant sign solutions in order to prove the existence of a sign-changing solution, also called nodal solution.

Hypotheses (H_1)(i), (iv) imply that

$$f(x,s)s \ge c_2|s|^{\tau(x)-1} - c_3|s|^{r(x)-1} \tag{3.17}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for some $c_3 > 0$. This unilateral growth condition on $f(x,\cdot)$ leads to the auxiliary Robin problem

$$\begin{aligned} -\Delta_{p(\cdot)}u + \xi(x)|u|^{p(x)-2}u &= c_2|u|^{\tau(x)-2}u - c_3|u|^{r(x)-2}u \quad && \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\nabla u \cdot \nu + \beta(x)|u|^{p(x)-2}u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.18}$$

For this problem we have the following existence and uniqueness result.

Proposition 3.3. *If hypothesis* (H_0) *holds, then problem* (3.18) *admits a unique positive solution* $\bar{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *and since problem* (3.18) *is odd,* $\bar{v} = -\bar{u} \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ *is the unique negative solution of* (3.18).

Proof. First we show the existence of a positive smooth solution for problem (3.18). To this end, we introduce the C^1-functional $\vartheta_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\vartheta_+(u) = \gamma_{p(\cdot)}(u) + \int_\Omega \frac{c_3}{r(x)}(u^+)^{r(x)}\,\mathrm{d}x - \int_\Omega \frac{c_2}{\tau(x)}(u^+)^{\tau(x)}\,\mathrm{d}x$$

for all $u \in W^{1,p(\cdot)}(\Omega)$.

Using Proposition 2.3 we have for all $\|u\| \geq 1$

$$\vartheta_+(u) \geq \hat{c}\|u\|^{p_-} - \frac{c_2}{\tau_-}\varrho_{\tau(\cdot)}(u) \geq \hat{c}\|u\|^{p_-} - c_4\|u\|^{\tau_+}$$

for some $c_4 > 0$; see also Proposition 2.1 and recall that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\tau(\cdot)}(\Omega)$.

Since $\tau_+ < p_-$, see hypothesis (H_1)(iv), we infer that $\vartheta_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is coercive. Since it is also sequentially weakly lover semicontinuous, we can find $\tilde{u} \in W^{1,p(\cdot)}(\Omega)$ such that

$$\vartheta_+(\tilde{u}) = \min\left[\vartheta_+(u) : u \in W^{1,p(\cdot)}(\Omega)\right]. \tag{3.19}$$

Since $\tau_+ < p_- \leq p(x) < r(x)$ for all $x \in \overline{\Omega}$, if $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $t \in (0,1)$ is sufficiently small, we have $\vartheta_+(tu) < 0$. Then, due to (3.19), it holds that $\vartheta_+(\tilde{u}) < 0 = \vartheta_+(0)$ and so, $\tilde{u} \neq 0$.

From (3.19) we know that $\vartheta_+'(\tilde{u}) = 0$ and so

$$\left\langle \gamma_{p(\cdot)}'(\tilde{u}), h \right\rangle = \int_\Omega c_2(\tilde{u}^+)^{\tau(x)-1}h\,\mathrm{d}x - \int_\Omega c_3(\tilde{u}^+)^{r(x)-1}h\,\mathrm{d}x \tag{3.20}$$

for all $h \in W^{1,p(\cdot)}(\Omega)$. Choosing $h = -\tilde{u}^- \in W^{1,p(\cdot)}(\Omega)$ in (3.20), we get $\varrho_0(\tilde{u}^-) = 0$ and so, $\tilde{u} \geq 0$ with $\tilde{u} \neq 0$; see Proposition 2.3.

Therefore, $\tilde{u}$ is a positive solution of (3.18) and as before, using the anisotropic regularity theory, see Winkert–Zacher [22] and Fan [7], and the anisotropic maximum principle, see Zhang [23], we infer that $\tilde{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Next we show that this positive solution of (3.18) is unique. For this purpose, we introduce the integral functional $j_+\colon L^1(\Omega) \to \overline{R} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j_+(u) = \begin{cases} \gamma_{p(\cdot)}(\nabla u^{\frac{1}{p_-}}) & \text{if } u \geq 0, u^{\frac{1}{p_-}} \in W^{1,p(\cdot)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Theorem 2.2 of Takáč–Giacomoni [20], see also Díaz–Saá [5] for the isotropic case, we have that $j_+\colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is convex.

Suppose that $\tilde{y} \in W^{1,p(\cdot)}(\Omega)$ is another positive solution of problem (3.18). As before, we have $\tilde{y} \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Using Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [16, p. 274], we see that

$$\frac{\tilde{u}}{\tilde{y}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\tilde{y}}{\tilde{u}} \in L^\infty(\Omega). \tag{3.21}$$

Let $h = \tilde{u}^{p_-} - \tilde{y}^{p_-} \in W^{1,p(\cdot)}(\Omega)$. Then, from (3.21) and if $|t| < 1$ is small enough, we conclude that

$$\tilde{u}^{p_-} + th \in \operatorname{dom} j_+ \quad \text{and} \quad \tilde{y}^{p_-} + th \in \operatorname{dom} j_+.$$

Hence, on account of the convexity of j_+, we infer that j_+ is Gateaux differentiable at $\tilde{u}^{p_-}$ and at $\tilde{y}^{p_-}$ in the direction h. Using Green's identity, see Takáč–Giacomoni [20, Remark 2.6], and the chain rule, we obtain

$$\begin{aligned} j_+'(\tilde{u}^{p_-})(h) &= \frac{1}{p_-}\int_\Omega \frac{-\Delta_{p(\cdot)}\tilde{u} + \xi(x)\tilde{u}^{p(x)-1}}{\tilde{u}^{p_- -1}} h \, dx \\ &= \frac{1}{p_-}\int_\Omega \Big[\frac{c_2}{\tilde{u}^{p_- - \tau(x)}} - c_3\tilde{u}^{r(x)-p_-}\Big] h \, dx, \\ j_+'(\tilde{y}^{p_-})(h) &= \frac{1}{p_-}\int_\Omega \frac{-\Delta_{p(\cdot)}\tilde{y} + \xi(x)\tilde{y}^{p(x)-1}}{\tilde{y}^{p_- -1}} h \, dx \\ &= \frac{1}{p_-}\int_\Omega \Big[\frac{c_2}{\tilde{y}^{p_- - \tau(x)}} - c_3\tilde{y}^{r(x)-p_-}\Big] h \, dx. \end{aligned}$$

The convexity of j_+ implies the monotonicity of j_+'. Therefore, we have

$$\begin{aligned} 0 \leq &\int_\Omega c_2\Big[\frac{1}{\tilde{u}^{p_- - \tau(x)}} - \frac{1}{\tilde{y}^{p_- - \tau(x)}}\Big](\tilde{u}^{p_-} - \tilde{y}^{p_-}) \, dx \\ &+ \int_\Omega c_3[\tilde{y}^{r(x)-p_-} - \tilde{u}^{r(x)-p_-}](\tilde{u}^{p_-} - \tilde{y}^{p_-}) \, dx. \end{aligned}$$

Recall that $\tau_+ < p_- < \tau(x)$ for all $x \in \overline{\Omega}$, we conclude that $\tilde{u} = \tilde{y}$. This proves the uniqueness of the positive solution $\tilde{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ for problem (3.18).

Since the problem is odd, $\tilde{v} = -\tilde{u} \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ is the unique negative solution of (3.18). ∎

We introduce the following two sets:

$$\mathcal{S}_+ = \{u : u \text{ is a positive solution of problem (1.1)}\},$$
$$\mathcal{S}_- = \{u : u \text{ is a negative solution of problem (1.1)}\}.$$

We have already seen in Proposition 3.1 that

$$\emptyset \neq \mathcal{S}_+ \subseteq \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and} \quad \emptyset \neq \mathcal{S}_- \subseteq -\operatorname{int}\left(C^1(\overline{\Omega})_+\right).$$

The solutions $\tilde{u}, \tilde{v}$ of (3.18) provide bounds for the sets $\mathcal{S}_+$ and $\mathcal{S}_-$, where $\tilde{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a lower bound for $\mathcal{S}_+$ and $\tilde{v} \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ is an upper bound for $\mathcal{S}_-$.

PROPOSITION 3.4. *If hypotheses* (H_0), (H_1) *hold, then* $\tilde{u} \leq u$ *for all* $u \in \mathcal{S}_+$ *and* $v \leq \tilde{v}$ *for all* $v \in \mathcal{S}_-$.

PROOF. Let $u \in \mathcal{S}_+$ and consider the Carathéodory function $k_+ \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_+(x,s) = \begin{cases} c_2(s^+)^{\tau(x)-1} - c_3(s^+)^{r(x)-1} & \text{if } s \leq u(x), \\ c_2\big(u(x)\big)^{\tau(x)-1} - c_3\big(u(x)\big)^{r(x)-1} & \text{if } u(x) < s. \end{cases} \tag{3.22}$$

We set $K_+(x,s) = \int_0^s k_+(x,t)\,\mathrm{d}t$ and consider the C^1-functional $\widehat{\vartheta}_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\vartheta}_+(u) = \gamma_{p(\cdot)}(u) - \int_\Omega K_+(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

Evidently, $\widehat{\vartheta}_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is coercive, see (3.22) and Proposition 2.3, and sequentially weakly lower semicontinuous. Hence, we find $\tilde{u}_* \in W^{1,p(\cdot)}(\Omega)$ such that

$$\widehat{\vartheta}_+(\tilde{u}_*) = \min\left[\widehat{\vartheta}_+(u) : u \in W^{1,p(\cdot)}(\Omega)\right]. \tag{3.23}$$

As before, see the proof of Proposition 3.3, if $w \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $t \in (0,1)$ sufficiently small, at least so that $tw \leq u$, we have $\widehat{\vartheta}_+(tw) < 0$, recall that $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$, and use Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [16, p. 274]. Then, due to (3.23), it follows that $\widehat{\vartheta}_+(\tilde{u}_*) < 0 = \widehat{\vartheta}_+(0)$. Hence, $\tilde{u}_* \neq 0$.

From (3.23) we have $(\widehat{\vartheta}_+)'(\tilde{u}_*) = 0$; that is,

$$\left\langle \gamma'_{p(\cdot)}(\tilde{u}_*), h \right\rangle = \int_\Omega k_+(x,\tilde{u}_*) h\,\mathrm{d}x \quad \text{for all } h \in W^{1,p(\cdot)}(\Omega). \tag{3.24}$$

First we choose $h = -\tilde{u}^- \in W^{1,p(\cdot)}(\Omega)$ and obtain $\tilde{u}_* \geq 0$ with $\tilde{u}_* \neq 0$; see (3.22). Next, we take $h = (\tilde{u}_* - u)^+ \in W^{1,p(\cdot)}(\Omega)$. Then, from (3.22), (3.17), and the fact that $u \in \mathcal{S}_+$, we obtain

$$\begin{aligned}\langle \gamma'_{p(\cdot)}(\tilde{u}_*), (\tilde{u}_* - u)^+ \rangle &= \int_\Omega [c_2 u^{\tau(x)-1} - c_3 u^{r(x)-1}](\tilde{u}_* - u)^+ \,\mathrm{d}x \\ &\leq \int_\Omega f(x,u)(\tilde{u}_* - u)^+ \,\mathrm{d}x \\ &= \langle \gamma'_{p(\cdot)}(u), (\tilde{u}_* - u)^+ \rangle.\end{aligned}$$

Hence, $\tilde{u}_* \leq u$. So, we have proved

$$\tilde{u}_* \in [0, u], \quad \tilde{u}_* \neq 0. \tag{3.25}$$

From (3.25), (3.22), (3.24), and Proposition 3.3, it follows that $\tilde{u}_* = \tilde{u}$. Thus, see (3.25), $\tilde{u} \leq u$ for all $u \in \mathcal{S}_+$.

Similarly, we show that $v \leq \tilde{v}$ for all $v \in \mathcal{S}_-$. ■

Now we are ready to produce extremal constant sign solutions for problem (1.1).

PROPOSITION 3.5. *If hypotheses* (H_0), (H_1) *hold, then problem* (1.1) *has a smallest positive solution* $u_* \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *and a greatest negative solution* $v_* \in -\operatorname{int}(C^1(\overline{\Omega})_+)$.

PROOF. From the proof of Proposition 7 of Papageorgiou–Rădulescu–Repovš [15], we know that $\mathcal{S}_+$ is downward directed, that is, if $u_1, u_2 \in \mathcal{S}_+$, then we can find $u \in \mathcal{S}_+$ such that $u \leq u_1$ and $u \leq u_2$. Then Lemma 3.10 of Hu–Papageorgiou [13, p. 178] implies that there exists a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ such that

$$\inf \mathcal{S}_+ = \inf_{n\in\mathbb{N}} u_n.$$

On account of Proposition 3.1 we have that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(\cdot)}(\Omega)$ is bounded. Hence, we may assume that

$$u_n \xrightarrow{\mathrm{w}} u_* \text{ in } W^{1,p(\cdot)}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^{p(\cdot)}(\Omega) \text{ and in } L^{p(\cdot)}(\partial\Omega). \tag{3.26}$$

Since $u_n \in \mathcal{S}_+$, we have

$$\langle \gamma'_{p(\cdot)}(u_n), h \rangle = \int_\Omega f(x,u_n) h \,\mathrm{d}x \tag{3.27}$$

for all $h \in W^{1,p(\cdot)}(\Omega)$ and for all $n \in \mathbb{N}$. Choosing $h = u_n - u_* \in W^{1,p(\cdot)}(\Omega)$ in (3.27), passing to the limit as $n \to \infty$ and using (3.26), we obtain

$$\lim_{n\to\infty} \langle A_{p(\cdot)}(u_n), u_n - u_* \rangle = 0.$$

Then, from Proposition 2.2, we infer that

$$u_n \to u_* \quad \text{in } W^{1,p(\cdot)}(\Omega). \tag{3.28}$$

So, passing to the limit as $n \to \infty$ in (3.27) and using (3.28), one gets

$$\left\langle \gamma'_{p(\cdot)}(u_*), h \right\rangle = \int_\Omega f(x, u_*) h \,\mathrm{d}x \quad \text{for all } h \in W^{1,p(\cdot)}(\Omega).$$

Furthermore, by Proposition 3.4, we conclude that $\tilde{u}_* \leq u_*$. It follows that $u_* \in \mathcal{S}_+$ and $u_* = \inf \mathcal{S}_+$.

Similarly, we show that there exists $v_* \in \mathcal{S}_-$ such that $v \leq v_*$ for all $v \in \mathcal{S}_-$. We mention that $\mathcal{S}_-$ is upward directed, that is, if $v_1, v_2 \in \mathcal{S}_-$, we can find $v \in \mathcal{S}_-$ such that $v_1 \leq v$ and $v_2 \leq v$. ■

4. Nodal solution

In this section, using the extremal constant sign solutions of problem (1.1) obtained in Proposition 3.5, we show the existence of a nodal solution located between them.

Proposition 4.1. *If hypotheses* (H$_0$), (H$_1$) *hold, then problem* (1.1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}).$$

Proof. Let $u_* \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $v_* \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ be the two extremal constant sign solutions produced in Proposition 3.5. We introduce the Carathéodory function $l \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$l(x,s) = \begin{cases} f\big(x, v_*(x)\big) & \text{if } s < v_*(x), \\ f(x,s) & \text{if } v_*(x) \leq s \leq u_*(x), \\ f\big(x, u_*(x)\big) & \text{if } u_*(x) < s. \end{cases} \tag{4.1}$$

We also consider the positive and negative truncations of $l(x,\cdot)$, namely the Carathéodory functions $l_\pm \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$l_\pm(x,s) = l(x, \pm s^\pm). \tag{4.2}$$

We set $L(x,s) = \int_0^s l(x,t)\,\mathrm{d}t$ and $L_\pm(x,s) = \int_0^s l_\pm(x,t)\,\mathrm{d}t$ and consider the C^1-functional $\mu, \mu_\pm \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \gamma_{p(\cdot)}(u) - \int_\Omega L(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega),$$

$$\mu_\pm(u) = \gamma_{p(\cdot)}(u) - \int_\Omega L_\pm(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

Using (4.1) and (4.2) we easily show that

$$K_\mu \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}),$$
$$K_{\mu_+} \subseteq [0, u_*] \cap C^1(\overline{\Omega})_+,$$
$$K_{\mu_-} \subseteq [v_*, 0] \cap \big(-C^1(\overline{\Omega})_+\big).$$

The extremality of the solutions u_* and v_* implies that

$$K_\mu \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}), \quad K_{\mu_+} = \{0, u_*\}, \quad K_{\mu_-} = \{v_*, 0\}. \tag{4.3}$$

Due to (4.1) and (4.2) it is clear that $\mu_+ \colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is coercive and it is also sequentially weakly lower semicontinuous. Hence, $\hat{u}_* \in W^{1,p(\cdot)}(\Omega)$ exists such that

$$\mu_+(\hat{u}_*) = \min\big[\mu_+(u) : u \in W^{1,p(\cdot)}(\Omega)\big] < 0 = \mu_+(0);$$

see the proof of Proposition 3.3. Hence, $\hat{u}_* \neq 0$ and so, $\hat{u}_* = u_*$; see (4.3).

It is clear that

$$\mu|_{C^1(\overline{\Omega})_+} = \mu_+|_{C^1(\overline{\Omega})_+}.$$

Since $u_* \in \operatorname{int}(C^1(\overline{\Omega})_+)$, it follows that u_* is a local $C^1(\overline{\Omega})$-minimizer of μ. Therefore,

$$u_* \text{ is a local } W^{1,p(\cdot)}(\Omega)\text{-minimizer of } \mu; \tag{4.4}$$

see Fan [8] and Gasiński–Papageorgiou [11].

Similarly, working this time with the functional μ_-, we show that

$$v_* \text{ is a local } W^{1,p(\cdot)}(\Omega)\text{-minimizer of } \mu. \tag{4.5}$$

We may assume that $\mu(v_*) \leq \mu(u_*)$. The reasoning is similar if the opposite inequality holds using (4.5) instead of (4.4). From (4.3) it is clear that we may assume that K_μ is finite. Otherwise, taking (4.1) and the extremality of the solutions u_* and v_* into account, we already have an infinity of smooth nodal solutions and so we are done. Then, from (4.4) and Theorem 5.7.6 of Papageorgiou–Rădulescu–Repovš [16, p. 449], we know that we can find $\rho \in (0,1)$ small enough such that

$$\mu(v_*) \leq \mu(u_*) < \inf\big[\mu(u) : \|u - u_*\| = \rho\big] = m_*, \quad \|u_* - v_*\| > \rho. \tag{4.6}$$

The coercivity of μ implies that μ satisfies the C-condition; see Papageorgiou–Rădulescu–Repovš [16, Proposition 5.1.15 on p. 369]. This fact along with (4.6) permit the use of the mountain pass theorem. So, there exists $y_0 \in W^{1,p(\cdot)}(\Omega)$ such that

$$y_0 \in K_\mu \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}), \quad m_* \leq \mu(y_0). \tag{4.7}$$

From (4.6) and (4.7) it follows that $y_0 \notin \{v_*, u_*\}$. Moreover, from Corollary 6.6.9 of Papageorgiou–Rădulescu–Repovš [16, p. 533] we have

$$C_1(\mu, y_0) \neq 0. \tag{4.8}$$

On the other hand, from hypothesis (H_1)(iv) and Proposition 3.7 of Papageorgiou–Rădulescu [14], we obtain

$$C_k(\mu, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.9}$$

Comparing (4.8) and (4.9), we conclude that $y_0 \neq 0$. Since $y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega})$ with $y_0 \notin \{0, u_*, v_*\}$, the extremality of u_* and v_* implies that y_0 is a smooth nodal solution of (1.1). ∎

Finally, we can state the following multiplicity theorem for problem (1.1); see Propositions 3.1, 3.2, and 4.1.

THEOREM 4.2. *If hypotheses* (H_0), (H_1) *hold, then problem* (1.1) *has at least five nontrivial smooth solutions*

$$u_0, \hat{u} \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and} \quad v_0, \hat{v} \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and} \quad y_0 \in C^1(\overline{\Omega}) \text{ nodal}$$

with $u_0 \neq \hat{u}$, $v_0 \neq \hat{v}$ *and*

$$\hat{v}(x) \leq v_0(x) \leq y_0(x) \leq u_0(x) \leq \hat{u}(x) \quad \text{for all } x \in \overline{\Omega}$$

as well as

$$\eta_- < v_0(x) < 0 < u_0(x) < \eta_+ \quad \text{for all } x \in \Omega.$$

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Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University,
Zografou Campus, Athens 15780, Greece
npapg@math.ntua.gr

Patrick Winkert
Institut für Mathematik, Technische Universität Berlin,
Straße des 17. Juni 136, 10623 Berlin, Germany
winkert@math.tu-berlin.de