



# Nonlinear systems with Hartman-type perturbations

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## Abstract

We consider a nonlinear Lienard-type system driven by a nonlinear, nonhomogeneous differential operator and a maximal monotone map. On the Carathéodory perturbation we do not impose any global growth condition. Instead we employ a Hartman-type hypotheses. Using tools from fixed point theory and the theory of operators of monotone type, we prove two existence theorems.

**Keywords** Nonlinear nonhomogeneous differential operator · Maximal monotone map · Hartman condition · Leray–Schauder alternative principle · Lienard system

**Mathematics Subject Classification** 34A60 · 34B15

## 1 Introduction

In 1960, Hartman [4], see also Hartman [5], proved that the semilinear Dirichlet system

$$u''(t) = f(t, u(t)) \quad \text{on } T = [0, b], \quad u(0) = u(b) = 0$$

with  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  being continuous, admits a solution provided that there exists  $M > 0$  such that

$$(f(t, x), x)_{\mathbb{R}^N} \geq 0 \quad \text{for all } t \in T \text{ and for all } x \in \mathbb{R}^N \text{ with } |x| = M.$$

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Later, Knobloch [6] extended the result to semilinear periodic systems under the assumption that the vector field  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is locally Lipschitz. More recently, Mawhin [8] extended the results of Hartman and Knobloch to nonlinear systems driven by the vector  $p$ -Laplacian and having a continuous vector field  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ .

In this paper we go well beyond the aforementioned works and deal with the following nonlinear system:

$$\begin{aligned}
 a(u'(t))' + \frac{d}{dt} \nabla G(u(t)) &\in A(u(t)) + f(t, u(t)) \quad \text{for a.a. } t \in T = [0, b], \\
 u &\in \text{BC},
 \end{aligned}
 \tag{1.1}$$

where we mean by  $u \in \text{BC}$  that  $u$  satisfies one of the following boundary conditions

- Dirichlet condition:  $u(0) = u(b) = 0$ ;
- Neumann condition:  $u'(0) = u'(b) = 0$ ;
- Periodic condition:  $u(0) = u(b), u'(0) = u'(b)$ .

We will do the proof for the periodic problem and the same reasoning, in fact in a simpler form, applies also to the other two boundary conditions.

In problem (1.1), the mapping  $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a suitable homeomorphism, in general nonhomogeneous, which includes many differential operators of interest as special cases such as the vector  $p$ -Laplacian. For  $G$  we suppose  $G \in C^2(\mathbb{R}^N, \mathbb{R})$  and on the right-hand side of (1.1),  $A : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$  is a maximal monotone map and  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory perturbation, that is,  $t \rightarrow f(t, x)$  is measurable for all  $x \in \mathbb{R}^N$  and  $x \rightarrow f(t, x)$  is continuous for a.a.  $t \in T$ . We do not assume that the domain of  $A$  is all of  $\mathbb{R}^N$  and this incorporates in our framework systems with unilateral constraints, namely differential variational inequalities. Moreover, we do not impose any global growth condition on the perturbation term  $f(t, \cdot)$ . Instead we employ the Hartman-type condition mentioned in the beginning of the paper. The particular form of (1.1) classifies the problem as a nonlinear Lienard system, see Hartman [5, p. 179].

Our approach uses tools from fixed point theory and from the theory of nonlinear operators of monotone type.

## 2 Preliminaries and hypotheses

Let  $X$  be a reflexive Banach space, let  $X^*$  be its topological dual and denote by  $\langle \cdot, \cdot \rangle$  the duality brackets for the pair  $(X^*, X)$ . We say that a map  $A : X \rightarrow 2^{X^*}$  is monotone if

$$\langle u^* - x^*, u - x \rangle \geq 0 \quad \text{for all } (u, u^*), (x, x^*) \in \text{Gr } A,$$

where

$$\text{Gr } A = \{(v, v^*) \in X \times X^* : v^* \in A(v)\}$$

denotes the graph of  $A$ . If  $A$  satisfies

$$\langle u^* - x^*, u - x \rangle > 0 \quad \text{for all } (u, u^*), (x, x^*) \in \text{Gr } A \text{ with } u \neq x,$$

then we say that  $A$  is strictly monotone. Finally we say that  $A : X \rightarrow 2^{X^*}$  is maximal monotone if

$$\langle u^* - x^*, u - x \rangle \geq 0 \quad \text{for all } (u, u^*) \in \text{Gr } A \quad \text{implies} \quad (x, x^*) \in \text{Gr } A.$$

This means that  $\text{Gr } A$  is maximal with respect to inclusion among the graphs of all monotone maps. By  $D(A)$  we denote the domain of  $A$ , that is,

$$D(A) = \{u \in X : A(u) \neq \emptyset\}.$$

For a maximal monotone map  $A$  we have that  $\text{Gr } A$  is sequentially closed in  $X_w \times X^*$  and in  $X \times X_w^*$ .

Now, let  $H$  be a Hilbert space. We identify  $H$  with its dual by the Fréchet–Riesz theorem, that is,  $H = H^*$ . Let  $A : H \rightarrow 2^H$  be a maximal monotone map. For  $\lambda > 0$  we define the following single-valued maps

$$\begin{aligned} \text{Resolvent of } A : \quad J_\lambda &= (I + \lambda A)^{-1}, \\ \text{Yosida approximation of } A : \quad A_\lambda &= \frac{1}{\lambda}[I - J_\lambda]. \end{aligned}$$

The next proposition summarizes the main properties of these two operators.

**Proposition 2.1** *If  $A : H \rightarrow 2^H$  is a maximal monotone map and  $\lambda > 0$ , then the following hold:*

- (a)  $J_\lambda : H \rightarrow H$  is nonexpansive, that is  $\|J_\lambda(u) - J_\lambda(x)\| \leq \|u - x\|$  for all  $u, x \in H$ ;
- (b)  $A_\lambda(u) \in A(J_\lambda(u))$  for all  $u \in H$ ;
- (c)  $A_\lambda$  is monotone and  $\|A_\lambda(u) - A(x)\| \leq \frac{1}{\lambda}\|u - x\|$  for all  $u, x \in H$ ;
- (d)  $\|A_\lambda(u)\| \leq \|A^0(u)\| = \min \{\|u^*\| : u^* \in A(u)\}$  and  $A_\lambda(u) \rightarrow A^0(u)$  as  $\lambda \rightarrow 0^+$  for all  $u \in D(A)$ ;
- (e)  $\overline{D(A)}$  is convex and  $J_\lambda(u) \rightarrow \text{proj}\left(u; \overline{D(A)}\right)$  for all  $u \in H$ .

**Remark 2.2** The maximal monotonicity of  $A$  implies that  $A(u) \subseteq H$  is nonempty, closed and convex for all  $u \in D(A)$ . Therefore, the minimal norm element  $A^0(u)$  exists. Moreover,  $\overline{D(A)}$  is convex and so the metric projection  $\text{proj}(\cdot, \overline{D(A)})$  is well-defined. For more about maps of monotone type we refer to Papageorgiou–Winkert [9].

Suppose that  $V, Z$  are Banach spaces and let  $K : V \rightarrow Z$ . We introduce the following two notions:

- We say that  $K$  is completely continuous if

$$v_n \xrightarrow{w} v \quad \text{in } V \quad \text{implies} \quad K(v_n) \rightarrow K(v) \quad \text{in } Z.$$

- We say that  $K$  is compact if it is continuous and maps bounded sets in  $V$  to relatively compact sets in  $Z$ .

From the fixed point theory, we will use the Leray–Schauder Alternative Principle which says the following.

**Theorem 2.3** *If  $V$  is a Banach space,  $K : V \rightarrow V$  is a compact map and*

$$S = \{v \in V : v = \mu K(v) \text{ for some } 0 < \mu < 1\},$$

*then one of the following two statements is true:*

- (a)  $S$  is unbounded;
- (b)  $K$  has a fixed point.

By  $\rho_M : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with  $M > 0$  with denote the map

$$\rho_M(u) = \begin{cases} u & \text{if } |u| \leq M, \\ \frac{Mu}{|u|} & \text{if } M < |u|, \end{cases}$$

for all  $u \in \mathbb{R}^N$ , where we denote by  $|u|$  the Euclidean norm of  $u$  for every  $u \in \mathbb{R}^N$ . It is easy to see that the map  $\rho_m$  is nonexpansive.

For notational simplicity, we will write  $W^{1,p}$  with  $1 < p < \infty$  for the space  $W^{1,p}((0, b), \mathbb{R}^N)$  and by  $\|\cdot\|$  we will denote the norm of  $W^{1,p}$  defined by

$$\|u\| = (\|u\|_p^p + \|u'\|_p^p)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}.$$

Given a function  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  we denote by  $N_f$  the Nemytskij operator corresponding to  $f$  defined by

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}.$$

Now we introduce the hypotheses on the data of (1.1).

H(a):  $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a strictly monotone, continuous map such that  $a(0) = 0$ ,

$$a(y) = c(|y|)y \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}$$

with a continuous function  $c : (0, +\infty) \rightarrow (0, +\infty)$  and there exist  $c_0 > 0$  and  $1 < p < \infty$  such that

$$c_0|y|^p \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N.$$

**Remark 2.4** Evidently,  $a$  is maximal monotone. Furthermore,  $a$  is a homeomorphism onto  $\mathbb{R}^N$  and  $|a^{-1}(y)| \rightarrow +\infty$  as  $|y| \rightarrow +\infty$ . We stress that no growth condition is imposed on  $a$ .

**Example 2.5** The following maps satisfy hypotheses H(a):

- $a(y) = |y|^{p-2}y$  with  $1 < p < \infty$ ,
- $a(y) = |y|^{p-2}y + |y|^{q-2}y$  with  $1 < q < p < \infty$ ,
- $a(y) = \left[1 + |y|^2\right]^{\frac{p-2}{2}} y$  with  $1 < p < \infty$ ,
- $a(y) = \left[ce^{|y|^p} - 1\right] |y|^{p-2}y$  with  $1 < p < \infty$  and  $c > 1$ ,

for all  $y \in \mathbb{R}^N$ . The first map corresponds to the vector  $p$ -Laplacian and the second one to the vector  $(p, q)$ -Laplacian.

The assumptions on  $G$  read as follows:

H(G):  $G \in C^2(\mathbb{R}^N, \mathbb{R})$  and  $\nabla G(x) = g_0(|x|)x$  for all  $x \in \mathbb{R}^N$  with  $g_0(r) > 0$  for all  $r > 0$ .

**Remark 2.6** As mentioned before, we do not assume any global growth condition on the function  $G$ .

**Example 2.7** The following maps fulfill H(G):

- $G(x) = \frac{1}{r}|x|^r$  with  $2 \leq r < \infty$ ,
- $G(x) = \frac{1}{r}|x|^r + \frac{1}{q}|x|^q$  with  $2 \leq q < r < \infty$ ,
- $G(x) = \frac{1}{2} \left[ e^{|x|^2} - 1 \right]$ ,

for all  $x \in \mathbb{R}^N$ .

Finally, we can state our assumptions on  $A : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$  and  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ .

H(A):  $A : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$  is a maximal monotone map with  $0 \in A(0)$ ;

H(f):  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory function such that

- (i) for every  $\eta > 0$  there exists  $a_\eta \in L^2(T)_+$  such that

$$|f(t, x)| \leq a_\eta(t) \quad \text{for a.a. } t \in T \text{ and for all } |x| \leq \eta;$$

- (ii) there exists  $M > 0$  such that

$$(f(t, x), x)_{\mathbb{R}^N} \geq 0$$

for a.a.  $t \in T$  and for all  $x \in \mathbb{R}^N$  with  $|x| = M$ .

### 3 Existence of solutions

For  $h \in L^1(T, \mathbb{R}^N)$  we consider the following system

$$\begin{aligned} -a(u'(t))' + |u(t)|^{p-2}u'(t) &= h(t) & \text{for a.a. } t \in T, \\ u(0) = u(b), \quad u'(0) &= u'(b). \end{aligned} \tag{3.1}$$

**Proposition 3.1** *If hypotheses  $H(a)$  hold, then problem (3.1) has a unique solution  $K(h) \in C^1(T, \mathbb{R}^N)$  for every  $h \in L^1(T, \mathbb{R}^N)$ .*

**Proof** Note that

$$\int_0^b \left[ h(t) - |u(t)|^{p-2}u(t) \right] dt = 0.$$

The existence of a solution  $K(h) \in C^1(T, \mathbb{R}^N)$  follows from Theorem 5.3 of Manásevich–Mawhin [7]. The uniqueness of this solution is a consequence of the strict monotonicity of the maps

$$\mathbb{R}^N \ni y \rightarrow a(y) \quad \text{and} \quad \mathbb{R}^N \ni x \rightarrow |x|^{p-2}x.$$

□

**Remark 3.2** The above proposition is stated in a little more general form than we will need it here. Indeed, it is enough to consider  $h \in L^2(T, \mathbb{R}^N)$ , see hypothesis H(f)(i). However, when  $D(A) = \mathbb{R}^N$ , then we can have  $a_\eta \in L^1(T)_+$  in hypothesis H(f)(i) and so we use Proposition 3.1. For the Dirichlet problem, on account of the Poincaré inequality, we consider instead of (3.1) the following problem

$$\begin{aligned} -a(u'(t))' &= h(t) && \text{for a.a. } t \in T, \\ u(0) &= u(b) = 0. \end{aligned}$$

Then, the existence and uniqueness of a solution  $K(h) \in C^1(T, \mathbb{R}^N)$  follows from Theorem 5.1 of Manásevich–Mawhin [7].

Now we can define the solution map  $K : L^1(T, \mathbb{R}^N) \rightarrow C^1(T, \mathbb{R}^N)$  and obtain the following property of this map.

**Proposition 3.3** *If hypotheses  $H(a)$  hold, then  $K$  is completely continuous.*

**Proof** Let  $h_n \xrightarrow{w} h$  in  $L^1(T, \mathbb{R}^N)$  and set  $u_n = K(h_n)$  for all  $n \in \mathbb{N}$ . We have for  $n \in \mathbb{N}$

$$\begin{aligned} -a(u'_n(t))' + |u_n(t)|^{p-2}u_n(t) &= h_n(t) && \text{for a.a. } t \in T, \\ u_n(0) &= u_n(b), \quad u'_n(0) = u'_n(b). \end{aligned} \tag{3.2}$$

We take the inner product with  $u_n(t)$ , integrate over  $T = [0, b]$  and perform integration by parts. This leads to

$$\int_0^b (a(u'_n), u'_n)_{\mathbb{R}^N} dt + \|u_n\|_p^p \leq c_1 \|u_n\| \quad \text{for some } c_1 > 0 \text{ and for all } n \in \mathbb{N}.$$

Taking hypotheses H(a) into account gives

$$c_0 \|u'_n\|_p^p + \|u_n\|_p^p \leq c_1 \|u_n\| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the sequence  $\{u_n\}_{n \geq 1} \subseteq W^{1,p}$  is bounded and since  $W^{1,p} \hookrightarrow C(T, \mathbb{R}^N)$  is compactly embedded, we conclude that

$$\{u_n\}_{n \geq 1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.3}$$

From (3.2) we have

$$a(u'_n(t)) = a(u'_n(0)) + \int_0^t [h_n(s) - |u_n(s)|^{p-2}u_n(s)] ds \tag{3.4}$$

for all  $t \in T$  and for all  $n \in \mathbb{N}$ . This gives

$$u'_n(t) = a^{-1} \left[ a(u'_n(0)) + \int_0^t [h_n(s) - |u_n(s)|^{p-2}u_n(s)] ds \right]$$

for all  $t \in T$  and for all  $n \in \mathbb{N}$ . If

$$k_n(t) = \int_0^t [h_n(s) - |u_n(s)|^{p-2}u_n(s)] ds$$

for  $n \in \mathbb{N}$ , then  $\{k_n\}_{n \in \mathbb{N}} \subseteq C(T, \mathbb{R}^N)$  is bounded. Moreover, note that  $\int_0^t u'_n(t) dt = 0$  for  $n \in \mathbb{N}$ . Therefore, Lemma 3.1 of Manásevich–Mawhin [7] implies that

$$\{a(u_n(0))\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^N \text{ is bounded.}$$

Then, from (3.4) and the Arzela–Ascoli theorem, we infer that

$$\{a(u'_n(\cdot))\}_{n \in \mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.5}$$

Let  $\hat{a}^{-1} : C(T, \mathbb{R}^N) \rightarrow C(T, \mathbb{R}^N)$  be defined by

$$\hat{a}^{-1}(u)(\cdot) = a^{-1}(u(\cdot)) \text{ for all } u \in C(T, \mathbb{R}^N).$$

Evidently,  $\hat{a}^{-1}$  is continuous and bounded, that is, it maps bounded sets to bounded sets. Hence, from (3.5) we have

$$\{u'_n\}_{n \in \mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.6}$$

From (3.3) and (3.6) it follows that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq C^1(T, \mathbb{R}^N) \text{ is relatively compact.}$$

We may assume, at least for a subsequence, that

$$u_n \rightarrow u \text{ in } C^1(T, \mathbb{R}^N). \tag{3.7}$$

We have

$$\int_0^b (a(u'_n), v')_{\mathbb{R}^N} dt + \int_0^b |u_n|^{p-2} (u_n, v)_{\mathbb{R}^N} dt = \int_0^b (h_n, v)_{\mathbb{R}^N} dt \tag{3.8}$$

for all  $v \in W^{1,p}$  and for all  $n \in \mathbb{N}$ . From (3.7) and the continuity of  $a$ , we obtain

$$|a(u'_n(t))| \leq c_2$$

for some  $c_2 > 0$ , for all  $t \in T$  and for all  $n \in \mathbb{N}$ . So, if we pass to the limit in (3.8) as  $n \rightarrow \infty$ , then one has

$$\int_0^b (a(u'), v')_{\mathbb{R}^N} dt + \int_0^b |u|^{p-2} (u, v)_{\mathbb{R}^N} dt = \int_0^b (h, v)_{\mathbb{R}^N} dt$$

for all  $v \in W^{1,p}$ . Hence,  $u = K(h)$ .

Therefore, we obtain for the original sequence that

$$u_n = K(h_n) \rightarrow K(h) = u \text{ in } C^1(T, \mathbb{R}^N),$$

which shows that  $K : L^1(T, \mathbb{R}^N) \rightarrow C^1(T, \mathbb{R}^N)$  is completely continuous. □

**Remark 3.4** In particular, we obtain that  $K : L^2(T, \mathbb{R}^N) \rightarrow C^1(T, \mathbb{R}^N)$  is completely continuous and then, due to the reflexivity of  $L^2(T, \mathbb{R}^N)$ , we have that  $K$  is compact, see Papageorgiou–Winkert [9, Proposition 3.7.7].

For every  $\lambda > 0$ , let  $\hat{A}_\lambda : W^{1,p} \rightarrow L^2(T, \mathbb{R}^N)$  be defined by  $\hat{A}_\lambda(u)(\cdot) = A_\lambda(u(\cdot))$ . In fact,  $\hat{A}_\lambda$  is  $L^\infty(T, \mathbb{R}^N)$ -valued. Then, let  $N_\lambda : W^{1,p} \rightarrow L^2(T, \mathbb{R}^N)$  be defined by

$$N_\lambda(u) = -\hat{A}_\lambda(u) - N_f(\rho_M(u)) + |\rho_M(u)|^{p-2} \rho_M(u) + \nabla G(\rho_M(u)).$$

The following proposition is an immediate consequence of the properties of  $A_\lambda$ , see Proposition 2.1, and of the hypotheses H(G) and H(f).

**Proposition 3.5** *If hypotheses H(A), H(G) and H(f) hold, then  $N_\lambda : W^{1,p} \rightarrow L^2(T, \mathbb{R}^N)$  is continuous.*

From Propositions 3.3 and 3.5 we easily conclude that the map  $K \circ N_\lambda : W^{1,p} \rightarrow W^{1,p}$  is compact. We define

$$S_\lambda = \left\{ u \in W^{1,p} : u = \mu(K \circ N_\lambda)(u), 0 < \mu < 1 \right\}.$$

**Proposition 3.6** *If hypotheses H(a), H(A), H(G), H(f) hold and  $\lambda > 0$ , then  $S_\lambda \subseteq W^{1,p}$  is bounded.*



**Proof** Let  $u \in S_\lambda$ . Then  $\frac{1}{\mu}u = K(N_\lambda(u))$  and so

$$\begin{aligned}
 & -a \left( \frac{1}{\mu}u' \right)' + \frac{1}{\mu^{p-1}}|u|^{p-2}u \\
 & = -\hat{A}_\lambda(u) - N_f(\rho_M(u)) + |\rho_M(u)|^{p-2}\rho_M(u) + \frac{d}{dt}\nabla G(\rho_M(u)) \tag{3.9}
 \end{aligned}$$

with  $u(0) = u(b)$  and  $u'(0) = u'(b)$ .

**Claim:**  $|u(t)| \leq M$  for all  $t \in T$

Let  $r(t) = \frac{1}{2}|u(t)|^2$  for all  $t \in T$ . Then we can find  $t_0 \in T$  such that  $r(t_0) = \max_T r$ . Arguing by contradiction, suppose that

$$r(t_0) > \frac{1}{2}M^2.$$

First we assume that  $t_0 \in (0, b)$ . Then

$$r'(t_0) = (u'(t_0), u(t_0))_{\mathbb{R}^N} = 0. \tag{3.10}$$

Let  $t_1 \in [0, t_0)$  be such that  $|u(t_1)| = M$  and  $|u(t)| > M$  for all  $(t_1, t_0]$ . Then

$$\begin{aligned}
 & -a \left( \frac{1}{\mu}u'(t) \right)' + \frac{1}{\mu^{p-1}}|u(t)|^{p-2}u(t) \\
 & = -\hat{A}_\lambda(u(t)) - f(t, \rho_M(u(t))) + |\rho_M(u(t))|^{p-2}\rho_M(u(t)) + \frac{d}{dt}\nabla G(\rho_M(u(t)))
 \end{aligned}$$

for a.a.  $t \in T$ . This implies

$$\begin{aligned}
 & -\frac{d}{dt} \left( a \left( \frac{1}{\mu}u'(t) \right), u(t) \right)_{\mathbb{R}^N} + \left( a \left( \frac{1}{\mu}u'(t) \right), u'(t) \right)_{\mathbb{R}^N} + \frac{1}{\mu^{p-1}}|u(t)|^p \\
 & = - (A_\lambda(u(t)), u(t))_{\mathbb{R}^N} - \frac{|u(t)|}{M} (f(t, \rho_M(u(t))), \rho_M(u(t)))_{\mathbb{R}^N} \\
 & \quad + |u(t)|M^{p-1} + \left( \frac{d}{dt}\nabla G(\rho_M(u(t))), u(t) \right)_{\mathbb{R}^N} \tag{3.11}
 \end{aligned}$$

for a.a.  $t \in [t_1, t_0]$ . Since  $A_\lambda$  is maximal monotone, see Proposition 2.1, and  $A_\lambda(0) = 0$ , see hypotheses H(a), we have

$$- (A_\lambda(u(t)), u(t))_{\mathbb{R}^N} \leq 0 \quad \text{for all } t \in T. \tag{3.12}$$

Furthermore, taking hypothesis H(f)(ii) into account, we obtain

$$-\frac{|u(t)|}{M} (f(t, \rho_M(u(t))), \rho_M(u(t)))_{\mathbb{R}^N} \leq 0 \quad \text{for all } t \in [t_1, t_0]. \tag{3.13}$$

Finally, applying hypotheses H(G), we have

$$\begin{aligned}
 & \left( \frac{d}{dt} \nabla G(\rho_M(u(t))), u(t) \right)_{\mathbb{R}^N} \\
 &= \frac{|u(t)|}{M} \left( \frac{d}{dt} \nabla G(\rho_M(u(t))), \rho_M(u(t)) \right)_{\mathbb{R}^N} \\
 &= \frac{|u(t)|}{M} \left[ \frac{d}{dt} (\nabla G(\rho_M(u(t))), \rho_M(u(t)))_{\mathbb{R}^N} \right. \\
 &\quad \left. - \left( \nabla G(\rho_M(u(t))), \frac{d}{dt} \rho_M(u(t)) \right)_{\mathbb{R}^N} \right] \\
 &= \frac{|u(t)|}{dt} \left[ \frac{d}{dt} (g_0(M)M^2) - g_0(M) \frac{d}{dt} |\rho_M(u(t))|^2 \right] = 0
 \end{aligned} \tag{3.14}$$

for all  $t \in [t_1, t_0]$ . We return to (3.11) and apply (3.12), (3.13), (3.14) and hypotheses H(a). This gives

$$|u(t)| \left[ \frac{1}{\mu^{p-1}} |u(t)|^{p-1} - M^{p-1} \right] \leq \frac{d}{dt} \left( a \left( \frac{1}{\mu} u'(t) \right), u(t) \right)_{\mathbb{R}^N}$$

for a.a.  $t \in (t_1, t_0]$  and so, since  $0 < \mu < 1$ ,

$$0 < \frac{d}{dt} \left( a \left( \frac{1}{\mu} u'(t) \right), u(t) \right)_{\mathbb{R}^N} \quad \text{for a.a. } t \in (t_1, t_0].$$

Therefore, the function

$$t \rightarrow \left( a \left( \frac{1}{\mu} u'(t) \right), u(t) \right)_{\mathbb{R}^N}$$

is strictly increasing on  $(t_1, t_0]$ . Hence, we have

$$\left( a \left( \frac{1}{\mu} u'(t) \right), u(t) \right)_{\mathbb{R}^N} < \left( a \left( \frac{1}{\mu} u'(t_0) \right), u(t_0) \right)_{\mathbb{R}^N} \quad \text{for all } t \in (t_1, t_0).$$

Based on hypotheses H(a) and (3.10) we obtain

$$c \left( \frac{1}{\mu} |u'(t)| \right) (u'(t), u(t))_{\mathbb{R}^N} < c \left( \frac{1}{\mu} |u'(t_0)| \right) (u'(t_0), u(t_0))_{\mathbb{R}^N} = 0.$$

Thus,  $r'(t) < 0$  for all  $t \in (t_1, t_0)$ .

Finally we have

$$M^2 < r(t_0) < r(t_1) = M^2,$$

a contradiction.

If  $t_0 = 0$  or  $t_0 = b$ , then  $r(0) = r(b)$  and  $r'(0) \leq 0 \leq r'(b)$ . But

$$r'(t) = (u'(t), u(t))_{\mathbb{R}^N} \quad \text{for all } t \in T,$$

which implies  $r'(0) = r'(b) = 0$  and so the previous argument applies. This proves the Claim.

Next we act on (3.9) with  $u$ , perform integration by parts and use hypotheses H(a), H(G), H(f)(i) and the Claim. This gives

$$\frac{1}{\mu^{p-1}} \left[ c_0 \|u'\|_p^p + \|u\|_p^p \right] \leq c_3 \quad \text{for some } c_3 > 0 \text{ and for all } u \in S.$$

Recall that  $0 < \mu < 1$ , we see that  $S \subseteq W^{1,p}$  is bounded. □

For  $\lambda > 0$  we consider the following approximation to problem (1.1)

$$\begin{aligned} a(u'(t))' + \frac{d}{dt} \nabla G(u(t)) &= A_\lambda(u(t)) + f(t, u(t)) \quad \text{for a.a. } t \in T = [0, b], \\ u(0) &= u(b), \quad u'(0) = u'(b). \end{aligned} \tag{3.15}$$

**Proposition 3.7** *If hypotheses H(a), H(G), H(A), H(f) hold and let  $\lambda > 0$ , then problem (3.15) has a unique solution  $\hat{u}_\lambda \in C^1(T, \mathbb{R}^N)$ .*

**Proof** The compactness of  $K \circ N_\lambda : W^{1,p} \rightarrow W^{1,p}$  and Proposition 3.6 permit the use of the Leray–Schauder Alternative Principle stated as Theorem 2.3. So, there exists  $\hat{u}_\lambda \in W^{1,p}$  such that

$$\hat{u} = (K \circ N_\lambda)(\hat{u}_\lambda).$$

This gives

$$\hat{u}_\lambda \in C^1(T, \mathbb{R}^N) \quad \text{and} \quad |\hat{u}_\lambda(t)| \leq M \quad \text{for all } t \in T,$$

see the proof of Proposition 3.6. Then  $\rho_M(\hat{u}_\lambda(t)) = \hat{u}_\lambda(t)$  and so we conclude that  $\hat{u}_\lambda \in C^1(T, \mathbb{R}^N)$  is a solution of (3.15), see (3.9) with  $\mu = 1$ . □

Let  $\alpha : L^2(T, \mathbb{R}^N) \rightarrow 2^{L^2(T, \mathbb{R}^N)}$  be defined by

$$\alpha(u) = \left\{ \vartheta \in L^2(T, \mathbb{R}^N) : \vartheta(t) \in A(u(t)) \text{ for a.a. } t \in T \right\}.$$

Since  $0 \in A(0)$  we see that  $D(\alpha) \neq \emptyset$ . From Brézis [2, p. 21] we have the following result.

**Proposition 3.8** *If hypotheses H(A) hold, then  $\alpha$  is maximal monotone.*

Now we are ready to produce a solution for problem (1.1).

**Theorem 3.9** *If hypotheses  $H(a)$ ,  $H(G)$ ,  $H(A)$ ,  $H(f)$  hold, then problem (1.1) has a solution  $\hat{u} \in C^1(T, \mathbb{R}^N)$ .*

**Proof** Let  $\lambda_n \rightarrow 0^+$  and let  $\hat{u}_n = \hat{u}_{\lambda_n} \in C^1(T, \mathbb{R}^N)$  for  $n \in \mathbb{N}$  be a solution of (3.15) based on Proposition 3.7. From the proof of Proposition 3.6, see the Claim in that proof, we have

$$|\hat{u}_n(t)| \leq M \quad \text{for all } t \in T \text{ and for all } n \in \mathbb{N}. \tag{3.16}$$

From (3.15) it follows that

$$\int_0^b (a(\hat{u}'_n), \hat{u}'_n)_{\mathbb{R}^N} dt \leq \int_0^b |f(t, \hat{u}_n)| M dt + \int_0^b \left| \frac{d}{dt} \nabla G(\hat{u}_n) \right| M dt,$$

where we recall that  $A_{\lambda_n}$  is monotone,  $A_{\lambda_n}(0) = 0$  and see (3.16). Applying hypotheses  $H(a)$ ,  $H(G)$  and  $H(f)(i)$  leads to

$$c_0 \|\hat{u}'_n\|_p^p \leq c_3 \quad \text{for some } c_3 > 0 \text{ and for all } n \in \mathbb{N}.$$

Therefore, the sequence  $\{\hat{u}'_n\}_{n \in \mathbb{N}} \subseteq L^p(T, \mathbb{R}^N)$  is bounded and so it is  $\{\hat{u}_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}$ , see (3.16). So, by passing to a subsequence if necessary, we can say that

$$\hat{u}_n \xrightarrow{w} \hat{u} \text{ in } W^{1,p} \quad \text{and} \quad \hat{u}_n \rightarrow \hat{u} \text{ in } C(T, \mathbb{R}^N).$$

Now we take the inner product with  $A_{\lambda_n}(\hat{u}_n(t))$  in (3.15) and integrate over  $T = [0, b]$ . After integration by parts and by applying hypotheses  $H(G)$ ,  $H(f)(i)$  and (3.16), we obtain

$$\int_0^b \left( a(\hat{u}'_n), \frac{d}{dt} A_{\lambda_n}(u_n) \right)_{\mathbb{R}^N} dt + \|A_{\lambda_n}(u_n)\|_2^2 \leq c_4 \|A_{\lambda_n}(u_n)\|_2 \tag{3.17}$$

for some  $c_4 > 0$  and for all  $n \in \mathbb{N}$ .

The map  $x \rightarrow A_{\lambda_n}(x)$  for  $n \in \mathbb{N}$  is Lipschitz continuous from  $\mathbb{R}^N$  into  $\mathbb{R}^N$ . So, by the Rademacher theorem, see Evans–Gariepy [3, p. 81], we know that  $A_{\lambda_n}$  is differentiable at all  $x \in \mathbb{R}^N \setminus D_n$  with  $|D_n|_N = 0$ , where  $|\cdot|_N$  denotes the Lebesgue measure on  $\mathbb{R}^N$ . Then, since  $A_{\lambda_n}$  is monotone, we have for all  $x \in \mathbb{R}^N \setminus D_n$  and for every  $h \in \mathbb{R}^N$ ,

$$\left( \frac{A_{\lambda_n}(x + \tau h) - A_{\lambda_n}(x)}{\tau}, h \right)_{\mathbb{R}^N} \geq 0.$$

This implies

$$(A'_{\lambda_n}(x)h, h)_{\mathbb{R}^N} \geq 0. \tag{3.18}$$

Then, from the chain rule for Sobolev functions, see Papageorgiou–Winkert [9, Theorem 4.5.18], we have

$$\frac{d}{dt} A_{\lambda_n}(\hat{u}_n(t)) = A'_{\lambda_n}(\hat{u}_n(t)) \hat{u}'_n(t) \quad \text{for a.a. } t \in T. \tag{3.19}$$

Applying (3.19), hypotheses H(a) and (3.18) gives

$$\begin{aligned} & \int_0^b \left( a(\hat{u}'_n(t)), \frac{d}{dt} A_{\lambda_n}(\hat{u}_n) \right)_{\mathbb{R}^N} dt \\ &= \int_0^b (a(\hat{u}'_n(t)), A'_{\lambda_n}(\hat{u}_n) \hat{u}'_n)_{\mathbb{R}^N} dt \\ &= \int_0^b c(|\hat{u}'_n|) (\hat{u}'_n, \hat{A}_{\lambda_n}(u_n) \hat{u}_n)_{\mathbb{R}^N} dt \geq 0. \end{aligned}$$

Returning to (3.17) and using (3.19) we obtain

$$\|A_{\lambda_n}(u_n)\|_2^2 \leq c_4 \|A_{\lambda_n}(u_n)\|_2 \quad \text{for all } n \in \mathbb{N},$$

which shows that

$$\{\hat{A}_{\lambda_n}(u_n)\}_{n \in \mathbb{N}} = \{A_{\lambda_n}(u_n(\cdot))\}_{n \in \mathbb{N}} \subseteq L^2(T, \mathbb{R}^N) \text{ is bounded.}$$

So, we may assume that

$$\hat{A}_{\lambda_n}(\hat{u}_n) \xrightarrow{w} y \quad \text{in } L^2(T, \mathbb{R}^N). \tag{3.20}$$

From (3.15) we have

$$u'_n(t) = a^{-1} \left( a(u'_n(0)) + \int_0^t \left[ A_{\lambda_n}(\hat{u}_n(s)) + f(s, \hat{u}_n(s)) - \frac{d}{dt} \nabla G(\hat{u}_n(s)) \right] ds \right) \tag{3.21}$$

for all  $n \in \mathbb{N}$ . We set

$$g_n(t) = \int_0^t \left[ A_{\lambda_n}(\hat{u}_n(s)) + f(s, \hat{u}_n(s)) - \frac{d}{dt} \nabla G(\hat{u}_n(s)) \right] ds$$

for all  $t \in T$  and for all  $n \in \mathbb{N}$ . The Arzela–Ascoli theorem implies that

$$\{g_n\}_{n \in \mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.}$$

Therefore, invoking Lemma 3.1 of Manásevich–Mawhin [7], we infer that

$$\{a(u'_n(0))\}_{n \geq 1} \subseteq \mathbb{R}^N \text{ is relatively compact.}$$

Recall that the map  $\hat{a}^{-1} : C(T, \mathbb{R}^N) \rightarrow C(T, \mathbb{R}^N)$  defined by  $\hat{a}^{-1}(u)(\cdot) = a^{-1}(u(\cdot))$  is continuous. Thus, from (3.21) it follows that

$$\{\hat{u}'_n\}_{n \in \mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact}$$

and because of the compact embedding  $W^{1,p} \hookrightarrow C(T, \mathbb{R}^N)$ ,

$$\{\hat{u}_n\}_{n \in \mathbb{N}} \subseteq C^1(T, \mathbb{R}^N) \text{ is relatively compact.}$$

So, we have

$$\hat{u}_n \rightarrow \hat{u} \text{ in } C^1(T, \mathbb{R}^N). \tag{3.22}$$

In the limit as  $n \rightarrow \infty$ , we obtain

$$\begin{aligned} & - \int_0^b (a(\hat{u}'), v')_{\mathbb{R}^N} dt + \int_0^b \left( \frac{d}{dt} \nabla G(\hat{u}), v \right)_{\mathbb{R}^N} dt \\ & = \int_0^b (y, v)_{\mathbb{R}^N} dt + \int_0^b (f(t, \hat{u}), v)_{\mathbb{R}^N} dt \text{ for all } v \in W^{1,p}, \end{aligned}$$

see (3.20) and (3.22). Therefore,

$$\begin{aligned} a(\hat{u}'(t))' + \frac{d}{dt} \nabla G(\hat{u}(t)) &= y(t) + f(t, \hat{u}(t)) \text{ for a.a. } t \in T, \\ \hat{u}(0) = \hat{u}(b), \hat{u}'(0) &= \hat{u}'(b). \end{aligned}$$

We will be done if we can show that  $y(t) \in A(\hat{u}(t))$  for a.a.  $t \in T$ .

Let  $\hat{J}_{\lambda_n}(\hat{u}_n)(\cdot) = J_{\lambda_n}(\hat{u}_n(\cdot))$  for all  $n \in \mathbb{N}$ . From Proposition 2.1 and the chain rule for Sobolev functions we have that  $\hat{J}_{\lambda_n}(\hat{u}_n) \in W^{1,2}$  for all  $n \in \mathbb{N}$  and

$$\{\hat{J}_{\lambda_n}(\hat{u}_n)\}_{n \in \mathbb{N}} \subseteq W^{1,2} \text{ is bounded.}$$

So, we may assume that  $\hat{J}_{\lambda_n}(\hat{u}_n) \xrightarrow{w} w$  in  $W^{1,2}$  and because of the compact embedding  $W^{1,2} \hookrightarrow C(T, \mathbb{R}^N)$ ,

$$\hat{J}_{\lambda_n}(\hat{u}_n) \rightarrow w \text{ in } C(T, \mathbb{R}^N). \tag{3.23}$$

We know that

$$\hat{J}_{\lambda_n}(\hat{u}_n) + \hat{\lambda}_n \hat{A}_{\lambda_n}(\hat{u}_n) = \hat{u}_n \text{ for all } n \in \mathbb{N},$$

which implies  $w = \hat{u}$ , see (3.23) and (3.22). Also, from (3.23) we see that

$$\hat{J}_{\lambda_n}(\hat{u}_n) \rightarrow \hat{u} \text{ in } C(T, \mathbb{R}^N). \tag{3.24}$$

Moreover, we have

$$\hat{A}_{\lambda_n}(u_n) \in \mathfrak{a} \left( \hat{J}_{\lambda_n}(u_n) \right) \quad \text{for all } n \in \mathbb{N}, \tag{3.25}$$

see Proposition 2.1. From Proposition 3.7 we know that  $\mathfrak{a}$  is maximal monotone. So, the graph of  $\mathfrak{a}$  is sequentially closed in  $L^2(T, \mathbb{R}^N) \times L^2(T, \mathbb{R}^N)_w$ . From (3.20), (3.24) and (3.25) we have  $y \in \mathfrak{a}(\hat{u})$ . This means that

$$y(t) \in A(\hat{u}(t)) \quad \text{for a.a. } t \in T.$$

Therefore,  $\hat{u} \in C^1(T, \mathbb{R}^N)$  is a solution of problem (1.1). □

When  $D(A) = \mathbb{R}^N$  we can avoid the approximation by problem (3.15) and can also relax a little hypothesis H(f)(i).

Now, the hypotheses on the map  $A$  are the following.

H(A)’:  $A : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$  is a maximal monotone map such that  $D(A) = \mathbb{R}^N$  and  $0 \in A(0)$ .

**Remark 3.10** In this case we know that  $A$  has nonempty, compact and convex values and as a multifunction it is upper semicontinuous from  $\mathbb{R}^N$  into  $\mathbb{R}^N$ , see Papageorgiou–Winkert [9, Proposition 6.1.13].

The more general conditions on the perturbation  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  read as follows.

H(f)’:  $f : T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory function such that

(i) for every  $\eta > 0$  there exists  $a_\eta \in L^1(T)_+$  such that

$$|f(t, x)| \leq a_\eta(t) \quad \text{for a.a. } t \in T \text{ and for all } |x| \leq \eta;$$

(ii) same as hypothesis H(f)(ii).

The method of the proof remains the same. Only since we work directly on the inclusion problem (1.1) and do not pass first from its single-valued approximation (3.15), we do not use Theorem 2.3, but its multivalued counterpart due to Bader [1]. Then we can have the following existence theorem.

**Theorem 3.11** *If hypotheses H(a), H(G), H(A)’ and H(f)’ hold, then problem (1.1) admits a solution  $\hat{u} \in C^1(T, \mathbb{R}^N)$ .*

## References

1. Bader, R.: A topological fixed-point index theory for evolution inclusions. *Z. Anal. Anwend.* **20**(1), 3–15 (2001)
2. Brézis, H.: *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. North-Holland Publishing Co., Amsterdam (1973)

3. Evans, L.C., Gariépy, R.F.: Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton (1992)
4. Hartman, P.: On boundary value problems for systems of ordinary, nonlinear, second order differential equations. *Trans. Am. Math. Soc.* **96**, 493–509 (1960)
5. Hartman, P.: Ordinary Differential Equations. Wiley, New York (1964)
6. Knobloch, H.-W.: On the existence of periodic solutions for second order vector differential equations. *J. Differ. Equ.* **9**, 67–85 (1971)
7. Manásevich, R., Mawhin, J.: Boundary value problems for nonlinear perturbations of vector  $p$ -Laplacian-like operators. *J. Korean Math. Soc.* **37**(5), 665–685 (2000)
8. Mawhin, J.: Some boundary value problems for Hartman-type perturbations of the ordinary vector  $p$ -Laplacian. *Nonlinear Anal.* **40**(1–8), 497–503 (2000)
9. Papageorgiou, N.S., Winkert, P.: Applied Nonlinear Functional Analysis. An Introduction. De Gruyter, Berlin (2018)

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