TWO SOLUTIONS FOR DIRICHLET DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENTS

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Abstract. This paper is devoted to the study of a double phase problem with variable exponents and Dirichlet boundary condition. Based on an abstract critical point theorem, we establish existence results under very general assumptions on the nonlinear term, such as a subcritical growth and a superlinear condition. In particular, we prove the existence of two bounded weak solutions with opposite energy sign and we state some special cases in which they turn out to be nonnegative.

1. Introduction

This paper deals with the following boundary value problem with a nonlinear differential equation involving the double phase operator with variable exponents and Dirichlet boundary condition

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)&=\lambda f(x,u) && \text{in } \Omega,\\ u&=0 && \text{on } \partial\Omega, \end{aligned} \tag{P_λ}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ such that

$$1 < p(x) < N, \quad p(x) < q(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega},$$

with $p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$, $0 \leq \mu(\cdot) \in L^\infty(\Omega)$, $\lambda > 0$ is a parameter and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies subcritical growth and a certain behavior at $\pm\infty$, see assumptions (H_f) in Section 2.

Problems of this type are widely studied in the literature because of their application in several topics. The double phase operator, i.e.

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right), \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

is related to the following two-phase integral functional

$$u \to \int_\Omega \left(|\nabla u|^{p(x)}+\mu(x)|\nabla u|^{q(x)}\right) \mathrm{d}x, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $W_0^{1,\mathcal{H}}(\Omega)$ denotes a Musielak-Orlicz Sobolev space that will be introduced in an appropriate way in Section 2. This functional changes ellipticity depending on the set where the weight function $\mu(\cdot)$ is zero, shifting in two different phases of

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elliptic behavior. Zhikov [35] was the first who studied this functional for constant exponents to describe the behavior of strongly anisotropic materials; indeed, in the elasticity theory, $\mu(\cdot)$ contains the information on the geometry of composites made of two different materials with power hardening exponents $p(\cdot)$ and $q(\cdot)$, see also Zikhov [36]. Furthermore, the double phase operator has several applications also in mathematical topics, as the Lavrentiev gap phenomenon and the duality theory. A first mathematical treatment of such two-phase integrals has been done by Baroni-Colombo-Mingione [3, 4, 5], Colombo-Mingione [12, 13] and Ragusa-Tachikawa [30], see also the work of De Filippis-Mingione [16] for nonautonomous integrals.

Many authors have shown existence and multiplicity results for double phase problems with constant exponents, see, for example, the papers of Biagi-Esposito-Vecchi [6], Colasuonno-Squassina [11], Gasiński-Winkert [20], Gasiński-Papageorgiou [19], Liu-Dai [24], Papageorgiou-Rădulescu-Repovš [27], Perera-Squassina [29], Zeng-Bai-Gasiński-Winkert [33] and the references therein. Whereas, in the variable exponent case there are only few results, we refer to Aberqi-Bennouna-Benslimane-Ragusa [1], Bahrouni-Rădulescu-Winkert [2], Cen-Kim-Kim-Zeng [9], Crespo-Blanco-Gasiński-Harjulehto-Winkert [14], Leonardi-Papageorgiou [23], Liu-Pucci [25], Kim-Kim-Oh-Zeng [22], Vetro-Winkert [32] and Zeng-Rădulescu-Winkert [34].

Motivated by the large interest on this differential operator in the current literature, our aim is to apply a two critical point theorem due to Bonanno-D'Aguì [8] to a more general class of problems. We observe that Theorem 2.1 of [8] gives the existence of two nontrivial critical points, one of local minimum type and the other of mountain pass type, without using standard techniques such as upper and lower methods and regularity theory. Our work extends the recent papers of Chinnì-Sciammetta-Tornatore [10] if $\mu \equiv 1$ and of Sciammetta-Tornatore-Winkert [31] if the exponents p and q are constants. Indeed, if $\inf_{\overline{\Omega}} \mu > 0$, the double phase operator reduces to the $(p(\cdot), q(\cdot))$-Laplacian and if $\mu \equiv 0$ to the $p(\cdot)$-Laplacian. Moreover, instead of supposing the typical Ambrosetti-Rabinowitz condition on the function f as required in [10], we only assume that the nonlinear term on the right-hand side of (P_λ) is $(q_+ - 1)$-superlinear at $\pm\infty$ (see (H$_f$)(iii)) and satisfies another appropriate behavior at $\pm\infty$. However, these conditions are weaker than the Ambrosetti-Rabinowitz condition. Under these assumptions, we are going to prove the existence of two weak solutions for problem (P_λ) that have opposite energy sign and belong to $L^\infty(\Omega)$, namely they are bounded.
We also point out that the exponents $p(\cdot)$ and $q(\cdot)$ do not need to verify a condition of the type

$$\frac{q(\cdot)}{p(\cdot)} < 1 + \frac{1}{N} \tag{1.1}$$

as it was needed, for example, in Kim-Kim-Oh-Zeng [22] or in Colasuonno-Squassina [11] and Liu-Dai [24] for the constant exponent case. Indeed, we only require assumption (H) (see Section 2), since Crespo-Blanco-Gasiński-Harjulehto-Winkert in [14, Proposition 2.19] recently proved that our function space $W_0^{1,\mathcal{H}}(\Omega)$ can be equipped with the equivalent norm $\|\nabla \cdot\|_{\mathcal{H}}$ without supposing (1.1).

The paper is organized as follows. In Section 2 we recall the main properties of the Musielak-Orlicz Sobolev spaces and the double phase operator, we present the variational framework and we formulate our assumptions. In Section 3 we present our main result (Theorem 3.2) on the existence of two nontrivial bounded weak solutions for problem (P_λ), we consider other hypotheses in order to get nonnegative solutions (Corollary 3.3 and Theorem 3.4) and we also provide an example.

2. Variational framework and preliminaries

In this section we present the main preliminaries in order to study problem (P_λ) and recall first the main results concerning variable exponent Lebesgue and Sobolev spaces as well as properties of Musielak-Orlicz Sobolev space and the corresponding double phase operator. These results can be mainly found in the books of Diening-Harjulehto-Hästö-Růžička [17], Harjulehto-Hästö [21] and Musielak [26], see also the papers of Colasuonno-Squassina [11], Crespo-Blanco-Gasiński-Harjulehto- Winkert [14] and Liu-Dai [24]. To this end, let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For $1 \leq r \leq \infty$ we denote by $L^r(\Omega)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ and for $1 \leq r < \infty$, $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ indicate the Sobolev spaces endowed with the usual norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0} = \|\nabla\cdot\|_r$, respectively. First, we introduce the Lebesgue and Sobolev spaces with variable exponents and some properties that will be useful in the sequel. For any $r \in C(\overline{\Omega})$, we put

$$r_+ := \max_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_- := \min_{x\in\overline{\Omega}} r(x),$$

and we define

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega}) \,:\, r_- > 1\}.$$

For any $r \in C_+(\overline{\Omega})$, we define the modular by

$$\rho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}\,\mathrm{d}x,$$

and the variable exponent Lebesgue space is given by

$$L^{r(\cdot)}(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ measurable} \mid \rho_{r(\cdot)}(u) < \infty\},$$

equipped with the related Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf\left\{\tau > 0 \,:\, \rho_{r(\cdot)}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

We can also introduce, for any $r \in C_+(\overline{\Omega})$, the corresponding Sobolev space with variable exponent, denoted by

$$W^{1,r(\cdot)}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{r(\cdot)}(\Omega)\right\},$$

endowed with the usual norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)},$$

where $\|\nabla u\|_{r(\cdot)} = \|\,|\nabla u|\,\|_{r(\cdot)}$. Furthermore, we denote by $W_0^{1,r(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,r(\cdot)}(\Omega)$ for which we know that a Poincaré inequality holds and we can equip the space with the equivalent norm $\|u\|_{1,r(\cdot),0} = \|\nabla u\|_{r(\cdot)}$. In particular, these spaces are uniformly convex, separable and reflexive Banach spaces and we refer to Diening-Harjulehto-Hästö-Růžička [17, Theorems 3.2.7, 3.4.7, 3.4.9, 8.1.6, 8.1.13 and Corollary 3.4.5]

Next, we recall some properties about the relation between the norm and the corresponding modular that will be needed in the sequel, see Fan-Zhao[18].

Proposition 2.1. *Let $r \in C_+(\overline{\Omega})$, $u \in L^{r(\cdot)}(\Omega)$ and $\lambda \in \mathbb{R}$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\|_{r(\cdot)} = \lambda \iff \rho_{r(\cdot)}(\frac{u}{\lambda}) = 1$;*
(ii) *$\|u\|_{r(\cdot)} < 1$ (resp. > 1, $= 1$) $\iff \rho_{r(\cdot)}(u) < 1$ (resp. > 1, $= 1$);*
(iii) *If $\|u\|_{r(\cdot)} < 1 \implies \|u\|_{r(\cdot)}^{r_+} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_-}$;*
(iv) *If $\|u\|_{r(\cdot)} > 1 \implies \|u\|_{r(\cdot)}^{r_-} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_+}$;*
(v) *$\|u\|_{r(\cdot)} \to 0 \iff \rho_{r(\cdot)}(u) \to 0$;*
(vi) *$\|u\|_{r(\cdot)} \to +\infty \iff \rho_{r(\cdot)}(u) \to +\infty$.*

Now, we introduce the Musielak-Orlicz space, the Musielak-Orlicz Sobolev space and the double phase operator, recalling some main and useful properties. We assume the following hypotheses:

(H) $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$ is the critical Sobolev exponent to $p(\cdot)$, and $\mu \in L^\infty(\Omega)$, with $\mu(\cdot) \geq 0$.

Let $\mathcal{H}\colon \Omega \times [0,\infty[\to [0,\infty[$ be the nonlinear function defined by

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x)t^{q(x)} \quad \text{for all } (x,t) \in \Omega \times [0,\infty[,$$

and let $\rho_{\mathcal{H}}(\cdot)$ be the corresponding modular defined by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right)\mathrm{d}x.$$

Then, we denote by $L^{\mathcal{H}}(\Omega)$ the Musielak-Orlicz space, given by

$$L^{\mathcal{H}}(\Omega) = \left\{u\colon \Omega \to \mathbb{R} \text{ measurable } \mid \rho_{\mathcal{H}}(u) < +\infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

Similar to Proposition 2.1, we have a certain relationship between the modular $\rho_{\mathcal{H}}(\cdot)$ and the norm $\|\cdot\|_{\mathcal{H}}$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [14, Proposition 2.13] for a detailed proof.

Proposition 2.2. *Let (H) be satisfied, $u \in L^{\mathcal{H}}(\Omega)$ and $\lambda \in \mathbb{R}$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\|_{\mathcal{H}} = \lambda \iff \rho_{\mathcal{H}}(\frac{u}{\lambda}) = 1$;*
(ii) *$\|u\|_{\mathcal{H}} < 1$ (resp. > 1, $= 1$) $\iff \rho_{\mathcal{H}}(u) < 1$ (resp. > 1, $= 1$);*
(iii) *If $\|u\|_{\mathcal{H}} < 1 \implies \|u\|_{\mathcal{H}}^{q_+} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{p_-}$;*
(iv) *If $\|u\|_{\mathcal{H}} > 1 \implies \|u\|_{\mathcal{H}}^{p_-} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{q_+}$;*
(v) *$\|u\|_{\mathcal{H}} \to 0 \iff \rho_{\mathcal{H}}(u) \to 0$;*
(vi) *$\|u\|_{\mathcal{H}} \to +\infty \iff \rho_{\mathcal{H}}(u) \to +\infty$;*
(vii) *$\|u\|_{\mathcal{H}} \to 1 \iff \rho_{\mathcal{H}}(u) \to 1$;*
(viii) *If $u_n \to u$ in $L^{\mathcal{H}}(\Omega)$, then $\rho_{\mathcal{H}}(u_n) \to \rho_{\mathcal{H}}(u)$.*

We denote by $W^{1,\mathcal{H}}(\Omega)$ the Musielak-Orlicz Sobolev space defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

with $\|\nabla u\|_{\mathcal{H}} = \| \, |\nabla u| \, \|_{\mathcal{H}}$ and by $W_0^{1,\mathcal{H}}(\Omega)$ we indicate the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. In [14, Proposition 2.12] the authors prove that $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are uniformly convex, so reflexive Banach spaces. Moreover, the following embedding results can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [14, Proposition 2.16].

Proposition 2.3. *Let* (H) *be satisfied. Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *are continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p(x)$ *for all* $x \in \overline{\Omega}$*;*

(ii) $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is compact for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*.*

Moreover, from Proposition 2.18 in [14], we have that $W_0^{1,\mathcal{H}}(\Omega)$ is compactly embedded in $L^{\mathcal{H}}(\Omega)$, so we can equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}}.$$

Now, for any $r \in C(\overline{\Omega})$ for which the continuous embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ hold (see Proposition 2.3), we denote by $\tilde{c}_r$ the best constant for which one has

$$\|u\|_{r(\cdot)} \leq \tilde{c}_r \|u\|_{1,\mathcal{H},0}. \tag{2.1}$$

Finally, we introduce the assumptions on the perturbation in problem (P_λ) and suppose the following hypotheses:

(H_f) Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ be such that the following hold:

(i) f is a Carathéodory function, that is, $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$ and $t \to f(x,t)$ is continuous for almost all (a.a.) $x \in \Omega$;

(ii) there exist $\ell \in C_+(\overline{\Omega})$ with $\ell_+ < (p_-)^*$ and $\kappa_1 > 0$ such that
$$|f(x,t)| \leq \kappa_1 \left(1 + |t|^{\ell(x)-1}\right)$$
for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(iii)
$$\lim_{t \to \pm\infty} \frac{F(x,t)}{|t|^{q_+}} = +\infty$$
uniformly for a.a. $x \in \Omega$;

(iv) there exists $\zeta \in C_+(\overline{\Omega})$ with
$$\zeta_- \in \left((\ell_+ - p_-)\frac{N}{p_-}, \ell_+ \right)$$
and $\zeta_0 > 0$ such that
$$0 < \zeta_0 \leq \liminf_{t \to \pm\infty} \frac{f(x,t)t - q_+ F(x,t)}{|t|^{\zeta(x)}}$$
uniformly for a.a. $x \in \Omega$.

Remark 2.4. *It should be noted that the condition on* ζ *in* (H_f)(iv) *is well defined since from* (H_f)(ii) *we have* $\ell_+ < (p_-)^*$ *and so it holds*

$$(\ell_+ - p_-)\frac{N}{p_-} = \ell_+ \frac{N}{p_-} - (p_-)^* \frac{N - p_-}{p_-} < \ell_+ \frac{N}{p_-} - \ell_+ \frac{N - p_-}{p_-} = \ell_+.$$

The differential operator in (P_λ) is the so-called double phase operator with variable exponents given by

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\qquad \text{for } u\in W_0^{1,\mathcal{H}}(\Omega).$$

It is well known that $u\in W_0^{1,\mathcal{H}}(\Omega)$ is called a weak solution of problem (P_λ) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x=\lambda\int_\Omega f(x,u)v\,\mathrm{d}x, \tag{2.2}$$

for all $v\in W_0^{1,\mathcal{H}}(\Omega)$. In order to establish results on the existence of two nontrivial weak solution for (P_λ), we define the functionals $\Phi,\Psi,I_\lambda\colon W_0^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$ by

$$\Phi(u)=\int_\Omega\left(\frac{|\nabla u|^{p(x)}}{p(x)}+\mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right)\mathrm{d}x,\quad \Psi(u)=\int_\Omega F(x,u(x))\,\mathrm{d}x,$$

and

$$I_\lambda(u)=\Phi(u)-\lambda\Psi(u),$$

where I_λ is the so-called energy functional. We know that Φ and Ψ are Gâteaux differentiable with its derivatives

$$\begin{aligned}
\langle\Phi'(u),v\rangle&=\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x,\\
\langle\Psi'(u),v\rangle&=\int_\Omega f(x,u)v\,\mathrm{d}x,\\
\langle I_\lambda'(u),v\rangle&=\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x-\lambda\int_\Omega f(x,u)v\,\mathrm{d}x,
\end{aligned}$$

for all $u,v\in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle\,\cdot\,,\cdot\,\rangle$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. Hence, from (2.2) it follows that u is a weak solution of (P_λ) if and only if u is a critical point of I_λ, i.e., $\langle I_\lambda'(u),v\rangle=0$ for all $v\in W_0^{1,\mathcal{H}}(\Omega)$. In the next proposition we summarize the properties of the operator $\Phi'\colon W_0^{1,\mathcal{H}}(\Omega)\to W_0^{1,\mathcal{H}}(\Omega)^*$, see [14, Theorem 3.3] which is a generalization of [24, Proposition 3.1] in the variable exponent case.

Proposition 2.5. *Let hypotheses* (H) *be satisfied. Then, the operator Φ' is bounded, continuous, strictly monotone and of type* (S_+)*, that is,*

$$\text{if}\quad u_n\rightharpoonup u\quad \text{in } W_0^{1,\mathcal{H}}(\Omega)\quad\text{and}\quad \limsup_{n\to\infty}\langle\Phi'(u_n),u_n-u\rangle\le 0,$$

then $u_n\to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

The main tool of our investigation is a two critical point theorem established by Bonanno-D'Aguì in [8, Theorem 2.1 and Remark 2.2], which is a nontrivial consequence of a local minimum theorem due to Bonanno [7, Theorem 2.3] in combination with the Ambrosetti-Rabinowitz Theorem. Here we recall the definition of the Cerami condition that will be needed. In the following, for X being a Banach space, we denote by X^* its topological dual space.

Definition 2.6. *Given $L\in C^1(X)$, we say that L satisfies the Cerami-condition,* (C)*-condition for short, if every sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq X$ such that*

(C_1) *$\{L(u_n)\}_{n\ge1}\subseteq\mathbb{R}$ is bounded,*
(C_2) *$(1+\|u_n\|_X)\,L'(u_n)\to 0$ in X^* as $n\to\infty$,*

admits a strongly convergent subsequence in X.

For the reader's convenience, we restate Theorem 2.1 [8] taking into account Remark 2.2 [8].

Theorem 2.7. *Let X be a real Banach space and let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that Φ is coercive and there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.3}$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the (C)*-condition and it is unbounded from below. Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.*

3. Main Result

In this section, we present our main result on the existence of two nontrivial solutions for the Dirichlet double phase problem with variational exponents given in (P_λ). For this purpose, let

$$R := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega).$$

Then there exists $x_0 \in \Omega$ such that the ball with center x_0 and radius $R > 0$ belongs to Ω, that is,

$$B(x_0, R) \subseteq \Omega.$$

We indicate with ω_R the the Lebesgue measure of $B(x_0, R)$ in $\mathbb{R}^N$ given by

$$\omega_R := |B(x_0, R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})} R^N,$$

and we put

$$\delta := \frac{\min\{R^{p_-}, R^{q_+}\}\, p_-}{\max\{1, \|\mu\|_\infty\}\, \omega_R\, (2^N - 1) 2^{q_+ + 1 - N}}.$$

Furthermore, for any $r, \eta \in \mathbb{R}^+$, we define

$$\alpha(r) := \kappa_1 \, \frac{\tilde{c}_1 \max\left\{ (q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}} \right\} + \bar{c}_\ell \max\left\{ (q_+ r)^{\frac{\ell_+}{p_-}}, (q_+ r)^{\frac{\ell_-}{q_+}} \right\}}{r}, \tag{3.1}$$

$$\beta(\eta) := \delta \, \frac{\displaystyle\int_{B\left(x_0, \frac{R}{2}\right)} F(x, \eta)\, \mathrm{d}x}{\max\{\eta^{p_-}, \eta^{q_+}\}}, \tag{3.2}$$

where $\bar{c}_\ell = \max\left\{\tilde{c}_\ell^{\ell_-}, \tilde{c}_\ell^{\ell_+}\right\}$ and $\tilde{c}_1$, $\tilde{c}_\ell$, κ_1, ℓ are given in (2.1) and (H_f)(ii), respectively. From now on, we put

$$X = W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \|\cdot\|_X = \|\cdot\|_{1,\mathcal{H},0} = \|\nabla\cdot\|_{\mathcal{H}}.$$

First, we present the following Lemma that we will use in the proof of the main result.

Lemma 3.1. *Let the assumptions* (H) *and* (H_f) *be satisfied. Then, the functional* I_λ *satisfies the* (C)*-condition for all* $\lambda > 0$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ be a sequence such that (C_1) and (C_2) from Definition 2.6 hold. We provide the proof in three steps.

Claim 1. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{\zeta_-}(\Omega)$.

First, it is easy to show that using (H_f)(i), (ii) and (iv) we get that

$$f(x,t)t - q_+F(x,t) \geq c_1|t|^{\zeta_-} - c_2 \quad \text{for a.a. } x\in\Omega \text{ and for all } t\in\mathbb{R}, \tag{3.3}$$

with some constants $c_1, c_2 > 0$. Moreover, from (C_1) we have that there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$ one has $|I_\lambda(u_n)| \leq M$, so

$$\left|\int_\Omega \left(\frac{|\nabla u_n|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u_n|^{q(x)}}{q(x)}\right) \mathrm{d}x - \lambda\int_\Omega F(x,u_n)\,\mathrm{d}x\right| \leq M,$$

which, multiplying by q_+, leads to

$$\rho_{\mathcal{H}}(\nabla u_n) - \lambda\int_\Omega q_+F(x,u_n)\,\mathrm{d}x \leq c_3, \tag{3.4}$$

for some $c_3 > 0$ and for all $n \in \mathbb{N}$. Besides, from (C_2), there exists $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0^+$ such that

$$|\langle I'_\lambda(u_n), v\rangle| \leq \frac{\varepsilon_n\|v\|_X}{1+\|u_n\|_X} \quad \text{for all } n\in\mathbb{N} \text{ and for all } v\in X. \tag{3.5}$$

Choosing $v = u_n$, one has

$$\left|\int_\Omega \left(|\nabla u_n|^{p(x)} + \mu(x)|\nabla u_n|^{q(x)}\right) \mathrm{d}x - \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x\right| < \varepsilon_n,$$

which implies

$$-\rho_{\mathcal{H}}(\nabla u_n) + \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x < \varepsilon_n \tag{3.6}$$

for all $n \in \mathbb{N}$. Adding (3.4) and (3.6) we obtain

$$\int_\Omega (f(x,u_n)u_n - q_+F(x,u_n))\,\mathrm{d}x < c_4$$

for all $n \in \mathbb{N}$ with some constant $c_4 > 0$. Using this along with (3.3) we derive

$$\int_\Omega \left(c_1|u_n|^{\zeta_-} - c_2\right) \mathrm{d}x < c_4,$$

which gives

$$\|u_n\|_{\zeta_-}^{\zeta_-} < c_5 \quad \text{for all } n\in\mathbb{N}$$

with some $c_5 > 0$. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{\zeta_-}(\Omega)$ and so Claim 1 is proved.

Claim 2. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.

From (H_f)(ii) and (iv), we have that

$$\zeta_- < \ell_+ < (p_-)^*.$$

Hence, there exists $s \in (0,1)$ such that

$$\frac{1}{\ell_+} = \frac{s}{(p_-)^*} + \frac{1-s}{\zeta_-}, \tag{3.7}$$

and using the interpolation inequality, see Papageorgiou-Winkert [28, Proposition 2.3.17 p.116], one has

$$\|u_n\|_{\ell_+} \le \|u_n\|_{(p_-)^*}^{s} \|u_n\|_{\zeta_-}^{1-s} \quad \text{for all } n \in \mathbb{N}.$$

From Claim 1, it follows that

$$\|u_n\|_{\ell_+} \le c_6 \|u_n\|_{(p_-)^*}^{s} \tag{3.8}$$

for some $c_6 > 0$ and for all $n \in \mathbb{N}$. Again, from (3.5) with $v = u_n$, we get

$$\rho_{\mathcal{H}}(\nabla u_n) - \lambda \int_\Omega f(x,u_n)u_n \,\mathrm{d}x < \varepsilon_n. \tag{3.9}$$

We may assume $\|u_n\|_X \ge 1$ for all $n \in \mathbb{N}$, otherwise we are done. Then, using Proposition 2.2(iv), (3.9), (H_f)(ii) and (3.8), we derive that

$$\begin{aligned}
\|u_n\|_X^{p_-} &\le \rho_{\mathcal{H}}(\nabla u_n) < \varepsilon_n + \lambda \int_\Omega f(x,u_n)u_n \,\mathrm{d}x \\
&\le \lambda \kappa_1 \left(\|u_n\|_1 + \rho_{\ell(\cdot)}(u_n)\right) + \varepsilon_n \\
&\le c_7 \left(1 + c_6^{\ell_+} \|u_n\|_{(p_-)^*}^{s\ell_+}\right) + \varepsilon_n
\end{aligned}$$

with $c_7 > 0$. Hence, taking the embedding $X \hookrightarrow L^{(p_-)^*}(\Omega)$ into account, we have

$$\|u_n\|_X^{p_-} \le c_8 \left(1 + \|u_n\|_X^{s\ell_+}\right) + \varepsilon_n,$$

for all $n \in \mathbb{N}$ and for some $c_8 > 0$. From (3.7) and (H_f)(iv), it follows that

$$\begin{aligned}
s\ell_+ &= \frac{(p_-)^*(\ell_+ - \zeta_-)}{(p_-)^* - \zeta_-} = \frac{Np_-(\ell_+ - \zeta_-)}{Np_- - N\zeta_- + \zeta_- p_-} \\
&< \frac{Np_-(\ell_+ - \zeta_-)}{Np_- - N\zeta_- + p_-(\ell_+ - p_-)\frac{N}{p_-}} = p_-,
\end{aligned}$$

and this shows our second claim.

Claim 3. $u_n \to u$ in X up to a subsequence.

Since $\{u_n\}_{n\in\mathbb{N}} \subset X$ is bounded (Claim 2) and X is a reflexive space, there exists a subsequence, not relabeled, that converges weakly in X and strongly in $L^{\ell_+}(\Omega)$, that is,

$$u_n \rightharpoonup u \quad \text{in X} \quad \text{and} \quad u_n \to u \quad \text{in } L^{\ell_+}(\Omega).$$

Using this to (3.5) with $v = u_n - u$ and passing to the limit as $n \to \infty$, we obtain

$$\langle \Phi'(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

Since Φ' satisfies the (S_+)-property, see Proposition 2.5, the proof is complete. □

Now, we state our main result.

Theorem 3.2. *Assume that* (H) *and* (H_f) *hold. Furthermore, suppose that there exist two positive constants r, η satisfying*

$$\max\{\eta^{p_-}, \eta^{q_+}\} < \delta r, \tag{3.10}$$

such that

(H_1) $F(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and for all* $t \in [0, \eta]$*;*

(H_2) $\alpha(r) < \beta(\eta)$,

as defined in (3.1) *and* (3.2)*. Then, for each* $\lambda \in \Lambda$*, where*

$$\Lambda := \left] \frac{1}{\beta(\eta)}, \frac{1}{\alpha(r)} \right[,$$

problem (P_λ) *admits at least two nontrivial bounded weak solutions* $u_{\lambda,1}$*,* $u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ *with opposite energy sign.*

Proof. Our aim is to apply Theorem 2.7. Let $(X, \|\cdot\|_X), \Phi, \Psi$ be as in Section 2 and note that they already fulfill the required assumptions needed in Theorem 2.7. In particular, from Proposition 2.2(vi) and (H_f)(iii) follows that Φ is coercive and I_λ is unbounded from below, respectively.

Now, fix $\lambda \in \Lambda$, which is nonempty because of (H_2), and consider $\tilde{u} \in X$ defined by

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \dfrac{2\eta}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right), \\ \eta & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

Clearly, $\tilde{u} \in X$. We show that $0 < \Phi(\tilde{u}) < r$. Indeed, using (3.10), it follows

$$\begin{aligned} \Phi(\tilde{u}) &= \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left(\frac{1}{p(x)} \left(\frac{2\eta}{R}\right)^{p(x)} + \frac{\mu(x)}{q(x)} \left(\frac{2\eta}{R}\right)^{q(x)} \right) \mathrm{d}x \\ &\leq \frac{2^{q_+}}{p_-} \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left(\left(\frac{\eta}{R}\right)^{p(x)} + \mu(x) \left(\frac{\eta}{R}\right)^{q(x)} \right) \mathrm{d}x \\ &\leq \frac{2^{q_+}}{p_-} \frac{\max\{1, \|\mu\|_\infty\}}{\min\{R^{p_-}, R^{q_+}\}} \max\{\eta^{p_-}, \eta^{q_+}\} \cdot 2 \cdot \left(\omega_R - \omega_{\frac{R}{2}}\right) \\ &= \frac{1}{\delta} \max\{\eta^{p_-}, \eta^{q_+}\} < r. \end{aligned}$$

Now, we prove (2.3). From assumption (H_1), we obtain

$$\begin{aligned} \Psi(\tilde{u}) &= \int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\, \mathrm{d}x + \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} F\left(x, \frac{2\eta}{R}(R - |x - x_0|)\right) \mathrm{d}x \\ &\geq \int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\, \mathrm{d}x. \end{aligned}$$

Hence,

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \delta\, \frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\, \mathrm{d}x}{\max\{\eta^{p_-}, \eta^{q_+}\}}. \tag{3.11}$$

On the other hand, fix $u \in X$ such that $\Phi(u) \leq r$. Then, one has

$$q_+ r \geq q_+ \Phi(u) > \rho_{\mathcal{H}}(\nabla u) \geq \min\left\{\|u\|_X^{p_-}, \|u\|_X^{q_+}\right\},$$

which implies that

$$\Phi^{-1}\left(]-\infty, r]\right) \subseteq \left\{u \in X \, : \, \|u\|_X < \max\left\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}}\right\}\right\}.$$

Furthermore, we have

$$\begin{aligned}
&\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)\\
&\leq \sup_{u \in \Phi^{-1}(]-\infty,r])} \kappa_1 \int_\Omega \left(|u| + |u|^{\ell(x)}\right) \mathrm{d}x\\
&= \sup_{u \in \Phi^{-1}(]-\infty,r])} \kappa_1 \left(\|u\|_1 + \rho_{\ell(\cdot)}(u)\right)\\
&\leq \sup_{u \in \Phi^{-1}(]-\infty,r])} \kappa_1 \left(\|u\|_1 + \max\left\{\|u\|_{\ell(\cdot)}^{\ell_-}, \|u\|_{\ell(\cdot)}^{\ell_+}\right\}\right)\\
&\leq \sup_{u \in \Phi^{-1}(]-\infty,r])} \kappa_1 \left(\tilde{c}_1 \|u\|_X + \max\left\{\tilde{c}_\ell^{\ell_-}, \tilde{c}_\ell^{\ell_+}\right\} \max\left\{\|u\|_X^{\ell_-}, \|u\|_X^{\ell_+}\right\}\right)\\
&\leq \kappa_1 \left(\tilde{c}_1 \max\left\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}}\right\} + \bar{c}_\ell \max\left\{(q_+ r)^{\frac{\ell_+}{p_-}}, (q_+ r)^{\frac{\ell_-}{q_+}}\right\}\right).
\end{aligned}$$

Then, taking (H$_2$) and (3.11) into account, we get

$$\begin{aligned}
&\frac{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{r}\\
&\leq \frac{\kappa_1 \left(\tilde{c}_1 \max\left\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}}\right\} + \bar{c}_\ell \max\left\{(q_+ r)^{\frac{\ell_+}{p_-}}, (q_+ r)^{\frac{\ell_-}{q_+}}\right\}\right)}{r}\\
&< \delta \, \frac{\displaystyle\int_{B\left(x_0, \frac{R}{2}\right)} F(x, \eta) \, \mathrm{d}x}{\max\left\{\eta^{p_-}, \eta^{q_+}\right\}} \leq \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},
\end{aligned}$$

namely hypothesis (2.3) is satisfied. Hence, along with Lemma 3.1, Theorem 2.7 ensures the existence of two nontrivial weak solutions $u_{\lambda,1}$, $u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$. Finally, from Crespo-Blanco-Winkert [15, Theorem 3.1] it follows that $u_{\lambda,1}$, $u_{\lambda,2}$ belong to $L^\infty(\Omega)$. This finishes the proof. $\square$

Corollary 3.3. *Suppose that all assumptions of Theorem 3.2 are satisfied. Moreover, assume that $f(x,0) \geq 0$ and $f(x,t) = f(x,0)$ for a.a. $x \in \Omega$ and for all $t < 0$. Then, problem (P_λ) admits at least two nontrivial and nonnegative bounded weak solutions $u_{\lambda,1}$, $u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ with opposite energy sign.*

Proof. Since all the assumptions are satisfied, we can apply Theorem 3.2. We only need to prove that the solutions $u_{\lambda,1}, u_{\lambda,2}$ are nonnegative. Since $u_{\lambda,1}$ is a weak solution of (P_λ), from (2.2) one has $\langle I'_\lambda(u_{\lambda,1}), v\rangle = 0$ for every $v \in X$. Choosing $v = -u_{\lambda,1}^- = -\max\{-u_{\lambda,1}, 0\} \in W_0^{1,\mathcal{H}}(\Omega)$, see [14, Proposition 2.17(iii)], we have

$$\int_\Omega \left(|\nabla u_{\lambda,1}|^{p(x)-2} \nabla u_{\lambda,1} + \mu(x) |\nabla u_{\lambda,1}|^{q(x)-2} \nabla u_{\lambda,1}\right) \cdot \nabla\left(-u_{\lambda,1}^-\right) \mathrm{d}x$$

$$= \lambda \int_\Omega f(x, u_{\lambda,1}) \left(-u_{\lambda,1}^-\right) \mathrm{d}x,$$

which leads to

$$-\rho_{\mathcal{H}}(\nabla u_{\lambda,1}^-) \geq 0.$$

But the previous inequality implies that

$$\min\left\{\|u_{\lambda,1}^-\|_X^{p_-}, \|u_{\lambda,1}^-\|_X^{q_+}\right\} \leq \rho_{\mathcal{H}}(\nabla u_{\lambda,1}^-) \leq 0,$$

which gives $\|u_{\lambda,1}^-\|_X = 0$. Then, $u_{\lambda,1}^- = 0$ and $u_{\lambda,1} \geq 0$. With the same argument we obtain $u_{\lambda,2} \geq 0$ and the proof is complete. □

Now we consider the special case when the nonlinear term is nonnegative.

Theorem 3.4. *Assume that* (H) *and* (H_f) *hold. Furthermore, suppose that f is nonnegative and*

$$\limsup_{t\to 0^+} \frac{\inf\limits_{x\in\Omega} F(x,t)}{t^{p_-}} = +\infty. \tag{H_3}$$

Then, for each $\lambda \in]0, \lambda^*[$, with

$$\lambda^* = \sup_{r>0} \frac{1}{\alpha(r)},$$

where $\alpha(r)$ is given in (3.1)*, problem* (P_λ) *admits at least two nontrivial and nonnegative bounded weak solutions $u_{\lambda,1}$, $u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ with opposite energy sign.*

Proof. We observe that (H_3) implies that

$$\begin{aligned}
\limsup_{\eta\to 0^+} \beta(\eta) &= \limsup_{\eta\to 0^+} \delta \, \frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\eta)\,\mathrm{d}x}{\max\left\{\eta^{p_-}, \eta^{q_+}\right\}} \\
&\geq \delta\, \omega_{\frac{R}{2}} \limsup_{\eta\to 0^+} \frac{\inf\limits_{x\in\Omega} F(x,\eta)}{\eta^{p_-}} = +\infty.
\end{aligned} \tag{3.12}$$

Then, fixing $\lambda \in]0, \lambda^*[$, there exists $r > 0$ such that

$$\lambda < \frac{1}{\alpha(r)} = \frac{r}{\kappa_1 \left(\tilde{c}_1 \max\left\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+}}\right\} + \bar{c}_\ell \max\left\{(q_+ r)^{\frac{\ell_+}{p_-}}, (q_+ r)^{\frac{\ell_-}{q_+}}\right\}\right)}.$$

Moreover, from (3.12), there is $\eta > 0$ small enough such that

$$\delta\, \omega_{\frac{R}{2}} \frac{\inf\limits_{x\in\Omega} F(x,\eta)}{\eta^{p_-}} > \frac{1}{\lambda},$$

implying that $\alpha(r) < \beta(\eta)$. Applying Theorem 3.2 and arguing as in the proof of Corollary 3.3, we achieve our goal. □

Finally, we provide an example of a function that satisfies our assumptions.

Example 3.5. *Consider* $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x,t) = \begin{cases} |t|^{\alpha(x)-2}t & \text{if } |t| < 1, \\ |t|^{\beta(x)-2}t\,(\log|t|+1) & \text{if } |t| \geq 1, \end{cases}$$

where $\alpha, \beta \in C(\overline{\Omega})$ *such that* $q_+ < \beta(x) < (p_-)^*$ *for all* $x \in \overline{\Omega}$ *and*

$$\frac{\beta_+}{p_-} - \frac{\beta_-}{N} < 1.$$

Then, f *satisfies assumptions* (H_f) *with* $\zeta(x) = \beta(x)$ *for all* $x \in \overline{\Omega}$ *and* $\ell(x) = \beta(x) + \sigma$ *for all* $x \in \overline{\Omega}$*, with* $\sigma > 0$ *small enough such that*

$$\ell_+ < (p_-)^*,$$
$$\frac{\ell_+}{p_-} - \frac{\beta_-}{N} < 1.$$

Moreover, we can apply Theorem 3.4 at $f_+(x,t) = |f(x,t)|$ *for every* $(x,t) \in \Omega \times \mathbb{R}$*, requiring also that* $\alpha(x) < p_-$ *for all* $x \in \overline{\Omega}$*.*

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