

SINGULAR FINSLER DOUBLE PHASE PROBLEMS WITH NONLINEAR BOUNDARY CONDITION

CSABA FARKAS, ALESSIO FISCELLA, AND PATRICK WINKERT

ABSTRACT. In this paper we study a singular Finsler double phase problem with a nonlinear boundary condition and perturbations that have a type of critical growth, even on the boundary. Based on variational methods in combination with truncation techniques we prove the existence of at least one weak solution for this problem under very general assumptions. Even in the case when the Finsler manifold reduces to the Euclidean norm, our work is the first one dealing with a singular double phase problem and nonlinear boundary condition.

1. INTRODUCTION

In this paper we consider singular Finsler double phase problems with nonlinear boundary condition. The Finsler double phase operator is defined by

$$\operatorname{div}(A(u)) := \operatorname{div} (F^{p-1}(\nabla u)\nabla F(\nabla u) + \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) \quad (1.1)$$

for $u \in W^{1,\mathcal{H}}(\Omega)$ with $W^{1,\mathcal{H}}(\Omega)$ being the Musielak-Orlicz Sobolev space and F is a positive homogeneous function such that $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix $\nabla^2(F^2/2)(x)$ is positive definite for all $x \neq 0$. Furthermore, μ is a nonnegative bounded function and $1 < p < q < N$. If F coincides with the Euclidean norm, that is, $F(\xi) = \left(\sum_{i=1}^N |\xi_i|^2\right)^{1/2}$ for $\xi \in \mathbb{R}^N$, then (1.1) reduces to the usual double phase operator given by

$$\operatorname{div} (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u). \quad (1.2)$$

Also, if $\mu \equiv 0$ or $\inf_{\overline{\Omega}} \mu > 0$, then (1.2) (similarly (1.1)) reduces to the (Finsler) p -Laplacian or the (Finsler) (q, p) -Laplacian, respectively. The study of such operators and corresponding energy functionals are motivated by physical phenomena, see, for example, the work of Zhikov [58] (see also the monograph of Zhikov-Kozlov-Oleĭnik [59]) in order to describe models for strongly anisotropic materials. Related functionals to (1.2) have been studied intensively with respect to regularity properties of local minimizers, see the works of Baroni-Colombo-Mingione [4, 5, 6], Baroni-Kuusi-Mingione [7], Byun-Oh [9], Colombo-Mingione [14, 15], De Filippis-Palatucci [18], Marcellini [40, 41], Ok [42, 43], Ragusa-Tachikawa [51] and the references therein.

2020 *Mathematics Subject Classification.* 35J15, 35J62, 58B20, 58J60.

Key words and phrases. Anisotropic double phase operator, critical type exponent, existence results, Minkowski space, nonlinear boundary condition, singular problems.

On the other hand, the minimization of the functional $E_F: H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$E_F(u) = \int_{\Omega} F^2(\nabla u) \, dx \quad \text{for } u \in H_0^1(\Omega),$$

under certain constraints on perimeter or volume occurs in many subjects of mathematical physics. Here the minimizer corresponds to an optimal shape (or configuration) of anisotropic tension-surface. The minimization of the functional E_F describes, for example, the specific polyhedral shape of crystal structures in solid crystals with sufficiently small grains, as shown by Dinghas [22] and Taylor [53]. It is clear that E_F is the energy functional to the Finsler Laplacian given by

$$\Delta_F u = \operatorname{div}(F(\nabla u) \nabla F(\nabla u)). \quad (1.3)$$

The Finsler Laplacian, given in (1.3), has been studied by several authors in the last decade. We refer, for example, to the papers of Cianchi-Salani [12] and Wang-Xia [54], both dealing with the Serrin-type overdetermined anisotropic problem, or to Farkas-Fodor-Kristály [27] who studied a sublinear Dirichlet problem of this type. Related works concerning anisotropic phenomena can be found in the works of Bellettini-Paolini [8], Belloni-Ferone-Kawohl [10], Della Pietra-Gavitone [21], Della Pietra-di Blasio-Gavitone [19], Della Pietra-Gavitone-Piscitelli [20], Farkas [26], Farkas-Kristály-Varga [28], Ferone-Kawohl [30] and the references therein.

In this paper we combine the effect of a Finsler manifold and a double phase operator along with a singular term and a nonlinear boundary condition. More precisely, we study the following problem

$$\begin{aligned} -\operatorname{div}(A(u)) + u^{p-1} + \mu(x)u^{q-1} &= u^{p^*-1} + \lambda(u^{\gamma-1} + g_1(x, u)) && \text{in } \Omega, \\ A(u) \cdot \nu &= u^{p_*-1} + g_2(x, u) && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega, \end{aligned} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\nu(x)$ is the outer unit normal of Ω at the point $x \in \partial\Omega$, λ is a positive parameter and the following assumptions hold true:

(H) (i) $0 < \gamma < 1$ and

$$1 < p < q < N, \quad q < p^*, \quad 0 \leq \mu(\cdot) \in L^\infty(\Omega); \quad (1.5)$$

(ii) $g_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $g_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions and there exist $1 < \theta_1 < p \leq \nu_1 < p^*$, $p < \nu_2 < p_*$ as well as nonnegative constants a_1, a_2, b_1 such that

$$\begin{aligned} g_1(x, s) &\leq a_1 s^{\nu_1-1} + b_1 s^{\theta_1-1} && \text{for a. a. } x \in \Omega \text{ and for all } s \geq 0, \\ g_2(x, s) &\leq a_2 s^{\nu_2-1} && \text{for a. a. } x \in \partial\Omega \text{ and for all } s \geq 0, \end{aligned}$$

where p^* and p_* are the critical exponents to p given by

$$p^* := \frac{Np}{N-p} \quad \text{and} \quad p_* := \frac{(N-1)p}{N-p}; \quad (1.6)$$

(iii) the function $F: \mathbb{R}^N \rightarrow [0, \infty)$ is a positively homogeneous Minkowski norm with finite reversibility $r_F = \max_{w \neq 0} \frac{F(-w)}{F(w)}$.

Since we are looking for positive solutions and hypothesis (H)(ii) concerns the positive semiaxis $\mathbb{R}_+ = [0, \infty)$, without any loss of generality, we may assume that $g_1(x, s) = g_2(x, s) = 0$ for all $s \leq 0$ and for a. a. $x \in \Omega$, respectively, $x \in \partial\Omega$. Moreover, note that we always have $r_F \geq 1$, see for example in Farkas-Kristály-Varga [28]. It is clear that the Euclidean norm has finite reversibility. Finally, we observe that (1.5) implies that $W^{1, \mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega)$ compactly, as shown in Section 2.

Definition 1.1. *A function $u \in W^{1, \mathcal{H}}(\Omega)$ is called a weak solution of problem (1.4) if $u^{\gamma-1}\varphi \in L^1(\Omega)$, $u > 0$ for a. a. $x \in \Omega$ and if*

$$\begin{aligned} & \int_{\Omega} (F^{p-1}(\nabla u) \nabla F(\nabla u) + \mu(x) F^{q-1}(\nabla u) \nabla F(\nabla u)) \cdot \nabla \varphi \, dx \\ & + \int_{\Omega} u^{p-1} \varphi \, dx + \int_{\Omega} \mu(x) u^{q-1} \varphi \, dx \\ & = \int_{\Omega} u^{p^*-1} \varphi \, dx + \lambda \int_{\Omega} (u^{\gamma-1} + g_1(x, u)) \varphi \, dx + \int_{\partial\Omega} (u^{p^*-1} + g_2(x, u)) \varphi \, d\sigma \end{aligned}$$

is satisfied for all $\varphi \in W^{1, \mathcal{H}}(\Omega)$.

From hypotheses (H) we know that the definition of a weak solution is well defined.

The main result in this paper is the following theorem.

Theorem 1.2. *Let hypotheses (H) be satisfied. Then there exists $\lambda_* > 0$ such that for every $\lambda \in (0, \lambda_*)$, problem (1.4) has a nontrivial weak solution.*

To the best of our knowledge, this is the first work on a singular double phase problem with nonlinear boundary condition even in the Euclidean case, that is when $F(\xi) = \left(\sum_{i=1}^N |\xi_i|^2 \right)^{1/2}$ for $\xi \in \mathbb{R}^N$. The novelty of our paper is not only due by the combination of the Finsler double phase operator with a singular term and nonlinear boundary condition. Indeed, in (1.4) we also deal with a type of critical Sobolev nonlinearities, even on the boundary, related to the lower exponent p , as explained in (1.6). Such critical terms make the study of compactness of the energy functional related to (1.4) more intriguing, since the embeddings $W^{1, \mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1, \mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$ are not compact. We overcome these difficulties with a local analysis on a suitable closed convex subset of $W^{1, \mathcal{H}}(\Omega)$ combined with a truncation argument.

We point out that p^* and p_* are not the critical exponents to the space $W^{1, \mathcal{H}}(\Omega)$. Indeed, from Fan [24] we know that $W^{1, \mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}_*}(\Omega)$ is continuous while \mathcal{H}_* is the Sobolev conjugate function of \mathcal{H} , see also Crespo-Blanco-Gasiński-Harjulehto-Winkert [16, Definition 2.18 and Proposition 2.18]. So far it is not known how \mathcal{H}_* explicitly looks like in the double phase setting. For the moment, p^* and p_* seem to be the best exponents (probably not optimal) and only continuous (in general noncompact) embeddings from $W^{1, \mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1, \mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$ are available. So we call it “types of critical growth”.

For singular double phase problems with Dirichlet boundary condition there exists only few works. Recently, Liu-Dai-Papageorgiou-Winkert [39] studied the singular problem

$$\begin{aligned} -\operatorname{div} (|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u) &= a(x) u^{-\gamma} + \lambda u^{r-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (1.7)$$

Based on the fibering method along with the Nehari manifold, the existence of at least two weak solutions with different energy sign is shown, see also [17] for the corresponding Neumann problem. Furthermore, under a different treatment, Chen-Ge-Wen-Cao [11] considered problems of type (1.7) and proved the existence of a weak solution having negative energy. Finally, the existence of at least one weak solution to the singular problem

$$\begin{aligned} -\operatorname{div}(A(u)) &= u^{p^*-1} + \lambda(u^{\gamma-1} + g(u)) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has been shown by the first and the third author in [29]. The current paper can be seen as a nontrivial extension of the one in [29] to the case of a nonlinear boundary condition including type of critical growth. In particular, we are able to cover the situation when $1 < p < 2$ and/or $1 < q < 2$, which has not been considered in [29] where $2 \leq p < q$.

Also for the p -Laplacian or the (q, p) -Laplacian only few works exist involving singular terms and Neumann/Robin boundary condition. We refer to Papageorgiou-Rădulescu-Repovš [46, 47] for singular homogeneous Neumann p -Laplace problems and for singular Robin (q, p) -Laplacian problems, respectively. Existence results for singular Neumann Laplace problems has been obtained by Lei [37] based on variational and perturbation methods.

Finally, existence results for double phase problems without singular term can be found in the papers of Colasuonno-Squassina [13], El-Manouni-Marino-Winkert [23], Fiscella [31], Fiscella-Pinamonti [32], Gasiński-Papageorgiou [33], Gasiński-Winkert [34, 35, 36], Liu-Dai [38], Papageorgiou-Rădulescu-Repovš [45], Perera-Squassina [50], Zeng-Bai-Gasiński-Winkert [56, 57] and the references therein. For related works dealing with certain types of double phase problems we refer to the works of Bahrouni-Rădulescu-Winkert [1], Barletta-Tornatore [2], Faraci-Farkas [25], Papageorgiou-Rădulescu-Repovš [44], Papageorgiou-Winkert [49] and Zeng-Bai-Gasiński-Winkert [55].

2. PRELIMINARIES

In this section we are going to mention the main facts about the Minkowski space (\mathbb{R}^N, F) and the properties about Musielak-Orlicz Sobolev spaces.

To this end, let $F: \mathbb{R}^N \rightarrow [0, \infty)$ be a positively homogeneous Minkowski norm, that is, F is a positive homogeneous function such that $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix $\nabla^2(F^2/2)(x)$ is positive definite for all $x \neq 0$. We point out that the pair (\mathbb{R}^N, F) is the simplest not necessarily reversible Finsler manifold whose flag curvature is identically zero, the geodesics are straight lines and the intrinsic distance between two points $x, y \in \mathbb{R}^N$ is given by

$$d_F(x, y) = F(y - x).$$

The pair (\mathbb{R}^N, d_F) is a quasi-metric space and in general, it holds $d_F(x, y) \neq d_F(y, x)$.

The so-called Randers metric is a typical example for a Minkowski norm with finite reversibility which is given by

$$F(x) = \sqrt{\langle Ax, x \rangle} + \langle b, x \rangle,$$

where A is a positive definite and symmetric $(N \times N)$ -type matrix and $b = (b_i) \in \mathbb{R}^N$ is a fixed vector such that $\sqrt{\langle A^{-1}b, b \rangle} < 1$. Note that

$$r_F = \frac{1 + \sqrt{\langle A^{-1}b, b \rangle}}{1 - \sqrt{\langle A^{-1}b, b \rangle}}.$$

The pair (\mathbb{R}^N, F) is often called Randers space which describes the electromagnetic field of the physical space-time in general relativity, see Randers [52]. They are deduced as the solution of the Zermelo navigation problem.

In the next proposition we recall some basic properties of F , see Bao-Chern-Shen [3, §1.2].

Proposition 2.1. *Let $F: \mathbb{R}^N \rightarrow [0, \infty)$ be a positively homogeneous Minkowski norm. Then, the following assertions hold true:*

- (i) *Positivity: $F(x) > 0$ for all $x \neq 0$;*
- (ii) *Convexity: F and F^2 are strictly convex;*
- (iii) *Euler's theorem: $x \cdot \nabla F(x) = F(x)$ and*

$$\nabla^2(F^2/2)(x)x \cdot x = F^2(x) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\};$$

- (iv) *Homogeneity: $\nabla F(tx) = \nabla F(x)$ and*

$$\nabla^2 F^2(tx) = \nabla^2 F^2(x) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\} \text{ and for all } t > 0.$$

Furthermore, $L^r(\Omega)$ and $L^r(\Omega; \mathbb{R}^N)$ stand for the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ for $1 \leq r < \infty$. The corresponding Sobolev spaces are denoted by $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ equipped with the norms

$$\|u\|_{1,r,F} = \|F(\nabla u)\|_r + \|u\|_r \quad \text{and} \quad \|u\|_{1,r,0,F} = \|F(\nabla u)\|_r,$$

respectively.

On the boundary $\partial\Omega$ of Ω we consider the $(N-1)$ -dimensional Hausdorff (surface) measure σ and denote by $L^r(\partial\Omega)$ the boundary Lebesgue space with norm $\|\cdot\|_{r,\partial\Omega}$. We know that the trace mapping $W^{1,r}(\Omega) \rightarrow L^{\tilde{r}}(\partial\Omega)$ is compact for $\tilde{r} < r_*$ and continuous for $\tilde{r} = r_*$, where r_* is the critical exponent of r on the boundary given by

$$r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ \text{any } \ell \in (r, \infty) & \text{if } r \geq N. \end{cases}$$

For simplification we will avoid the notation of the trace operator throughout the paper.

Let us now introduce the Musielak-Orlicz Sobolev spaces. For this purpose, let $\mathcal{H}: \Omega \times [0, \infty) \rightarrow [0, \infty)$ be the function defined by

$$(x, t) \mapsto t^p + \mu(x)t^q,$$

where (1.5) is satisfied. Then, the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \mid u: \Omega \rightarrow \mathbb{R} \text{ is measurable and } \rho_{\mathcal{H}}(u) < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 : \rho_{\mathcal{H}} \left(\frac{u}{\tau} \right) \leq 1 \right\},$$

where the modular function $\rho_{\mathcal{H}}: L^{\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ is given by

$$\rho_{\mathcal{H}}(u) := \int_{\Omega} \mathcal{H}(x, |u|) dx = \int_{\Omega} (|u|^p + \mu(x)|u|^q) dx.$$

From Colasuonno-Squassina [13, Proposition 2.14] we know that the space $L^{\mathcal{H}}(\Omega)$ is a reflexive Banach space.

Furthermore, we define the seminormed space

$$L_{\mu}^q(\Omega) = \left\{ u \mid u: \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_{\Omega} \mu(x)|u|^q dx < \infty \right\},$$

which is endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_{\Omega} \mu(x)|u|^q dx \right)^{\frac{1}{q}}.$$

Similarly we define $L_{\mu}^q(\Omega; \mathbb{R}^N)$ with seminorm $\|F(\cdot)\|_{q,\mu}$.

The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \{u \in L^{\mathcal{H}}(\Omega) : F(\nabla u) \in L^{\mathcal{H}}(\Omega)\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H},F} = \|F(\nabla u)\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}.$$

Finally, we mention the main embedding results between Musielak-Orlicz Sobolev spaces and usual Lebesgue and Sobolev spaces. We refer to Gasiński-Winkert [36, Proposition 2.2] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [16, Proposition 2.17].

Proposition 2.2. *Let (1.5) be satisfied and let p^* and p_* be the critical exponents to p , see (1.6). Then the following embeddings hold:*

- (i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ are continuous for all $r \in [1, p]$;
- (ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [1, p^*]$ and compact for all $r \in [1, p^*)$;
- (iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is continuous for all $r \in [1, p_*]$ and compact for all $r \in [1, p_*)$;
- (iv) $L^{\mathcal{H}}(\Omega) \hookrightarrow L_{\mu}^q(\Omega)$ is continuous;
- (v) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous.

Let $B: W^{1,\mathcal{H}}(\Omega) \rightarrow W^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear operator defined by

$$\langle B(u), \varphi \rangle_{\mathcal{H},F} := \int_{\Omega} (F^{p-1}(\nabla u) \nabla F(\nabla u) + \mu(x) F^{q-1}(\nabla u) \nabla F(\nabla u)) \cdot \nabla \varphi dx, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H},F}$ is the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$. The operator $B: W^{1,\mathcal{H}}(\Omega) \rightarrow W^{1,\mathcal{H}}(\Omega)^*$ has the following properties, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [16, Proposition 3.4(ii)], by taking the properties of F into account.

Proposition 2.3. *The operator B defined by (2.1) is bounded, continuous and monotone (hence maximal monotone).*

3. PROOF OF THE MAIN RESULT

Let $J_\lambda: W^{1,\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ be the functional given by

$$J_\lambda(u) = \frac{1}{p} \|F(\nabla u)\|_p^p + \frac{1}{q} \|F(\nabla u)\|_{q,\mu}^q + \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|u\|_{q,\mu}^q - \frac{1}{p^*} \|u_+\|_{p^*}^{p^*} \\ - \frac{\lambda}{\gamma} \int_\Omega (u_+)^{\gamma} dx - \lambda \int_\Omega G_1(x, u_+) dx - \frac{1}{p_*} \|u_+\|_{p_*,\partial\Omega}^{p_*} - \int_{\partial\Omega} G_2(x, u_+) d\sigma,$$

where $u_\pm = \max(\pm u, 0)$ and

$$G_1(x, s) = \int_0^s g_1(x, t) dt \quad \text{as well as} \quad G_2(x, s) = \int_0^s g_2(x, t) dt.$$

Due to the presence of the singular term, it is easy to see that J_λ is not C^1 .

Throughout the paper we denote by c_{p^*} and c_{p_*} the inverses of the Sobolev embedding constants of $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$, respectively. This means, in particular,

$$(c_{p^*})^{-1} = \inf_{\substack{u \in W^{1,p}(\Omega), \\ u \neq 0}} \frac{\|u\|_{1,p,F}}{\|u\|_{p^*}} \quad \text{and} \quad (c_{p_*})^{-1} = \inf_{\substack{u \in W^{1,p}(\Omega), \\ u \neq 0}} \frac{\|u\|_{1,p,F}}{\|u\|_{p_*,\partial\Omega}}. \quad (3.1)$$

Moreover, we define the function $\Psi: (0, \infty) \rightarrow \mathbb{R}$ given by

$$\Psi(s) := \frac{1}{p2^{p-1}r_F^p} - \frac{2^{p^*-1}c_{p^*}^{p^*}}{p^*} s^{p^*-p} - \frac{2^{p_*-1}c_{p_*}^{p_*}}{p_*} s^{p_*-p}, \quad (3.2)$$

where $r_F = \max_{w \neq 0} \frac{F(-w)}{F(w)}$ is finite by (H)(iii). Since Ψ is strictly decreasing, we know there exists a unique $\varrho^* > 0$ such that $\Psi(\varrho^*) = 0$. In addition, $\Psi(s) \geq 0$ for all $s \in (0, \varrho^*)$.

We start with the study of the functional $I: W^{1,\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{p} \|F(\nabla u)\|_p^p + \frac{1}{q} \|F(\nabla u)\|_{q,\mu}^q + \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|u\|_{q,\mu}^q \\ - \frac{1}{p^*} \|u\|_{p^*}^{p^*} - \frac{1}{p_*} \|u\|_{p_*,\partial\Omega}^{p_*}.$$

The next proposition shows the sequentially weakly lower semicontinuity of the functional $I: W^{1,\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ on closed convex subsets of $W^{1,\mathcal{H}}(\Omega)$.

Proposition 3.1. *Let hypotheses (H) be satisfied. For every $\varrho \in (0, \varrho^*)$ the restriction of I to the closed convex set B_ϱ which is given by*

$$B_\varrho := \left\{ u \in W^{1,\mathcal{H}}(\Omega) : \|u\|_{1,p,F} \leq \varrho \right\},$$

is sequentially weakly lower semicontinuous.

Proof. Let $\varrho \in (0, \varrho^*)$ and let $\{u_n\}_{n \in \mathbb{N}} \subseteq B_\varrho$ be such that $u_n \rightharpoonup u$ in $W^{1,\mathcal{H}}(\Omega)$. We are going to prove that

$$\liminf_{n \rightarrow \infty} (I(u_n) - I(u)) \geq 0.$$

For $\kappa \geq 1$ we consider the following truncation functions $T_\kappa, R_\kappa: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$T_\kappa(s) = \begin{cases} -\kappa & \text{if } s < -\kappa, \\ s & \text{if } -\kappa \leq s \leq \kappa, \\ \kappa & \text{if } s > \kappa, \end{cases} \quad R_\kappa(s) = \begin{cases} s + \kappa & \text{if } s < -\kappa, \\ 0 & \text{if } -\kappa \leq s \leq \kappa, \\ s - \kappa & \text{if } s > \kappa. \end{cases}$$

Note that $T_\kappa(s) + R_\kappa(s) = s$ for all $s \in \mathbb{R}$.

First, we observe that

$$\begin{aligned} \|F(\nabla u)\|_p^p &= \int_{\{|u| \leq \kappa\}} F^p(\nabla u) \, dx + \int_{\{|u| > \kappa\}} F^p(\nabla u) \, dx \\ &= \int_{\{|u| \leq \kappa\}} F^p(\nabla(T_\kappa(u))) \, dx + \int_{\{|u| > \kappa\}} F^p(\nabla(R_\kappa(u))) \, dx \\ &= \|F(\nabla(T_\kappa(u)))\|_p^p + \|F(\nabla(R_\kappa(u)))\|_p^p. \end{aligned} \quad (3.3)$$

The same argument leads to

$$\|F(\nabla u)\|_{q,\mu}^q = \|F(\nabla(T_\kappa(u)))\|_{q,\mu}^q + \|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q. \quad (3.4)$$

Since $\|\cdot\|_p$ is sequentially weakly lower semicontinuous and considering that $F(\nabla(T_\kappa(u_n))) \rightharpoonup F(\nabla(T_\kappa(u)))$ in $L^q_\mu(\Omega)$ due to the weak convergence of $u_n \rightharpoonup u$ in $W^{1,\mathcal{H}}(\Omega)$ we have for every $\kappa \geq 1$

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left(\frac{1}{p} \|F(\nabla(T_\kappa(u_n)))\|_p^p - \frac{1}{p} \|F(\nabla(T_\kappa(u)))\|_p^p \right) &\geq 0, \\ \lim_{n \rightarrow \infty} \left(\frac{1}{q} \|F(\nabla(T_\kappa(u_n)))\|_{q,\mu}^q - \frac{1}{q} \|F(\nabla(T_\kappa(u)))\|_{q,\mu}^q \right) &= 0. \end{aligned} \quad (3.5)$$

Applying the triangle inequality for the Minkowski norm F , see Bao-Chern-Shen [3, Theorem 1.2.2], along with the convexity of the function $s \mapsto s^r$, $r > 1$, we get the following inequality

$$\frac{1}{2^{r-1} r^r} F^r(w_1 - w_2) - 2F^r(w_2) \leq F^r(w_1) - F^r(w_2) \quad \text{for all } w_1, w_2 \in \mathbb{R}^N. \quad (3.6)$$

From (3.6) by taking $w_1 = \nabla(R_\kappa(u_n))$ and $w_2 = \nabla(R_\kappa(u))$, respectively, we get

$$\begin{aligned} &\|F(\nabla(R_\kappa(u_n)))\|_p^p - \|F(\nabla(R_\kappa(u)))\|_p^p \\ &\geq \frac{1}{2^{p-1} r^p} \|F(\nabla(R_\kappa(u_n)) - \nabla(R_\kappa(u)))\|_p^p - 2\|F(\nabla(R_\kappa(u)))\|_p^p. \\ &\|F(\nabla(R_\kappa(u_n)))\|_{q,\mu}^q - \|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q \\ &\geq \frac{1}{2^{q-1} r^q} \|F(\nabla(R_\kappa(u_n)) - \nabla(R_\kappa(u)))\|_{q,\mu}^q - 2\|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q. \end{aligned} \quad (3.7)$$

On the other hand, by the Brezis-Lieb lemma, see, for example, Papageorgiou-Winkert [48, Lemma 4.1.22], we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left(\|u_n\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*} \right) &= \liminf_{n \rightarrow \infty} \|u_n - u\|_{p^*}^{p^*}, \\ \liminf_{n \rightarrow \infty} \left(\|u_n\|_{p^*, \partial\Omega}^{p^*} - \|u\|_{p^*, \partial\Omega}^{p^*} \right) &= \liminf_{n \rightarrow \infty} \|u_n - u\|_{p^*, \partial\Omega}^{p^*}. \end{aligned} \quad (3.8)$$

Claim: $\|h\|_p^p \geq \|R_\kappa(h)\|_p^p$ for all $h \in W^{1,\mathcal{H}}(\Omega)$ and for all $\kappa \geq 1$.

First, we have

$$\begin{aligned}
\|h\|_p^p &= \|T_\kappa(h) + R_\kappa(h)\|_p^p \\
&= \int_{\{h < -\kappa\}} |-\kappa + R_\kappa(h)|^p dx + \int_{\{|h| \leq \kappa\}} |u + R_\kappa(h)|^p dx \\
&\quad + \int_{\{h > \kappa\}} |\kappa + R_\kappa(h)|^p dx \\
&\geq \int_{\{h < -\kappa\}} |-\kappa + R_\kappa(h)|^p dx + \int_{\{h > \kappa\}} |R_\kappa(h)|^p dx.
\end{aligned} \tag{3.9}$$

Applying the inequality

$$|w_2|^p > |w_1|^p + p|w_1|^{p-2}w_1(w_2 - w_1) \quad \text{for all } w_1, w_2 \in \mathbb{R}^N,$$

with $w_2 = R_\kappa(h) - \kappa$ and $w_1 = R_\kappa(h)$, we get

$$\begin{aligned}
&\int_{\{h < -\kappa\}} |-\kappa + R_\kappa(h)|^p dx \\
&\geq \int_{\{h < -\kappa\}} [|R_\kappa(h)|^p + p|R_\kappa(h)|^{p-2}R_\kappa(h) \cdot (-\kappa)] dx \\
&\geq \int_{\{h < -\kappa\}} |R_\kappa(h)|^p dx
\end{aligned} \tag{3.10}$$

since $R_\kappa(h) < 0$ if $h < -\kappa$. Combining (3.9) and (3.10) leads to

$$\|u\|_p^p \geq \int_{\{h < -\kappa\}} |R_\kappa(h)|^p dx + \int_{\{h > \kappa\}} |R_\kappa(h)|^p dx = \|R_\kappa(h)\|_p^p$$

because $R_\kappa(h) = 0$ if $|h| \leq \kappa$. This proves the Claim.

Thus, we may apply the Brezis-Lieb lemma along with the Claim in order to obtain

$$\begin{aligned}
\liminf_{n \rightarrow \infty} (\|u_n\|_p^p - \|u\|_p^p) &= \liminf_{n \rightarrow \infty} \|u_n - u\|_p^p \\
&\geq \liminf_{n \rightarrow \infty} \|R_\kappa(u_n) - R_\kappa(u)\|_p^p \\
&\geq \frac{1}{2^{p-1}r_F^p} \liminf_{n \rightarrow \infty} \|R_\kappa(u_n) - R_\kappa(u)\|_p^p
\end{aligned} \tag{3.11}$$

since $r_F \geq 1$ and so $2^{p-1}r_F^p \geq 1$.

Note that

$$\begin{aligned}
\|F(\nabla(R_\kappa(u)))\|_p^p &\rightarrow 0 \quad \text{as } \kappa \rightarrow \infty, \\
\|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q &\rightarrow 0 \quad \text{as } \kappa \rightarrow \infty, \\
\|u_n\|_{q,\mu}^q &\rightarrow \|u\|_{q,\mu}^q \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{3.12}$$

The last convergence in (3.12) follows from Proposition 2.2 (ii) since $q < p^*$ and due to the boundedness of $\mu(\cdot)$, as given in (1.5).

Hence, for κ large enough, taking (3.3), (3.4), (3.5), (3.7), (3.8), (3.11) and (3.12) into account, we have that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} (I(u_n) - I(u)) \\ & \geq \liminf_{n \rightarrow \infty} \left(\frac{1}{p2^{p-1}r_F^p} \|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \right. \\ & \quad \left. - \frac{1}{p^*} \|u_n - u\|_{p^*}^{p^*} - \frac{1}{p_*} \|u_n - u\|_{p_*,\partial\Omega}^{p_*} \right). \end{aligned} \quad (3.13)$$

We observe that

$$\begin{aligned} & \|u_n - u\|_{p^*}^{p^*} \\ & \leq 2^{p^*-1} \|T_\kappa(u_n) - T_\kappa(u)\|_{p^*}^{p^*} + 2^{p^*-1} \|R_\kappa(u_n) - R_\kappa(u)\|_{p^*}^{p^*}, \\ & \|u_n - u\|_{p_*,\partial\Omega}^{p_*} \\ & \leq 2^{p^*-1} \|T_\kappa(u_n) - T_\kappa(u)\|_{p_*,\partial\Omega}^{p_*} + 2^{p^*-1} \|R_\kappa(u_n) - R_\kappa(u)\|_{p_*,\partial\Omega}^{p_*}. \end{aligned} \quad (3.14)$$

By Lebesgue's dominated convergence theorem we get that

$$\lim_{n \rightarrow \infty} \|T_\kappa(u_n) - T_\kappa(u)\|_{p^*}^{p^*} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|T_\kappa(u_n) - T_\kappa(u)\|_{p_*,\partial\Omega}^{p_*} = 0. \quad (3.15)$$

Finally, combining (3.13), (3.14) and (3.15) we arrive at

$$\begin{aligned} & \liminf_{n \rightarrow \infty} (I(u_n) - I(u)) \\ & \geq \liminf_{n \rightarrow \infty} \left(\frac{1}{p2^{p-1}r_F^p} \|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \right. \\ & \quad \left. - \frac{2^{p^*-1}}{p^*} \|R_\kappa(u_n) - R_\kappa(u)\|_{p^*}^{p^*} - \frac{2^{p^*-1}}{p_*} \|R_\kappa(u_n) - R_\kappa(u)\|_{p_*,\partial\Omega}^{p_*} \right). \end{aligned}$$

Using this along with (3.1) and the fact that $\psi(s) \geq 0$ for all $s \in (0, \varrho^*)$ (see (3.2)), it follows that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} (I(u_n) - I(u)) \\ & \geq \liminf_{n \rightarrow \infty} \left(\frac{1}{p2^{p-1}r_F^p} \|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \right. \\ & \quad - \frac{2^{p^*-1}c_{p^*}^{p^*}}{p^*} \|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^{p^*} \\ & \quad \left. - \frac{2^{p^*-1}c_{p_*}^{p_*}}{p_*} \|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^{p_*} \right) \\ & \geq \liminf_{n \rightarrow \infty} \left(\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \Psi(\varrho) \right) \geq 0, \end{aligned}$$

which proves the assertion of the proposition. \square

Taking into account the assumption (H)(ii) together with the compact embeddings $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r_1}(\Omega)$ for $r_1 < p^*$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r_2}(\partial\Omega)$ for $r_2 < p_*$, see

Proposition 2.2 (ii), (iii), it is quite standard to prove that the functional

$$u \mapsto \frac{\lambda}{\gamma} \int_{\Omega} (u_+)^{\gamma} dx + \lambda \int_{\Omega} G(x, u_+) dx + \int_{\partial\Omega} G_2(x, u_+) d\sigma,$$

is sequentially weakly lower semicontinuous on $W^{1,\mathcal{H}}(\Omega)$ for every $\lambda > 0$. This fact along with Proposition 3.1 leads to the following corollary.

Corollary 3.2. *Let hypotheses (H) be satisfied. For every $\lambda > 0$ and for every $\varrho \in (0, \varrho^*)$ the restriction of J_{λ} to the closed convex set B_{ϱ} is sequentially weakly lower semicontinuous.*

Now we are going to prove Theorem 1.2. For this purpose, we introduce the functionals $I_1: W^{1,\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ and $I_2: L^{\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ given by

$$\begin{aligned} I_1(u) &= -\frac{1}{q} \|F(\nabla u)\|_{q,u}^q - \frac{1}{q} \|u\|_{q,\mu}^q + \frac{1}{p^*} \|u_+\|_{p^*}^{p^*} + \frac{\lambda}{\gamma} \int_{\Omega} (u_+)^{\gamma} dx \\ &\quad + \lambda \int_{\Omega} G_1(x, u_+) dx + \frac{1}{p_*} \|u_+\|_{p_*,\partial\Omega}^{p_*} + \int_{\partial\Omega} G_2(x, u_+) d\sigma \end{aligned}$$

and

$$\begin{aligned} I_2(u) &= \frac{1}{p^*} \|u_+\|_{p^*}^{p^*} + \frac{\lambda}{\gamma} \int_{\Omega} (u_+)^{\gamma} dx + \lambda \int_{\Omega} G(x, u_+) dx \\ &\quad + \frac{1}{p_*} \|u_+\|_{p_*,\partial\Omega}^{p_*} + \int_{\partial\Omega} G_2(x, u_+) d\sigma. \end{aligned}$$

Proof of Theorem 1.2. Let $\lambda > 0$ and $\varrho \in (0, \varrho^*)$ be as in Corollary 3.2. First we define

$$\varphi_{\lambda}(\varrho) := \inf_{\|u\|_{1,p,F} < \varrho} \frac{\sup_{B_{\varrho}} I_1 - I_1(u)}{\varrho^p - \|u\|_{1,p,F}^p} \quad \text{and} \quad \psi_{\lambda}(\varrho) := \sup_{B_{\varrho}} I_1.$$

Claim: *There exist $\lambda, \varrho > 0$ small enough such that*

$$\varphi_{\lambda}(\varrho) < \frac{1}{p}. \quad (3.16)$$

In order to prove (3.16), it is enough to find $\lambda, \varrho > 0$ such that

$$\inf_{\xi < \varrho} \frac{\psi_{\lambda}(\varrho) - \psi_{\lambda}(\xi)}{\varrho^p - \xi^p} < \frac{1}{p}. \quad (3.17)$$

Taking $\xi = \varrho - \varepsilon$ for some $\varepsilon \in (0, \varrho)$ we easily see that

$$\begin{aligned} \frac{\psi_{\lambda}(\varrho) - \psi_{\lambda}(\xi)}{\varrho^p - \xi^p} &= \frac{\psi_{\lambda}(\varrho) - \psi_{\lambda}(\varrho - \varepsilon)}{\varrho^p - (\varrho - \varepsilon)^p} \\ &= \frac{\psi_{\lambda}(\varrho) - \psi_{\lambda}(\varrho - \varepsilon)}{\varepsilon} \cdot \frac{-\varepsilon}{\varrho^{p-1}[(1 - \frac{\varepsilon}{\varrho})^p - 1]}. \end{aligned}$$

Therefore, if we pass to the limit as $\varepsilon \rightarrow 0$, then (3.17) holds if

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{\psi_{\lambda}(\varrho) - \psi_{\lambda}(\varrho - \varepsilon)}{\varepsilon} < \varrho^{p-1} \quad (3.18)$$

is satisfied.

Thus we have to verify (3.18) to get our Claim. First note that

$$\frac{1}{\varepsilon} |\psi_{\lambda}(\varrho) - \psi_{\lambda}(\varrho - \varepsilon)|$$

$$\begin{aligned}
&= \frac{1}{\varepsilon} \left| \sup_{v \in B_1} I_1(\varrho v) - \sup_{v \in B_1} I_1((\varrho - \varepsilon)v) \right| \\
&\leq \frac{1}{\varepsilon} \sup_{v \in B_1} |I_1(\varrho v) - I_1((\varrho - \varepsilon)v)| \\
&\leq \frac{1}{\varepsilon} \sup_{v \in B_1} \left| \frac{(\varrho - \varepsilon)^q - \varrho^q}{q} \left[\|F(\nabla v)\|_{q,\mu}^q + \|v\|_{q,\mu}^q \right] + I_2(\varrho v) - I_2((\varrho - \varepsilon)v) \right|.
\end{aligned}$$

The growth conditions in (H)(ii), along with the continuous embeddings $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ as well as $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, yield

$$\begin{aligned}
&\frac{\psi_\lambda(\varrho) - \psi_\lambda(\varrho - \varepsilon)}{\varepsilon} \\
&\leq \frac{1}{\varepsilon} \sup_{\|v\|_{1,p}, F \leq 1} \int_{\Omega} \left| \int_{(\varrho - \varepsilon)v_+(x)}^{\varrho v_+(x)} \left[t^{p^*-1} + \lambda t^{\gamma-1} + \lambda g_1(x, t) \right] dt \right| dx \\
&\quad + \frac{1}{\varepsilon} \sup_{\|v\|_{1,p}, F \leq 1} \int_{\partial\Omega} \left| \int_{(\varrho - \varepsilon)v_+(x)}^{\varrho v_+(x)} \left[t^{p^*-1} + g_2(x, t) \right] dt \right| d\sigma \\
&\leq \frac{c_{p^*}^{p^*}}{p^*} \left| \frac{\varrho^{p^*} - (\varrho - \varepsilon)^{p^*}}{\varepsilon} \right| + \lambda \frac{c_{p^*}^\gamma |\Omega|^{\frac{p^*-\gamma}{p^*}}}{\gamma} \left| \frac{\varrho^\gamma - (\varrho - \varepsilon)^\gamma}{\varepsilon} \right| \\
&\quad + \lambda a_1 \frac{c_{p^*}^{\nu_1} |\Omega|^{\frac{p^*-\nu_1}{p^*}}}{\nu_1} \left| \frac{\varrho^{\nu_1} - (\varrho - \varepsilon)^{\nu_1}}{\varepsilon} \right| + \lambda b_1 \frac{c_{p^*}^{\theta_1} |\Omega|^{\frac{p^*-\theta_1}{p^*}}}{\theta_1} \left| \frac{\varrho^{\theta_1} - (\varrho - \varepsilon)^{\theta_1}}{\varepsilon} \right| \\
&\quad + \frac{c_{p^*}^{p^*}}{p^*} \left| \frac{\varrho^{p^*} - (\varrho - \varepsilon)^{p^*}}{\varepsilon} \right| + a_2 \frac{c_{p^*}^{\nu_2} |\partial\Omega|^{\frac{p^*-\nu_2}{p^*}}}{\nu_2} \left| \frac{\varrho^{\nu_2} - (\varrho - \varepsilon)^{\nu_2}}{\varepsilon} \right|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\limsup_{\varepsilon \rightarrow 0^+} \frac{\psi_\lambda(\varrho) - \psi_\lambda(\varrho - \varepsilon)}{\varepsilon} \\
&\leq c_{p^*}^{p^*} \varrho^{p^*-1} + \lambda c_{p^*}^\gamma |\Omega|^{\frac{p^*-\gamma}{p^*}} \varrho^{\gamma-1} + \lambda a_1 c_{p^*}^{\nu_1} |\Omega|^{\frac{p^*-\nu_1}{p^*}} \varrho^{\nu_1-1} \\
&\quad + \lambda b_1 c_{p^*}^{\theta_1} |\Omega|^{\frac{p^*-\theta_1}{p^*}} \varrho^{\theta_1-1} + c_{p^*}^{p^*} \varrho^{p^*-1} + a_2 c_{p^*}^{\nu_2} |\partial\Omega|^{\frac{p^*-\nu_2}{p^*}} \varrho^{\nu_2-1}.
\end{aligned}$$

Now we consider the function $\Lambda: (0, \infty) \rightarrow \mathbb{R}$ given by

$$\Lambda(s) = \frac{s^{p-\gamma} - c_{p^*}^{p^*} s^{p^*-\gamma} - c_{p^*}^{p^*} s^{p^*-\gamma} - a_2 c_{p^*}^{\nu_2} |\partial\Omega|^{\frac{p^*-\nu_2}{p^*}} s^{\nu_2-\gamma}}{c_{p^*}^\gamma |\Omega|^{\frac{p^*-\gamma}{p^*}} + a_1 c_{p^*}^{\nu_1} |\Omega|^{\frac{p^*-\nu_1}{p^*}} s^{\nu_1-\gamma} + b_1 c_{p^*}^{\theta_1} |\Omega|^{\frac{p^*-\theta_1}{p^*}} s^{\theta_1-\gamma}}.$$

We easily see that $\lim_{s \rightarrow 0} \Lambda(s) = 0$ and from L'Hospital's rule we verify that $\lim_{s \rightarrow \infty} \Lambda(s) = -\infty$. Moreover, since $\nu_2 > p$, see (H)(ii), and due to the continuity of Λ , we know that there exists $s_0 > 0$ small enough such that $\Lambda(s) > 0$ for all $s \in (0, s_0)$. Hence, we find $s_{\max} > 0$ such that

$$\Lambda(s_{\max}) = \max_{s > 0} \Lambda(s).$$

Let us set

$$\lambda_* := \Lambda(\min\{s_{\max}, \varrho^*\}).$$

If we now take $\lambda < \lambda_*$ and $\varrho < \min\{s_{\max}, \varrho^*\}$, then (3.18) is satisfied and so (3.16). This proves the Claim.

From the Claim we know that there exists an element $\hat{u} \in W^{1,\mathcal{H}}(\Omega)$ with $\|\hat{u}\|_{1,p,F} \leq \varrho$ such that

$$J_\lambda(\hat{u}) < \frac{1}{p}\varrho^p - I_1(u_1) \quad \text{for all } u_1 \in B_\varrho. \quad (3.19)$$

From Corollary 3.2 we know that $J_\lambda|_{B_\varrho}$ is sequentially weakly lower semicontinuous. Therefore, $J_\lambda: W^{1,\mathcal{H}}(\Omega) \rightarrow \mathbb{R}$ restricted to B_ϱ has a global minimizer $u \in W^{1,\mathcal{H}}(\Omega)$ with $\|u\|_{1,p,F} \leq \varrho$. Suppose that $\|u\|_{1,p,F} = \varrho$. Then we have from (3.19) that

$$J_\lambda(u) = \frac{1}{p}\varrho^p - I_1(u) > J_\lambda(\hat{u}),$$

which is a contradiction. We conclude that $u \in B_\varrho$ is a local minimizer for J_λ with $\|u\|_{1,p,F} < \varrho$ for $\lambda < \lambda_*$.

We claim that $u \neq 0$. Let $v \in W^{1,\mathcal{H}}(\Omega)$ be such that $v > 0$ and let $t > 0$. Then we have

$$\begin{aligned} J_\lambda(tv) &= \frac{t^p}{p} \|F(\nabla v)\|_p^p + \frac{t^q}{q} \|F(\nabla v)\|_{q,\mu}^q + \frac{t^p}{p} \|v\|_p^p + \frac{t^q}{q} \|v\|_{q,\mu}^q - \frac{t^{p^*}}{p^*} \|v\|_{p^*}^{p^*} \\ &\quad - \lambda \frac{t^\gamma}{\gamma} \int_\Omega v^\gamma \, dx - \lambda \frac{a_1 t^{\nu_1}}{\nu_1} \|v\|_{\nu_1}^{\nu_1} - \lambda \frac{b_1 t^{\theta_1}}{\theta_1} \|v\|_{\theta_1}^{\theta_1} \\ &\quad - \frac{t^{p^*}}{p^*} \|v\|_{p^*,\partial\Omega}^{p^*} - \frac{a_2 t^{\nu_2}}{\nu_2} \|v\|_{\nu_2,\partial\Omega}^{\nu_2}, \end{aligned}$$

which implies $J_\lambda(tv) < 0$ for $t > 0$ sufficiently small. Thus, $u \neq 0$.

Let us now prove that $u \in W^{1,\mathcal{H}}(\Omega)$ is nonnegative a. e. in Ω . First we observe that $u + tu_- \in B_\varrho$ and $(u + tu_-)_+ = u_+$ for $t > 0$ sufficiently small. Using this fact we have

$$\begin{aligned} 0 &\leq \frac{J_\lambda(u + tu_-) - J_\lambda(u)}{t} \\ &= \frac{1}{p} \int_\Omega \frac{F^p(\nabla(u + tu_-)) - F^p(\nabla u)}{t} \, dx + \frac{1}{q} \int_\Omega \mu(x) \frac{F^q(\nabla(u + tu_-)) - F^q(\nabla u)}{t} \, dx \\ &\quad + \frac{1}{p} \int_\Omega \frac{|u + tu_-|^p - |u|^p}{t} \, dx + \frac{1}{q} \int_\Omega \mu(x) \frac{|u + tu_-|^q - |u|^q}{t} \, dx. \end{aligned}$$

From this we conclude

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 0^+} \frac{J_\lambda(u + tu_-) - J_\lambda(u)}{t} \\ &= \int_\Omega F^{p-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla u_- \, dx + \int_\Omega \mu(x) F^{q-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla u_- \, dx \\ &\quad + \int_\Omega |u|^{p-2} u u_- \, dx + \int_\Omega \mu(x) |u|^{q-2} u u_- \, dx. \end{aligned}$$

However, from Proposition 2.1 (iii), we know that

$$\begin{aligned} \int_\Omega F(\nabla u)^{p-1} \nabla F(\nabla u) \cdot \nabla u_- \, dx &= - \int_\Omega F^{p-1}(\nabla u_-) \nabla F(\nabla u_-) \cdot \nabla u_- \, dx \\ &= - \|F(\nabla u_-)\|_p^p \end{aligned}$$

and

$$\int_\Omega \mu(x) F^{q-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla u_- \, dx = - \|F(\nabla u_-)\|_{q,\mu}^q.$$

This leads to

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 0} \frac{J_\lambda(u + tu_-) - J_\lambda(u)}{t} \\ &= -\|F(\nabla u_-)\|_p^p - \|F(\nabla u_-)\|_{q,\mu}^q - \|u_-\|_p^p - \|u_-\|_{q,\mu}^q \leq 0. \end{aligned}$$

Therefore, $u_- = 0$ and so $u \geq 0$ a. e. in Ω .

Let us now show that u is positive in Ω . We argue indirectly and suppose there is a set C with positive measure such that $u = 0$ in C . Let $\varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\varphi > 0$ and let $t > 0$ small enough such that $u + t\varphi \in B_\sigma$ and $(u + t\varphi)^\gamma > u^\gamma$ a. e. in Ω . We obtain

$$\begin{aligned} 0 &\leq \frac{J_\lambda(u + t\varphi) - J_\lambda(u)}{t} \\ &= \frac{1}{p} \frac{\|F(\nabla(u + t\varphi))\|_p^p - \|F(\nabla u)\|_p^p}{t} + \frac{1}{q} \frac{\|F(\nabla(u + t\varphi))\|_{q,\mu}^q - \|F(\nabla u)\|_{q,\mu}^q}{t} \\ &\quad + \frac{1}{p} \frac{\|u + t\varphi\|_p^p - \|u\|_p^p}{t} + \frac{1}{q} \frac{\|u + t\varphi\|_{q,\mu}^q - \|u\|_{q,\mu}^q}{t} - \frac{1}{p^*} \frac{\|u + t\varphi\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*}}{t} \\ &\quad - \frac{\lambda}{\gamma t^{1-\gamma}} \int_C \varphi^\gamma dx - \frac{\lambda}{\gamma} \int_{\Omega \setminus C} \frac{(u + t\varphi)^\gamma - u^\gamma}{t} dx \\ &\quad - \lambda \int_\Omega \frac{G_1(x, u + t\varphi) - G_1(x, u)}{t} dx \\ &\quad - \frac{1}{p_*} \frac{\|u + t\varphi\|_{p_*,\partial\Omega}^{p_*} - \|u\|_{p_*,\partial\Omega}^{p_*}}{t} - \int_{\partial\Omega} \frac{G_2(x, u + t\varphi) - G_2(x, u)}{t} d\sigma \\ &< \frac{1}{p} \frac{\|F(\nabla(u + t\varphi))\|_p^p - \|F(\nabla u)\|_p^p}{t} + \frac{1}{q} \frac{\|F(\nabla(u + t\varphi))\|_{q,\mu}^q - \|F(\nabla u)\|_{q,\mu}^q}{t} \\ &\quad + \frac{1}{p} \frac{\|u + t\varphi\|_p^p - \|u\|_p^p}{t} + \frac{1}{q} \frac{\|u + t\varphi\|_{q,\mu}^q - \|u\|_{q,\mu}^q}{t} - \frac{1}{p^*} \frac{\|u + t\varphi\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*}}{t} \\ &\quad - \frac{\lambda}{\gamma t^{1-\gamma}} \int_C \varphi^\gamma dx - \lambda \int_\Omega \frac{G_1(x, u + t\varphi) - G_1(x, u)}{t} dx \\ &\quad - \frac{1}{p_*} \frac{\|u + t\varphi\|_{p_*,\partial\Omega}^{p_*} - \|u\|_{p_*,\partial\Omega}^{p_*}}{t} - \int_{\partial\Omega} \frac{G_2(x, u + t\varphi) - G_2(x, u)}{t} d\sigma. \end{aligned}$$

This yields

$$0 \leq \frac{J_\lambda(u + t\varphi) - J_\lambda(u)}{t} \rightarrow -\infty \quad \text{as } t \rightarrow 0^+,$$

a contradiction. Hence, $u > 0$ a. e. in Ω .

Next we want to show that

$$u^{\gamma-1}\varphi \in L^1(\Omega) \quad \text{for all } \varphi \in W^{1,\mathcal{H}}(\Omega) \quad (3.20)$$

and

$$\begin{aligned} &\int_\Omega (F(\nabla u)^{p-1} + \mu(x)F(\nabla u)^{q-1}) \nabla F(\nabla u) \cdot \nabla \varphi dx + \int_\Omega u^{p-1}\varphi dx \\ &+ \int_\Omega \mu(x)u^{q-1}\varphi dx - \int_\Omega u^{p^*-1}\varphi dx - \lambda \int_\Omega u^{\gamma-1}\varphi dx - \lambda \int_\Omega g_1(x, u)\varphi dx \quad (3.21) \\ &- \int_{\partial\Omega} u^{p_*-1}\varphi d\sigma - \int_{\partial\Omega} g_2(x, u)\varphi d\sigma \geq 0 \end{aligned}$$

for all $\varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$.

We choose $\varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$ and fix a decreasing sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq (0, 1]$ such that $\lim_{n \rightarrow \infty} t_n = 0$. It is clear that the functions

$$h_n(x) = \frac{(u(x) + t_n \varphi(x))^\gamma - u(x)^\gamma}{t_n}, \quad n \in \mathbb{N}$$

are measurable and nonnegative. Moreover, we have

$$\lim_{n \rightarrow \infty} h_n(x) = \gamma u(x)^{\gamma-1} \varphi(x) \quad \text{for a. a. } x \in \Omega.$$

Applying Fatou's lemma gives

$$\int_{\Omega} u^{\gamma-1} \varphi \, dx \leq \frac{1}{\gamma} \liminf_{n \rightarrow \infty} \int_{\Omega} h_n \, dx. \quad (3.22)$$

Then, for $n \in \mathbb{N}$ large enough, we obtain

$$\begin{aligned} 0 &\leq \frac{J_\lambda(u + t\varphi) - J_\lambda(u)}{t} \\ &= \frac{1}{p} \frac{\|F(\nabla(u + t_n\varphi))\|_p^p - \|F(\nabla u)\|_p^p}{t_n} + \frac{1}{q} \frac{\|F(\nabla(u + t_n\varphi))\|_{q,\mu}^q - \|F(\nabla u)\|_{q,\mu}^q}{t_n} \\ &\quad + \frac{1}{p} \frac{\|u + t_n\varphi\|_p^p - \|u\|_p^p}{t_n} + \frac{1}{q} \frac{\|u + t_n\varphi\|_{q,\mu}^q - \|u\|_{q,\mu}^q}{t_n} - \frac{1}{p^*} \frac{\|u + t_n\varphi\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*}}{t_n} \\ &\quad - \frac{\lambda}{\gamma} \int_{\Omega} h_n \, dx - \lambda \int_{\Omega} \frac{G_1(x, u + t_n\varphi) - G_1(x, u)}{t_n} \, dx \\ &\quad - \frac{1}{p_*} \frac{\|u + t_n\varphi\|_{p_*,\partial\Omega}^{p_*} - \|u\|_{p_*,\partial\Omega}^{p_*}}{t_n} - \int_{\partial\Omega} \frac{G_2(x, u + t_n\varphi) - G_2(x, u)}{t_n} \, d\sigma. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ in the inequality above and using (3.22) we derive (3.20) and we have

$$\begin{aligned} \lambda \int_{\Omega} u^{\gamma-1} \varphi \, dx &\leq \int_{\Omega} (F(\nabla u)^{p-1} + \mu(x)F(\nabla u)^{q-1}) \nabla F(\nabla u) \cdot \nabla \varphi \, dx \\ &\quad + \int_{\Omega} u^{p-1} \varphi \, dx + \int_{\Omega} \mu(x) u^{q-1} \varphi \, dx - \int_{\Omega} u^{p^*-1} \varphi \, dx \\ &\quad - \lambda \int_{\Omega} u^{\gamma-1} \varphi \, dx - \lambda \int_{\Omega} g_1(x, u) \varphi \, dx \\ &\quad - \int_{\partial\Omega} u^{p_*-1} \varphi \, d\sigma - \int_{\partial\Omega} g_2(x, u) \varphi \, d\sigma, \end{aligned}$$

which shows (3.21). Note that it is sufficient to prove the integrability in (3.20) for nonnegative test functions $\varphi \in W^{1,\mathcal{H}}(\Omega)$.

Now, let $\varepsilon \in (0, 1)$ be such that $(1+t)u \in B_\sigma$ for all $t \in [-\varepsilon, \varepsilon]$. Note that the function $\beta(t) := J_\lambda((1+t)u)$ has a local minimum in zero. We apply again Proposition 2.1 (iii) in order to get

$$\begin{aligned} 0 &= \beta'(0) = \lim_{t \rightarrow 0} \frac{J_\lambda((1+t)u) - J_\lambda(u)}{t} \\ &= \|F(\nabla u)\|_p^p + \|F(\nabla u)\|_{q,\mu}^q + \|u\|_p^p + \|u\|_{q,\mu}^q - \|u\|_{p^*}^{p^*} \\ &\quad - \lambda \int_{\Omega} u^\gamma \, dx - \lambda \int_{\Omega} g_1(x, u) u \, dx - \|u\|_{p_*,\partial\Omega}^{p_*} - \int_{\partial\Omega} g_2(x, u) u \, d\sigma. \end{aligned} \quad (3.23)$$

Finally, we need to show that u is a positive weak solution of (1.4). To this end, let $v \in W^{1,\mathcal{H}}(\Omega)$ and take the test function $\varphi = (u + \varepsilon v)_+ \in W^{1,\mathcal{H}}(\Omega)$ in (3.21). Taking (3.23) into account we have

$$\begin{aligned}
0 &\leq \int_{\{u+\varepsilon v \geq 0\}} (F^{p-1}(\nabla u) + \mu(x)F^{q-1}(\nabla u)) \nabla F(\nabla u) \cdot \nabla(u + \varepsilon v) \, dx \\
&\quad + \int_{\{u+\varepsilon v \geq 0\}} (u^{p-1} + \mu(x)u^{q-1}) (u + \varepsilon v) \, dx - \int_{\Omega} u^{p^*-1}(u + \varepsilon v) \, dx \\
&\quad - \lambda \int_{\Omega} u^{\gamma-1}(u + \varepsilon v) \, dx - \lambda \int_{\Omega} g_1(x, u)(u + \varepsilon v) \, dx \\
&\quad - \int_{\partial\Omega} (u^{p^*-1} + g_2(x, u)) (u + \varepsilon v) \, d\sigma \\
&= \|F(\nabla u)\|_p^p + \|F(\nabla u)\|_{q,\mu}^q + \|u\|_p^p + \|u\|_{q,\mu}^q - \|u\|_{p^*}^{p^*} - \lambda \int_{\Omega} u^{\gamma} \, dx \\
&\quad - \lambda \int_{\Omega} g_1(x, u)u \, dx - \|u\|_{p^*,\partial\Omega}^{p^*} - \int_{\partial\Omega} g_2(x, u) \, u \, d\sigma \\
&\quad + \varepsilon \int_{\Omega} F^{p-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla v \, dx + \varepsilon \int_{\Omega} \mu(x)F^{q-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla v \, dx \\
&\quad + \varepsilon \int_{\Omega} u^{p-1}v \, dx + \varepsilon \int_{\Omega} \mu(x)u^{q-1}v \, dx - \varepsilon \int_{\Omega} u^{p^*-1}v \, dx - \varepsilon \lambda \int_{\Omega} u^{\gamma-1}v \, dx \\
&\quad - \varepsilon \lambda \int_{\Omega} g_1(x, u)v \, dx - \varepsilon \int_{\partial\Omega} u^{p^*-1}v \, d\sigma - \varepsilon \int_{\partial\Omega} g_2(x, u)v \, d\sigma \\
&\quad - \int_{\{u+\varepsilon v < 0\}} F^p(\nabla u) \, dx - \varepsilon \int_{\{u+\varepsilon v < 0\}} F^{p-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla v \, dx \quad (3.24) \\
&\quad - \int_{\{u+\varepsilon v < 0\}} \mu(x)F^q(\nabla u) \, dx - \varepsilon \int_{\{u+\varepsilon v < 0\}} \mu(x)F^{q-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla v \, dx \\
&\quad - \int_{\{u+\varepsilon v < 0\}} u^{p-1}(u + \varepsilon v) \, dx - \int_{\{u+\varepsilon v < 0\}} \mu(x)u^{q-1}(u + \varepsilon v) \, dx \\
&\quad + \int_{\{u+\varepsilon v < 0\}} u^{p^*-1}(u + \varepsilon v) \, dx + \lambda \int_{\{u+\varepsilon v < 0\}} u^{\gamma-1}(u + \varepsilon v) \, dx \\
&\quad + \lambda \int_{\{u+\varepsilon v < 0\}} g_1(x, u)(u + \varepsilon v) \, dx + \int_{\partial\Omega} u^{p^*-1}(u + \varepsilon v) \, d\sigma \\
&\quad + \int_{\partial\Omega} g_2(x, u)(u + \varepsilon v) \, d\sigma \\
&\leq \varepsilon \left[\int_{\Omega} (F^{p-1}(\nabla u) + \mu(x)F^{q-1}(\nabla u)) \nabla F(\nabla u) \cdot \nabla v \, dx + \int_{\Omega} u^{p-1}v \, dx \right. \\
&\quad + \int_{\Omega} \mu(x)u^{q-1}v \, dx - \int_{\Omega} u^{p^*-1}v \, dx - \lambda \int_{\Omega} u^{\gamma-1}v \, dx - \lambda \int_{\Omega} g_1(x, u)v \, dx \\
&\quad \left. - \int_{\partial\Omega} u^{p^*-1}v \, d\sigma - \int_{\partial\Omega} g_2(x, u)v \, d\sigma \right] \\
&\quad - \varepsilon \int_{\{u+\varepsilon v < 0\}} F^{p-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla v \, dx
\end{aligned}$$

$$\begin{aligned}
& - \varepsilon \int_{\{u+\varepsilon v < 0\}} \mu(x) F^{q-1}(\nabla u) \nabla F(\nabla u) \cdot \nabla v \, dx \\
& - \varepsilon \int_{\{u+\varepsilon v < 0\}} u^{p-1} v \, dx - \varepsilon \int_{\{u+\varepsilon v < 0\}} \mu(x) u^{q-1} v \, dx.
\end{aligned}$$

Note that the measure of the set $\{u + \varepsilon v < 0\}$ goes to 0 as $\varepsilon \rightarrow 0$. Hence,

$$\begin{aligned}
& \int_{\{u+\varepsilon v < 0\}} F^{p-1}(\nabla u) \nabla F(\nabla u) \nabla v \, dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \\
& \int_{\{u+\varepsilon v < 0\}} \mu(x) F^{q-1}(\nabla u) \nabla F(\nabla u) \nabla v \, dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \\
& \int_{\{u+\varepsilon v < 0\}} u^{p-1} v \, dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \\
& \int_{\{u+\varepsilon v < 0\}} \mu(x) u^{q-1} v \, dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

Therefore, dividing the inequality (3.24) by ε and passing to the limit as $\varepsilon \rightarrow 0$ we conclude that

$$\begin{aligned}
& \int_{\Omega} (F(\nabla u)^{p-1} + \mu(x) F(\nabla u)^{q-1}) \nabla F(\nabla u) \cdot \nabla v \, dx + \int_{\Omega} u^{p-1} v \, dx \\
& + \int_{\Omega} \mu(x) u^{q-1} v \, dx - \int_{\Omega} u^{p^*-1} v \, dx - \lambda \int_{\Omega} u^{\gamma-1} v \, dx - \lambda \int_{\Omega} g_1(x, u) v \, dx \\
& - \int_{\partial\Omega} u^{p^*-1} v \, d\sigma - \int_{\partial\Omega} g_2(x, u) v \, d\sigma \geq 0.
\end{aligned}$$

Since $v \in W^{1, \mathcal{H}}(\Omega)$ was arbitrary chosen, we see from the last inequality that equality must hold. Therefore, $u \in W^{1, \mathcal{H}}(\Omega)$ is a weak solution of problem (1.4) in the sense of Definition 1.1. \square

ACKNOWLEDGMENTS

C. Farkas was supported by the National Research, Development and Innovation Fund of Hungary, financed under the K_18 funding scheme, Project No. 127926 and by the Sapientia Foundation - Institute for Scientific Research, Romania, Project No. 17/11.06.2019.

A. Fiscella is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica "G. Severi" (INdAM). A. Fiscella realized the manuscript within the auspices of the INdAM-GNAMPA project titled "Equazioni alle derivate parziali: problemi e modelli" (Prot.20191219-143223-545), of the FAPESP Project titled "Operators with non standard growth" (2019/23917-3), of the FAPESP Thematic Project titled "Systems and partial differential equations" (2019/02512-5) and of the CNPq Project titled "Variational methods for singular fractional problems" (3787749185990982).

REFERENCES

- [1] A. Bahrouni, V.D. Rădulescu, P. Winkert, *Double phase problems with variable growth and convection for the Baouendi-Grushin operator*, Z. Angew. Math. Phys. **71** (2020), no. 6, 183.
- [2] G. Barletta, E. Tornatore, *Elliptic problems with convection terms in Orlicz spaces*, J. Math. Anal. Appl. **495** (2021), no. 2, 124779, 28 pp.

- [3] D. Bao, S.-S. Chern, Z. Shen, “An Introduction to Riemann-Finsler Geometry”, Springer-Verlag, New York, 2000.
- [4] P. Baroni, M. Colombo, G. Mingione, *Harnack inequalities for double phase functionals*, *Nonlinear Anal.* **121** (2015), 206–222.
- [5] P. Baroni, M. Colombo, G. Mingione, *Non-autonomous functionals, borderline cases and related function classes*, *St. Petersburg Math. J.* **27** (2016), 347–379.
- [6] P. Baroni, M. Colombo, G. Mingione, *Regularity for general functionals with double phase*, *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 62, 48 pp.
- [7] P. Baroni, T. Kuusi, G. Mingione, *Borderline gradient continuity of minima*, *J. Fixed Point Theory Appl.* **15** (2014), no. 2, 537–575.
- [8] G. Bellettini, M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, *Hokkaido Math. J.* **25** (1996), no. 3, 537–566.
- [9] S.-S. Byun, J. Oh, *Regularity results for generalized double phase functionals*, *Anal. PDE* **13** (2020), no. 5, 1269–1300.
- [10] M. Belloni, V. Ferone, B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators*, *Z. Angew. Math. Phys.* **54** (2003), no. 5, 771–783.
- [11] Z.-Y. Chen, B. Ge, W.-S. Yuan, X.-F. Cao, *Existence of solution for double-phase problem with singular weights*, *Adv. Math. Phys.* **2020** (2020), Art. ID 5376013, 7 pp.
- [12] A. Cianchi, P. Salani, *Overdetermined anisotropic elliptic problems*, *Math. Ann.* **345** (2009), no. 4, 859–881.
- [13] F. Colasuonno, M. Squassina, *Eigenvalues for double phase variational integrals*, *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 6, 1917–1959.
- [14] M. Colombo, G. Mingione, *Bounded minimisers of double phase variational integrals*, *Arch. Ration. Mech. Anal.* **218** (2015), no. 1, 219–273.
- [15] M. Colombo, G. Mingione, *Regularity for double phase variational problems*, *Arch. Ration. Mech. Anal.* **215** (2015), no. 2, 443–496.
- [16] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, preprint 2021, arXiv: 2103.08928.
- [17] Á. Crespo-Blanco, N.S. Papageorgiou, P. Winkert, *Parametric superlinear double phase problems with singular term and critical growth on the boundary*, preprint 2021, arXiv: 2106.15511.
- [18] C. De Filippis, G. Palatucci, *Hölder regularity for nonlocal double phase equations*, *J. Differential Equations* **267** (2019), no. 1, 547–586.
- [19] F. Della Pietra, G. di Blasio, N. Gavitone, *Sharp estimates on the first Dirichlet eigenvalue of nonlinear elliptic operators via maximum principle*, *Adv. Nonlinear Anal.* **9** (2020), no. 1, 278–291.
- [20] F. Della Pietra, N. Gavitone, G. Piscitelli, *On the second Dirichlet eigenvalue of some nonlinear anisotropic elliptic operators* *Bull. Sci. Math.* **155** (2019), 10–32.
- [21] F. Della Pietra, N. Gavitone, *Sharp estimates and existence for anisotropic elliptic problems with general growth in the gradient*, *Z. Anal. Anwend.* **35** (2016), no. 1, 61–80.
- [22] A. Dinghas, *Über einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen*, *Z. Kristallogr., Mineral. Petrogr.* **105** (1944), Abt. A., 304–314.
- [23] S. El Manouni, G. Marino, P. Winkert, *Existence results for double phase problems depending on Robin and Steklov eigenvalues for the p -Laplacian*, *Adv. Nonlinear Anal.* **11** (2022), no. 1, 304–320.
- [24] X. Fan, *An imbedding theorem for Musielak-Sobolev spaces*, *Nonlinear Anal.* **75** (2012), no. 4, 1959–1971.
- [25] F. Faraci, C. Farkas, *A quasilinear elliptic problem involving critical Sobolev exponents*, *Collect. Math.* **66** (2015), no. 2, 243–259.
- [26] C. Farkas, *Critical elliptic equations on non-compact Finsler manifolds*, preprint, 2020, arXiv: 2010.07686.
- [27] C. Farkas, J. Fodor and A. Kristály, *Anisotropic elliptic problems involving sublinear terms*, 2015 IEEE 10th Jubilee International Symposium on Applied Computational Intelligence and Informatics, Timisoara, 2015, pp. 141–146.
- [28] C. Farkas, A. Kristály, C. Varga, *Singular Poisson equations on Finsler-Hadamard manifolds*, *Calc. Var. Partial Differential Equations* **54** (2015), no. 2, 1219–1241.

- [29] C. Farkas, P. Winkert, *An existence result for singular Finsler double phase problems*, J. Differential Equations **286**, 455–473.
- [30] V. Ferone, B. Kawohl, *Remarks on a Finsler-Laplacian*, Proc. Amer. Math. Soc. **137** (2009), no. 1, 247–253.
- [31] A. Fiscella, *A double phase problem involving Hardy potentials*, preprint, 2020, arXiv: abs/2008.00117.
- [32] A. Fiscella, A. Pinamonti, *Existence and multiplicity results for Kirchhoff type problems on a double phase setting*, preprint, 2020, arXiv: 2008.00114.
- [33] L. Gasiński, N.S. Papageorgiou, *Constant sign and nodal solutions for superlinear double phase problems*, Adv. Calc. Var., <https://doi.org/10.1515/acv-2019-0040>.
- [34] L. Gasiński, P. Winkert, *Constant sign solutions for double phase problems with superlinear nonlinearity*, Nonlinear Anal. **195** (2020), 111739.
- [35] L. Gasiński, P. Winkert, *Existence and uniqueness results for double phase problems with convection term*, J. Differential Equations **268** (2020), no. 8, 4183–4193.
- [36] L. Gasiński, P. Winkert, *Sign changing solution for a double phase problem with nonlinear boundary condition via the Nehari manifold*, J. Differential Equations **274** (2021), 1037–1066.
- [37] C.-Y. Lei, *Existence and multiplicity of positive solutions for Neumann problems involving singularity and critical growth*, J. Math. Anal. Appl. **459** (2018), no. 2, 959–979.
- [38] W. Liu, G. Dai, *Existence and multiplicity results for double phase problem*, J. Differential Equations **265** (2018), no. 9, 4311–4334.
- [39] W. Liu, G. Dai, N.S. Papageorgiou, P. Winkert, *Existence of solutions for singular double phase problems via the Nehari manifold method*, preprint, 2020, arXiv: 2101.00593.
- [40] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q -growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.
- [41] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284.
- [42] J. Ok, *Partial regularity for general systems of double phase type with continuous coefficients*, Nonlinear Anal. **177** (2018), 673–698.
- [43] J. Ok, *Regularity for double phase problems under additional integrability assumptions*, Nonlinear Anal. **194** (2020), 111408.
- [44] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Double-phase problems and a discontinuity property of the spectrum*, Proc. Amer. Math. Soc. **147** (2019), no. 7, 2899–2910.
- [45] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Ground state and nodal solutions for a class of double phase problems*, Z. Angew. Math. Phys. **71** (2020), no. 1, 15 pp.
- [46] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Positive solutions for nonlinear Neumann problems with singular terms and convection*, J. Math. Pures Appl. (9) **136** (2020), 1–21.
- [47] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Robin double-phase problems with singular and superlinear terms*, Nonlinear Anal. Real World Appl. **58** (2021), 103217, 20 pp.
- [48] N.S. Papageorgiou, P. Winkert, “Applied Nonlinear Functional Analysis. An Introduction”, De Gruyter, Berlin, 2018.
- [49] N.S. Papageorgiou, P. Winkert, *Singular p -Laplacian equations with superlinear perturbation*, J. Differential Equations **266** (2019), no. 2-3, 1462–1487.
- [50] K. Perera, M. Squassina, *Existence results for double-phase problems via Morse theory*, Commun. Contemp. Math. **20** (2018), no. 2, 1750023, 14 pp.
- [51] M.A. Ragusa, A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.
- [52] G. Randers, *On an asymmetrical metric in the four-space of general relativity* Phys. Rev. (2) **59** (1941), 195–199.
- [53] J.E. Taylor, *Crystalline variational methods*, Proc. Natl. Acad. Sci. USA **99** (2002), no. 24, 15277–15280.
- [54] G. Wang, C. Xia, *A characterization of the Wulff shape by an overdetermined anisotropic PDE*, Arch. Ration. Mech. Anal. **199** (2011), no. 1, 99–115.
- [55] S.D. Zeng, Y.R. Bai, L. Gasiński, P. Winkert, *Convergence analysis for double phase obstacle problems with multivalued convection term*, Adv. Nonlinear Anal. **10** (2021), no. 1, 659–672.
- [56] S.D. Zeng, Y.R. Bai, L. Gasiński, P. Winkert, *Existence results for double phase implicit obstacle problems involving multivalued operators*, Calc. Var. Partial Differential Equations **59** (2020), no. 5, 176.

- [57] S.D. Zeng, L. Gasiński, P. Winkert, Y.R. Bai, *Existence of solutions for double phase obstacle problems with multivalued convection term*, J. Math. Anal. Appl. **501** (2021), no. 1, 123997, 12 pp.
- [58] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.
- [59] V. V. Zhikov, S. M. Kozlov, O. A. Oleĭnik, “Homogenization of Differential Operators and Integral Functionals”, Springer-Verlag, Berlin, 1994.

(C. Farkas) SAPIENTIA HUNGARIAN UNIVERSITY OF TRANSYLVANIA, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, TÂRGU MUREȘ, ROMANIA
Email address: `farkascs@ms.sapientia.ro`, `farkas.csaba2008@gmail.com`

(A. Fiscella) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE ESTADUAL DE CAMPINAS, IMECC, RUA SÉRGIO BUARQUE DE HOLANDA, 651, CAMPINAS, SP CEP 13083-859, BRAZIL
Email address: `fiscella@ime.unicamp.br`

(P. Winkert) TECHNISCHE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, STRASSE DES 17. JUNI 136, 10623 BERLIN, GERMANY
Email address: `winkert@math.tu-berlin.de`