MULTIPLE SIGN-CHANGING SOLUTIONS FOR SUPERLINEAR (p,q)-EQUATIONS IN SYMMETRICAL EXPANDING DOMAINS

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ABSTRACT. In this paper we study quasilinear elliptic equations defined on symmetrical expanding domains driven by the (p,q)-Laplacian and with a superlinear right-hand side. Based on the Lusternik-Schnirelmann category we prove the existence of at least $\gamma(\Omega_\lambda \setminus \{0\})$ pairs $(\pm u)$ of odd weak solutions with precisely two nodal domains, where γ stands for the genus.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $\Omega_\lambda := \lambda\Omega$ be an expanding domain, where λ is a positive parameter. In this paper we consider the following problem

$$\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(u) - |u|^{p-2}u && \text{in } \Omega_\lambda,\\ u &= 0 && \text{on } \partial\Omega_\lambda,\\ u(-x) &= -u(x) && \text{for a. a. } x \in \Omega_\lambda, \end{aligned} \tag{1.1}$$

where we suppose the following assumptions:

(H1) $\mu > 0$ and $1 < q < p < N$.

(H2) $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous and odd function with primitive $F(s) = \int_0^s f(t)\,\mathrm{d}t$ satisfying the following conditions:

  (i) there exist $r \in (p, p^*)$ and a constant $C > 0$ such that
  $$|f(s)| \leq C\left(1 + |s|^{r-1}\right) \quad \text{for all } s \in \mathbb{R},$$
  where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent to p;

  (ii) $\displaystyle\lim_{s\to 0} \frac{f(s)}{|s|^{q-2}s} = 0$;

  (iii) $\displaystyle\lim_{|s|\to+\infty} \frac{F(s)}{|s|^p} = +\infty$;

  (iv) $\displaystyle\frac{f(s)}{|s|^{p-1}}$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

A function $u \in W_0^{1,p}(\Omega_\lambda)$ is said to be a weak solution of problem (1.1) if $u(-x) = -u(x)$ for a. a. $x \in \Omega_\lambda$ and if

$$\int_{\Omega_\lambda} \left(|\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x = \int_{\Omega_\lambda} \left(f(u) - |u|^{p-2}u\right) v \,\mathrm{d}x$$

is satisfied for all $v \in W_0^{1,p}(\Omega_\lambda)$. The corresponding energy functional $J_\lambda\colon W_0^{1,p}(\Omega_\lambda) \to \mathbb{R}$ for problem (1.1) is given by

$$J_\lambda(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|\nabla u\|_q^q - \int_{\Omega_\lambda} F(u)\,\mathrm{d}x \quad \text{for all } u \in W_0^{1,p}(\Omega_\lambda). \tag{1.2}$$

Under the assumptions in (H1) and (H2), it is clear that J_λ is well-defined and of class C^1.

The following theorem is our main result.

2020 *Mathematics Subject Classification.* 35J15, 35J62, 35J92, 35P30.

Key words and phrases. Lusternik-Schnirelmann category, (p,q)-equation, sign-changing solution, superlinear problem, symmetrical expanding domain.

Theorem 1.1. *Let hypotheses* (H1) *and* (H2) *be satisfied and let* Ω *be symmetric with respect to the origin, that is,* $\Omega = -\Omega$. *Then there exists* $\lambda^* > 0$ *such that, for any* $\lambda \geqslant \lambda^*$, *problem* (1.1) *has at least* $\gamma(\Omega_\lambda \setminus \{0\})$ *pairs* $(\pm u)$ *of odd weak solutions with precisely two nodal domains, where* γ *stands for the genus.*

The proof of Theorem 1.1 relies on the Lusternik-Schnirelmann category in combination with the odd symmetry invariant Nehari submanifold. As far as we know this is the first work dealing with a superlinear (p,q)-equation in expanding domains that has multiple sign-changing solutions obtained via the Lusternik-Schnirelmann category.

A starting point in the direct application of the Lusternik-Schnirelmann category to elliptic equations was the work of Benci-Cerami [11] who studied the problem

$$\begin{aligned} -\Delta u + \lambda u &= u^{p-1} && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $p \in (2, 2^*)$. It is shown that problem (1.3) has at least $\operatorname{cat}(\Omega)$ solutions when p is close to 2^*, where $\operatorname{cat}(\Omega)$ denotes the Lusternik-Schnirelmann category of Ω. Motivated by this work and its used methods, Bartsch-Wang [9] treated nonlinear Schrödinger equations of the form

$$-\Delta u + (\lambda a(x) + 1)u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

with $1 < p < 2^* - 1$ and showed the existence of at least $\operatorname{cat}(\Omega)$ solutions of (1.4) when the parameter $\lambda > 0$ is large enough, see also [8] of the same authors. Afterwards, the Lusternik-Schnirelmann category has been applied to several type of problems. We mention, for example, the works of Alves [2] for p-Laplace equations with expanding domains, Alves-Ding [3] for critical p-Laplace equations, Alves-Figueiredo-Furtado [4] for multiple solutions for nonlinear Schrödinger equations with magnetic fields, Benci-Bonanno-Micheletti [10] for elliptic equations on Riemannian manifolds, Cingolani [16] for nonlinear Schrödinger equations with an external magnetic field, Cingolani-Lazzo [17] for nonlinear Schrödinger equations, Figueiredo-Pimenta-Siciliano [20] for fractional Laplacian in expanding domains, Figueiredo-Siciliano [21] for fractional Schrödinger equations in $\mathbb{R}^N$ and Wang-Tian-Xu-Zhang [26] for Kirchhoff type problems, see also the references therein. All these works are dealing with constant sign solutions.

For sign-changing solutions via the Lusternik-Schnirelmann category we refer to the paper of Castro-Clapp [14] in which the problem

$$\begin{aligned} \Delta u + \lambda u + |u|^{2^*-2}u &= 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\\ u(\tau x) &= -u(x) && \text{for all } x \in \Omega \end{aligned} \tag{1.5}$$

was studied where τ is a nontrivial orthogonal involution. For $\lambda > 0$ to be small, the existence of pairs of sign-changing solutions which change the sign exactly once has been shown for problem (1.5). These results have been improved by Cano-Clapp [13]. Finally, we mention some results concerning problems with expanding domains, see, for example the papers of Ackermann-Clapp-Pacella [1] for alternating sign multibump solutions in expanding tubular domains, Alves-Figueiredo-Furtado [5] for complex equations, Bartsch-Clapp-Grossi-Pacella [7] for asymptotically radial solutions in expanding domains, Byeon-Tanaka [12] for multibump positive solutions in expanding tubular domains, Catrina-Wang [15] for Dirichlet Laplace problems in an expanding annulus, Dancer-Yan [18] for multibump solutions and Feireisl-Nečasová-Sun [19] for inviscid incompressible limits on expanding domains.

The paper is organized as follows. In Section 2 we recall some basic definitions and investigate the relation between the unit sphere and the odd symmetry invariant Nehari manifold. Section 3 is devoted to the (PS)-condition property and some needed estimates and in Section 4 we prove Theorem 1.1. Our results are combining ideas from the work of Alves [2], Castro-Clapp [14] and Catrina-Wang [15].

2. The mapping between $\mathcal{S}_\pm^\circ$ and $\mathcal{N}_\pm^\circ$

We denote by $L^s(\Omega)$ (resp. $L^s(\Omega;\mathbb{R}^N)$) and $L^s(\Omega_\lambda)$ (resp. $L^s(\Omega_\lambda;\mathbb{R}^N)$) the usual Lebesgue spaces equipped with the norm $\|\cdot\|_s$ for every $1\le s<\infty$. For $1<s<\infty$, $W^{1,s}(\Omega)$ and $W_0^{1,s}(\Omega_\lambda)$ stand for the Sobolev spaces endowed with the norm $\|\cdot\|_{1,s}$.

Let X be a Banach space and let $\mathcal{A}$ be the class of all closed subsets B of $X\setminus\{0\}$ which are symmetric, that is, $u\in B$ implies $-u\in B$.

Definition 2.1. *Let $B\in\mathcal{A}$. The genus $\gamma(B)$ of B is defined as the least integer n such that there exists $\varphi\in C(X,\mathbb{R}^n)$ such that φ is odd and $\varphi(x)\neq 0$ for all $x\in B$. We set $\gamma(B)=+\infty$ if there are no integers with the above property and $\gamma(\emptyset)=0$.*

Remark 2.2. *An equivalent way to define $\gamma(B)$ is to take the minimal integer n such that there exists an odd map $\varphi\in C(B,\mathbb{R}^n\setminus\{0\})$.*

For a function u, from now on, we denote by u^+ (resp. u^-) the positive (resp. negative) part of u, that is

$$u^+=\max(u,0),\quad u^-=\min(u,0). \tag{2.1}$$

Let

$$W_0^{1,p}(\Omega_\lambda)^\circ:=\left\{u\in W_0^{1,p}(\Omega_\lambda):\ u(-x)=-u(x)\right\}.$$

We denote the Nehari manifold corresponding to (1.1) by

$$\mathcal{N}_\lambda:=\left\{u\in W_0^{1,p}(\Omega_\lambda)\setminus\{0\}:\langle J_\lambda'(u),u\rangle=0\right\}$$

and the odd symmetry invariant Nehari submanifold by

$$\mathcal{N}_\lambda^\circ:=\{u\in\mathcal{N}_\lambda:\ u(-x)=-u(x)\}.$$

It is clear that

$$\mathcal{N}_\lambda^\circ=\mathcal{N}_\lambda\cap W_0^{1,p}(\Omega_\lambda)^\circ.$$

Note that $J_\lambda\colon W_0^{1,p}(\Omega_\lambda)^\circ\to\mathbb{R}$ is an even functional with $(J_\lambda(-u))'=-J_\lambda'(u)$. Therefore, if $J_\lambda\in C^2$, then the nontrivial solutions of (1.1) are the critical points of the restriction of J_λ to the odd symmetry invariant Nehari submanifold $\mathcal{N}_\lambda^\circ$. However, we only assume that f is continuous. This leads to $J_\lambda\in C^1$ and the non-differentiability of $\mathcal{N}_\lambda^\circ$. To overcome these difficulties, we need the following two lemmas.

We write

$$\mathcal{S}^\circ=\left\{u\in W_0^{1,p}(\Omega_\lambda)^\circ\ :\ \|u\|_{1,p}=1\right\},\ \mathcal{S}_\pm^\circ=\left\{u^\pm:u\in\mathcal{S}^\circ\right\}\ \text{and}\ \mathcal{N}_\pm^\circ=\left\{u^\pm:u\in\mathcal{N}_\lambda^\circ\right\}.$$

Then we can set up a one-to-one correspondence between $\mathcal{S}_\pm^\circ$ and $\mathcal{N}_\pm^\circ$ as follows.

Lemma 2.3. *Let hypotheses (H1) and (H2) be satisfied.*

(i) *For each $w\in W_0^{1,p}(\Omega_\lambda)^\circ\setminus\{0\}$, set $h_{w^\pm}(t)=J_\lambda(tw^\pm)$ for $t\ge 0$. Then there exists a unique $t_{w^\pm}>0$ such that $h'_{w^\pm}(t)>0$ if $0<t<t_{w^\pm}$ and $h'_{w^\pm}(t)<0$ if $t>t_{w^\pm}$, that is, $\max\limits_{t\in[0,+\infty)}h_{w^\pm}(t)$ is achieved at $t=t_{w^\pm}$ and $t_{w^\pm}w^\pm\in\mathcal{N}_\pm^\circ$.*

(ii) *There exists $\delta>0$ such that $t_{w^\pm}\geqslant\delta$ for $w\in\mathcal{S}_\pm^\circ$ and for each compact subset $\mathcal{W}^\circ\subseteq\mathcal{S}_\pm^\circ$ there exists a constant $C_{\mathcal{W}^\circ}$ such that $t_{w^\pm}\leqslant C_{\mathcal{W}^\circ}$ for all $w\in\mathcal{W}^\circ$.*

Proof. (i) Let $w\in W_0^{1,p}(\Omega_\lambda)^\circ\setminus\{0\}$ be fixed and define $h_{w^\pm}(t)=J_\lambda(tw^\pm)$ on $[0,\infty)$. It is clear that $h_{w^\pm}(0)=0$. From (H2)(i) and (H2)(ii) we know that for given $\varepsilon>0$ we can find $C_\varepsilon>0$ such that

$$|F(s)|\le\varepsilon|s|^q+C_\varepsilon|s|^r\quad\text{for a. a. }x\in\Omega\text{ and for all }s\in\mathbb{R}. \tag{2.2}$$

Using (2.2) and the embedding $W_0^{1,q}(\Omega_\lambda)\to L^q(\Omega_\lambda)$ with embedding constant $C_q>0$ we get for $t>0$

$$h_{w^\pm}(t)=J_\lambda(tw^\pm)=\frac{t^p}{p}\|w^\pm\|_{1,p}^p+\frac{\mu t^q}{q}\|\nabla w^\pm\|_q^q-\int_{\Omega_\lambda}F(tw^\pm)\,\mathrm{d}x$$

$$\begin{aligned}&\geq \frac{t^p}{p}\|w^\pm\|_{1,p}^p+\frac{\mu t^q}{q}\|\nabla w^\pm\|_q^q-\int_{\Omega_\lambda}\left(\varepsilon t^q|w^\pm|^q+C_\varepsilon t^r|w^\pm|^r\right)\mathrm{d}x\\&\geq \frac{t^p}{p}\|w^\pm\|_{1,p}^p+\left(\frac{\mu}{q}-C_q^q\varepsilon\right)t^q\|\nabla w^\pm\|_q^q-C_\varepsilon t^r\|w^\pm\|_r^r\\&=C_1t^p+C_2t^q-C_3t^r\quad\text{for } 0<\varepsilon<\frac{\mu}{qC_q^q}\end{aligned}$$

with $C_1, C_2, C_3 > 0$. Hence, for $t > 0$ small enough we see that $h_{w^\pm}(t) > 0$ due to $q < p < r$.

From hypothesis (H2)(iii) there exists for any $M > 0$ a number $T_M > 0$ such that

$$F(s) \geq M|s|^p \quad \text{for a. a. } x\in\Omega \text{ and for all } |s|>T_M. \tag{2.3}$$

Taking (2.3) into account, we have for $t > 0$ large

$$\begin{aligned}h_{w^\pm}(t)=J_\lambda(tw^\pm)&\leq \frac{t^p}{p}\|w^\pm\|_{1,p}^p+\frac{\mu t^q}{q}\|\nabla w^\pm\|_q^q-M\int_{\Omega_\lambda}t^p|w^\pm|^p\,\mathrm{d}x\\&=C_1t^p+C_2t^q-C_3Mt^p\\&\leqslant -C_4t^p+C_2t^q\quad\text{for } M>\frac{C_1}{C_3},\end{aligned}$$

with $C_1, C_2, C_3, C_4 > 0$. This implies that $h_{w^\pm}(t) < 0$ for t large enough. Hence there exists $t_{w^\pm} > 0$ such that $h'_{w^\pm}(t_{w^\pm}) = 0$. Note that

$$0=h'_{w^\pm}(t)=t^{p-1}\|w^\pm\|_{1,p}^p+\mu t^{q-1}\|\nabla w^\pm\|_q^q-\int_{\Omega_\lambda}f(tw^\pm)w^\pm\,\mathrm{d}x$$

implies $tw^\pm \in \mathcal{N}_\pm^\circ$ and

$$\begin{aligned}\|w^\pm\|_{1,p}^p&=\int_{\Omega_\lambda}\frac{f(tw^\pm)w^\pm}{t^{p-1}}\,\mathrm{d}x-\frac{\mu}{t^{p-q}}\|\nabla w^\pm\|_q^q\\&=\begin{cases}\displaystyle\int_{\Omega_\lambda^>}\frac{f(tw^+)w^+}{t^{p-1}}\,\mathrm{d}x-\frac{\mu}{t^{p-q}}\|\nabla w^\pm\|_q^q,\\ \displaystyle\int_{\Omega_\lambda^<}\frac{f(tw^-)w^-}{t^{p-1}}\,\mathrm{d}x-\frac{\mu}{t^{p-q}}\|\nabla w^\pm\|_q^q,\end{cases}\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}\Omega_\lambda^>&=\{x\in\Omega_\lambda\,:\,w(x)>0\},\\\Omega_\lambda^<&=\{x\in\Omega_\lambda\,:\,w(x)<0\}\end{aligned}$$

and w^+ (resp. w^-) is the positive (resp. negative) part of w, given in (2.1). By (H2)(iv), the right-hand side of (2.4) is a strictly increasing function in t. It follows that $h_{w^\pm}(t)$ has a unique critical point. Therefore $\max_{t\in[0,+\infty)} h_{w^\pm}(t)$ is achieved at the unique point $t = t_{w^\pm} > 0$ so that $h'_{w^\pm}(t_{w^\pm}) = 0$ and $t_{w^\pm}w^\pm \in \mathcal{N}_\pm^\circ$.

(ii) First, we prove that there exists $\delta > 0$ such that $t_{w^\pm} > \delta$ for any $w \in \mathcal{S}_\pm^\circ$. From (H2)(i) and (H2)(ii) we know that for given $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that

$$|f(s)|\leq\varepsilon|s|^{q-1}+C_\varepsilon|s|^{r-1}\quad\text{for a. a. } x\in\Omega \text{ and for all } s\in\mathbb{R}. \tag{2.5}$$

Let $w^\pm \in \mathcal{S}_\pm^\circ$. Using $t_{w^\pm}w^\pm \in \mathcal{N}_\pm^\circ$, (2.5) and the embeddings $W_0^{1,q}(\Omega_\lambda) \to L^q(\Omega_\lambda)$, $W_0^{1,p}(\Omega_\lambda) \to L^r(\Omega_\lambda)$ with embedding constants $C_q, C_p > 0$ we obtain

$$\begin{aligned}t_{w^\pm}^p\|w^\pm\|_{1,p}^p+\mu t_{w^\pm}^q\|\nabla w^\pm\|_q^q&=\int_{\Omega_\lambda}f(t_{w^\pm}w^\pm)t_{w^\pm}w^\pm\,\mathrm{d}x\\&\leq \varepsilon t_{w^\pm}^q\int_{\Omega_\lambda}|w^\pm|^q\,\mathrm{d}x+C_\varepsilon t_{w^\pm}^r\int_{\Omega_\lambda}|w^\pm|^r\,\mathrm{d}x\\&\leq C_q^q\varepsilon t_{w^\pm}^q\|\nabla w^\pm\|_q^q+C_p^rC_\varepsilon t_{w^\pm}^r\|w^\pm\|_{1,p}^r.\end{aligned}$$

Choosing $\varepsilon \in (0, \frac{\mu}{C_q^q})$ and using the fact that $\|w^{\pm}\|_{1,p} = 1/2$, it follows that

$$\frac{t_{w^{\pm}}^p}{2^p} \le t_w^p \|w\|_{1,p}^p + \left(\mu - C_q^q \varepsilon\right) t_w^q \|\nabla w\|_q^q \le C_p^r C_\varepsilon \frac{t_{w^{\pm}}^r}{2^r}.$$

We take $\delta = 2\left(\frac{1}{C_p^r C_\varepsilon}\right)^{\frac{1}{r-p}} > 0$ in order to get the desired assertion.

Next, let $\mathcal{W}^\circ \subseteq \mathcal{S}_{\pm}^\circ$ be compact. Suppose by contradiction that there is a sequence $\{w_n^{\pm}\}_{n\in\mathbb{N}} \subseteq \mathcal{W}^\circ$ such that $t_n := t_{w_n^{\pm}} \to +\infty$. By (i), we know that $J_\lambda(t_n w_n^{\pm}) = \max_{t\in[0,+\infty)} J_\lambda(t w_n^{\pm}) \geqslant 0$.

Using $\|\cdot\|_{1,q}^q \le C_{pq}\|\cdot\|_{1,p}^q$ along with (H2)(iii), we deduce that

$$0 \leqslant \frac{J_\lambda(t_n w_n^{\pm})}{t_n^p} \leqslant \frac{1}{p} + \frac{\mu C_{pq}}{q} - \int_{\Omega_\lambda} \frac{F(t_n w_n^{\pm})}{t_n^p}\,\mathrm{d}x \to -\infty \quad \text{as } n \to \infty,$$

which yields a contradiction. Thus there exists $C_{\mathcal{W}^\circ}$ such that $t_{w^{\pm}} \leqslant C_{\mathcal{W}^\circ}$. □

We define

$$\hat{m}_{\pm} : \left\{w^{\pm} : w \in W_0^{1,p}(\Omega_\lambda)^\circ \setminus \{0\}\right\} \to \mathcal{N}_{\pm}^\circ, \quad w^{\pm} \mapsto \hat{m}_{\pm}(w^{\pm}) := t_{w^{\pm}} w^{\pm},$$

where $t_{w^{\pm}}$ is defined in Lemma 2.3. For simplification we write $m_{\pm} := \hat{m}_{\pm}|_{\mathcal{S}_{\pm}^\circ}$. Next, we are going to prove that $m_{\pm}$ is a one-to-one correspondence between $\mathcal{S}_{\pm}^\circ$ and $\mathcal{N}_{\pm}^\circ$.

Lemma 2.4. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) *The mapping $\hat{m}_{\pm}$ is continuous.*
(ii) *The mapping $m_{\pm}$ is a homeomorphism between $\mathcal{S}_{\pm}^\circ$ and $\mathcal{N}_{\pm}^\circ$ and the inverse of $m_{\pm}$ is given by*

$$m_{\pm}^{-1}(u^{\pm}) = \frac{u^{\pm}}{\|u^{\pm}\|_{1,p}} \quad \textit{for all } u \in \mathcal{N}_{\pm}^\circ$$

Proof. (i) Assume that $w_n^{\pm} \to w^{\pm}$. From Lemma 2.3 (ii) it follows that $\{t_{w_n^{\pm}}\}_{n\in\mathbb{N}}$ is uniformly bounded. Hence, there exists a subsequence of $\{t_{w_n^{\pm}}\}_{n\in\mathbb{N}}$, not relabeled, which converges to a limit t_0. From (2.4) we conclude that $t_0 = t_{w^{\pm}}$. But then $t_{w_n^{\pm}} \to t_{w^{\pm}}$. Thus $\hat{m}_{\pm}$ is continuous.

(ii) From (i) we know that $m_{\pm}(\mathcal{S}_{\pm}^\circ)$ is a bounded set in $W_0^{1,p}(\Omega_\lambda)$ and for any $u^{\pm} \in m_{\pm}(\mathcal{S}_{\pm}^\circ) \subseteq \mathcal{N}_{\pm}^\circ$, there exists $\delta > 0$ such that $\|u^{\pm}\|_{1,p} \ge \delta$. Indeed, similar to the proof of Lemma 2.3 (i), by using $u \in \mathcal{N}_{\pm}^\circ \subseteq \mathcal{N}_\lambda$, (2.3) and the embeddings $W_0^{1,q}(\Omega_\lambda) \to L^q(\Omega_\lambda)$, $W_0^{1,p}(\Omega_\lambda) \to L^r(\Omega_\lambda)$ with embedding constants $C_q, C_p > 0$ we have

$$\begin{aligned}\|u^{\pm}\|_{1,p}^p + \mu\|\nabla u^{\pm}\|_q^q &= \int_{\Omega_\lambda} f(u^{\pm})u^{\pm}\,\mathrm{d}x \le \varepsilon \int_{\Omega_\lambda} |u^{\pm}|^q\,\mathrm{d}x + C_\varepsilon \int_{\Omega_\lambda} |u^{\pm}|^r\,\mathrm{d}x \\ &\le C_q^q \varepsilon \|\nabla u^{\pm}\|_q^q + C_p^r C_\varepsilon \|u^{\pm}\|_{1,p}^r.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we obtain from this

$$\|u^{\pm}\|_{1,p}^p \le \|u^{\pm}\|_{1,p}^p + \left(\mu - C_q^q\varepsilon\right)\|\nabla u^{\pm}\|_q^q \le C_p^r C_\varepsilon \|u^{\pm}\|_{1,p}^r.$$

Taking $\delta = 2\left(\frac{1}{C_p^r C_\varepsilon}\right)^{\frac{1}{r-p}} > 0$ we have $\|u^{\pm}\|_{1,p} \ge \delta$. From the continuity of $\hat{m}_{\pm}$ and its definition, we know that the map $m_{\pm} \colon \mathcal{S}_{\pm}^\circ \to \mathcal{N}_{\pm}^\circ$ is continuous and one-to-one. It is clear that the inverse function of $m_{\pm}$ is given by $m_{\pm}^{-1}(u^{\pm}) = \frac{u^{\pm}}{\|u^{\pm}\|_{1,p}}$ for any $u^{\pm} \in \mathcal{N}_{\pm}^\circ$. To reach the desired conclusion, it is enough to show that $m_{\pm}^{-1}$ is continuous. Indeed, we have

$$\begin{aligned}\left\|m_{\pm}^{-1}(u^{\pm}) - m_{\pm}^{-1}(v^{\pm})\right\|_{1,p} &= \left\|\frac{u^{\pm}}{\|u^{\pm}\|_{1,p}} - \frac{v^{\pm}}{\|v^{\pm}\|_{1,p}}\right\|_{1,p} \\ &= \left\|\frac{u^{\pm} - v^{\pm}}{\|u\|_{1,p}} + \frac{v^{\pm}\left(\|v^{\pm}\|_{1,p} - \|u^{\pm}\|_{1,p}\right)}{\|u^{\pm}\|_{1,p}\|v^{\pm}\|_{1,p}}\right\|_{1,p} \\ &\le \frac{2\|u^{\pm} - v^{\pm}\|_{1,p}}{\|u^{\pm}\|_{1,p}} \le \frac{2}{\delta}\|u^{\pm} - v^{\pm}\|_{1,p},\end{aligned}$$

that is, $m_{\pm}^{-1}$ is Lipschitz continuous. □

We write $\hat{\Psi}(w^{\pm}) := J_\lambda(\hat{m}_{\pm}(w^{\pm}))$. In the next lemma, we are going to show that the problem of finding critical points of $\hat{\Psi}|_{\mathcal{S}_{\pm}^{\circ}}$ is equivalent to the problem of finding critical points of $J_\lambda|_{\mathcal{N}_{\pm}^{\circ}}$. Recall that a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ is called a $(\mathrm{PS})_c$-sequence if $J(u_n) \to c$ and $J'(u_n) \to 0$. We say that J_λ satisfies the (PS)-condition on $\mathcal{M}$, if every $(\mathrm{PS})_c$-sequence has a converging subsequence.

Lemma 2.5. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) $\hat{\Psi} \in C^1\left(\left\{w^{\pm} : w \in W_0^{1,p}(\Omega_\lambda)^{\circ} \setminus \{0\}\right\}, \mathbb{R}\right)$ *and*

$$\left\langle \hat{\Psi}'(w^{\pm}), z \right\rangle = \left\langle J_\lambda'(m_{\pm}(w^{\pm})), \|m_{\pm}(w^{\pm})\|_{1,p} z \right\rangle \quad \text{for all } w^{\pm} \in \mathcal{S}_{\pm}^{\circ} \text{ and for all } z \in T_{w^{\pm}}(\mathcal{S}_{\pm}^{\circ}),$$

where $T_{w^{\pm}}(\mathcal{S}_{\pm}^{\circ})$ denote the tangent space to $\mathcal{S}_{\pm}^{\circ}$ at $w^{\pm}$.

(ii) *If $\{w_n^{\pm}\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_{\pm}^{\circ}$ is a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$, then $\{m_{\pm}(w_n^{\pm})\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_{\pm}^{\circ}$ is a $(\mathrm{PS})_c$-sequence for J_λ. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_{\pm}^{\circ}$ is a bounded $(\mathrm{PS})_c$-sequence for J_λ, then $\{m_{\pm}^{-1}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_{\pm}^{\circ}$ is a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$.*

(iii) *$w^{\pm} \in \mathcal{S}_{\pm}^{\circ}$ is a critical point of $\hat{\Psi}$ if and only if $m_{\pm}(w^{\pm}) \in \mathcal{N}_{\pm}^{\circ}$ is a nontrivial critical point of J_λ. Moreover, $\inf_{\mathcal{S}_{\pm}^{\circ}} \hat{\Psi} = \inf_{\mathcal{N}_{\pm}^{\circ}} J_\lambda$.*

(iv) *If J_λ is even, then so is $\hat{\Psi}$.*

Proof. The lemma follows from Szulkin-Weth [25, Proposition 9 and Corollary 10] and Lemmas 2.3 and 2.4. We omit the details. □

Remark 2.6.

(i) *Set*

$$c^{\circ}(\Omega_\lambda) = \inf_{u \in \mathcal{N}_\lambda^{\circ}} J_\lambda(u).$$

Then it follows from Lemma 2.5 (iii) *that*

$$c^{\circ}(\Omega_\lambda) = \inf_{w \in \mathcal{S}^{\circ}} \hat{\Psi}(w).$$

From Lemmas 2.3 and 2.4 it is easy to see that $c^{\circ}(\Omega_\lambda)$ has the following minimax characterization:

$$c^{\circ}(\Omega_\lambda) = \inf_{w \in W_0^{1,p}(\Omega_\lambda)^{\circ} \setminus \{0\}} \max_{t>0} J_\lambda(tw) = \inf_{w \in \mathcal{S}^{\circ}} \max_{t>0} J_\lambda(tw).$$

We know from the proof of Lemma 2.3 that there exists a unique $t_w > 0$ such that $\max_{t>0} J_\lambda(tw) = J(t_w w)$ for $w \in \mathcal{S}^{\circ}$. Lemma 2.3 (ii) *implies that there exists $\delta > 0$ such that $t_w \geqslant \delta$ uniformly for $w \in \mathcal{S}^{\circ}$. Thus, for any $w \in \mathcal{S}^{\circ}$, we have*

$$J(t_w w) = \max_{t>0} J_\lambda(tw) \geqslant \sigma,$$

for some $\sigma > 0$ independent of w and consequently

$$\inf_{w \in \mathcal{S}^{\circ}} \max_{t>0} J_\lambda(tw) \geqslant \sigma,$$

that is

$$c^{\circ}(\Omega_\lambda) \geqslant \sigma > 0.$$

(ii) *Set*

$$c(\Omega_\lambda) = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u). \tag{2.6}$$

By an argument similar to that of (i), *we can show that $c(\Omega_\lambda) > 0$. We can also show that $c^{\circ}(\Omega_\lambda) \geq 2c(\Omega_\lambda)$. It is similar to the proof of Lemma 3.2 and we omit it.*

3. (PS)-condition and some estimates

Our first result is that $\hat{\Psi}$ satisfies the (PS)-condition on $\mathcal{S}^\circ_\pm$. We set

$$I_\lambda(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|\nabla u\|_q^q \quad \text{and} \quad K_\lambda(u) = \int_{\Omega_\lambda} F(u)\,\mathrm{d}x.$$

Then $J_\lambda(u) = I_\lambda(u) - K_\lambda(u)$. We denote the derivative operator of I_λ in the weak sense by A_λ. It is well known that the operator A_λ is of type (S_+). We also denote by $\partial\mathcal{S}^\circ_\pm$ the boundary of $\mathcal{S}^\circ_\pm$.

Lemma 3.1. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) *Let* $\{w_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{S}^\circ_\pm$ *be a sequence such that* $\operatorname{dist}(w_n^\pm, \partial\mathcal{S}^\circ_\pm) \to 0$ *as* $n \to +\infty$. *Then* $\|m(w_n^\pm)\| \to +\infty$ *and* $\hat{\Psi}(w_n^\pm) \to +\infty$ *as* $n \to +\infty$.

(ii) *For any* $\lambda > 0$, $\hat{\Psi}$ *satisfies the* (PS)*-condition on* $\mathcal{S}^\circ_\pm$.

Proof. (i) Recall that we denote u^+ (resp. u^-) the positive (resp. negative) part of u, given in (2.1) and write

$$\mathcal{S}^\circ_\pm = \left\{u^\pm : u \in \mathcal{S}^\circ\right\}.$$

Let $w \in \mathcal{S}^\circ_\pm$ and $\gamma \in [1, p^*]$. By the embedding theorem, we have

$$\begin{aligned}\|w^+\|_{L^\gamma(\Omega_\lambda)} &= \inf_{v\in\overline{\mathcal{S}^\circ_\pm}} \|w-v\|_{L^\gamma(\Omega_\lambda)} \le \inf_{v\in\partial\mathcal{S}^\circ_\pm} \|w-v\|_{L^\gamma(\Omega_\lambda)}\\ &\le C_\gamma \inf_{v\in\partial\mathcal{S}^\circ_\pm} \|w-v\|_{1,p} = C_\gamma \operatorname{dist}\left(w, \partial\mathcal{S}^\circ_\pm\right).\end{aligned}$$

Here we denote by $\overline{\mathcal{S}^\circ_\pm}$ the closure of $\mathcal{S}^\circ_\pm$.

Similarly, it holds

$$\|w^-\|_{L^\gamma(\Omega_\lambda)} \le C_\gamma \operatorname{dist}\left(w, \partial\mathcal{S}^\circ_\pm\right).$$

Let $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}^\circ_\pm$ be a sequence such that $\operatorname{dist}(w_n, \partial\mathcal{S}^\circ_\pm) \to 0$ as $n \to +\infty$ and let

$$\begin{aligned}\Omega_\lambda^> &= \{x \in \Omega_\lambda : w_n(x) > 0\},\\ \Omega_\lambda^< &= \{x \in \Omega_\lambda : w_n(x) < 0\},\\ \Omega_\lambda^= &= \{x \in \Omega_\lambda : w_n(x) = 0\}.\end{aligned}$$

For every $t > 0$, using (2.2), we have

$$\begin{aligned}|K_\lambda(tw_n)| &= \left|\int_{\Omega_\lambda^<} F(tw_n)\,\mathrm{d}x + \int_{\Omega_\lambda^>} F(tw_n)\,\mathrm{d}x + \int_{\Omega_\lambda^=} F(tw_n)\,\mathrm{d}x\right|\\ &= \left|\int_{\Omega_\lambda} F(tw_n^+)\,\mathrm{d}x + \int_{\Omega_\lambda} F(tw_n^-)\,\mathrm{d}x\right|\\ &\le \varepsilon t^q \left(\|w_n^+\|_{L^q(\Omega_\lambda)}^q + \|w_n^-\|_{L^q(\Omega_\lambda)}^q\right) + C_\varepsilon t^r \left(\|w_n^+\|_{L^r(\Omega_\lambda)}^r + \|w_n^-\|_{L^r(\Omega_\lambda)}^r\right)\\ &\le C\left[t^q \left(\operatorname{dist}(w_n, \partial\mathcal{S}^\circ_\pm)\right)^q + t^r \left(\operatorname{dist}(w_n, \partial\mathcal{S}^\circ_\pm)\right)^r\right] \to 0 \text{ as } n \to +\infty.\end{aligned}$$

Note that for any $t > 1$,

$$\left(\frac{1}{p} + \frac{\mu C_{pq}}{q}\right)\|tw_n\|_{1,p}^p + |K_\lambda(tw_n)| \ge J_\lambda(tw_n) \ge \frac{1}{p}\|tw_n\|_{1,p}^p - |K_\lambda(tw_n)| = \frac{t^p}{p} - |K_\lambda(tw_n)|.$$

Consequently

$$\liminf_{n\to+\infty}\left(\frac{1}{p} + \frac{\mu C_{pq}}{q}\right)\|m(w_n)\|_{1,p}^p \ge \liminf_{n\to+\infty}\hat{\Psi}(w_n) \ge \liminf_{n\to+\infty} J_\lambda(tw_n) \ge \frac{t^p}{p}$$

for every $t > 1$. Hence, $\|m(w_n)\| \to +\infty$ and $\hat{\Psi}(w_n) \to +\infty$ as $n \to +\infty$.

(ii) For any $c > 0$, let $\{w_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{S}^\circ_\pm$ be a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$. Let $u_n^\pm := m_\pm(w_n^\pm)$ for all $n \in \mathbb{N}$. It follows from Lemma 2.5 that $\{u_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{N}^\circ_\pm$ is a $(\mathrm{PS})_c$-sequence for J_λ. First we will prove that $\{u_n^\pm\}_{n\in\mathbb{N}}$ is bounded. Let us assume this is not the case, so there exists a subsequence (still denoted by $u_n^\pm$) such that $\|u_n^\pm\|_{1,p} \to +\infty$. We define $v_n^\pm := \frac{u_n^\pm}{\|u_n^\pm\|_{1,p}}$, then $\|v_n^\pm\|_{1,p} = 1$. Thus we may assume that

$$v_n^\pm \rightharpoonup v^\pm \quad \text{in } W_0^{1,p}(\Omega_\lambda).$$

If $v^\pm = 0$, then it follows from Lemma 2.3 and Remark 2.6 that

$$c + o(1) \geqslant J_\lambda(u_n^\pm) = J_\lambda(t_{v_n^\pm} v_n^\pm) \geqslant J_\lambda(tv_n^\pm) \quad \text{for all } t > 0.$$

Recalling that K_λ is weakly continuous, we have that

$$J_\lambda(tv_n^\pm) \geq \frac{1}{p} t^p - \int_{\Omega_\lambda} F(tv_n^\pm)\,\mathrm{d}x \to \frac{1}{p} t^p \quad \text{as } n \to +\infty.$$

Choosing $t > 2(pc)^{\frac{1}{p}}$ yields a contradiction. If $v^\pm \neq 0$, then we know from (H2)(iii) that

$$0 \leq \frac{J_\lambda(u_n^\pm)}{\|u_n^\pm\|_{1,p}^p} \leq \frac{1}{p} + \frac{\mu C_{pq}}{q} - \int_{\Omega_\lambda} \frac{F(\|u_n^\pm\|_{1,p} v_n^\pm)}{\|u_n^\pm\|_{1,p}^p}\,\mathrm{d}x \to -\infty \quad \text{as } n \to +\infty.$$

This is again a contradiction. Hence $\{u_n^\pm\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega_\lambda)$ and so there exists a subsequence of $\{u_n^\pm\}_{n\in\mathbb{N}}$ (not relabeled) such that

$$u_n^\pm \rightharpoonup u^\pm \quad \text{in } W_0^{1,p}(\Omega_\lambda).$$

It is clear that $K_\lambda'(u_n^\pm) \to K_\lambda'(u^\pm)$, see Liu-Dai [22]. Since

$$J_\lambda'(u_n^\pm) = A_\lambda(u_n^\pm) - K_\lambda'(u_n^\pm) \to 0 \quad \text{as } n \to +\infty,$$

one has

$$A_\lambda(u_n^\pm) \to K_\lambda'(u^\pm) \quad \text{as } n \to +\infty.$$

Therefore, we conclude that $u_n^\pm \to u^\pm$ since A_λ is a mapping of type (S_+). Consequently, $m_\pm^{-1}(u_n^\pm) \to m_\pm^{-1}(u^\pm)$ by Lemma 2.4, that is, $w_n^\pm \to w^\pm$. Therefore, $\hat{\Psi}$ satisfies the $(\mathrm{PS})_c$-condition on $\mathcal{S}_\pm^\circ$. ☐

We say that u changes sign m times if the set $\{x \in \Omega_\lambda : u(x) \neq 0\}$ has $m+1$ connected components. It is clear that a solution of problem (1.1) changes sign an odd number of times. Following the ideas of Castro-Clapp [14], we can show the following energy estimate.

Lemma 3.2. *Let hypotheses* (H1) *and* (H2) *be satisfied. If u is a solution of problem* (1.1) *which changes sign $2m-1$ times, then $J_\lambda(u) \geq mc^\circ(\Omega_\lambda)$.*

Proof. From the assumptions we know that the set $\{x \in \Omega : u(x) > 0\}$ has m connect components $\Omega_1, \Omega_2, \cdots, \Omega_m$. Let

$$u_i(x) = \begin{cases} u(x), & \text{if } x \in -\Omega_i \cup \Omega_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since u is a solution of problem (1.1), it is a critical point of J_λ. This gives

$$\begin{aligned} 0 &= \langle J_\lambda'(u), u_i \rangle \\ &= \int_{\Omega_\lambda} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla u_i + |u|^{p-2} u u_i \right) \mathrm{d}x + \mu \int_{\Omega_\lambda} |\nabla u|^{q-2} \nabla u \cdot \nabla u_i \,\mathrm{d}x - \int_{\Omega_\lambda} f(u) u_i \,\mathrm{d}x \\ &= \|u_i\|_{1,p}^p + \mu \|\nabla u_i\|_{1,q}^q - \int_{\Omega_\lambda} f(u_i) u_i \,\mathrm{d}x, \end{aligned}$$

which implies that $u_i \in \mathcal{N}_\lambda^\circ$ for all $i = 1, 2, \cdots, m$. Consequently

$$J_\lambda(u) = J_\lambda(u_1) + J_\lambda(u_2) + \cdots + J_\lambda(u_m) \geqslant mc^\circ(\Omega_\lambda).$$

☐

We denote the limiting energy functional by

$$J_\infty(u) := \int_{\mathbb{R}^N} \left(\frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p + \frac{\mu}{q}|\nabla u|^q - F(u) \right) \mathrm{d}x.$$

The corresponding Nehari manifold is

$$\mathcal{N}_\infty := \left\{ u \in W^{1,p}_r(\mathbb{R}^N) \setminus \{0\} \, : \, \langle J'_\infty(u), u \rangle = 0 \right\},$$

where

$$W^{1,p}_r(\mathbb{R}^N) := \left\{ u \in W^{1,p}(\mathbb{R}^N) \, : \, u \text{ is radially symmetric} \right\}.$$

The least energy level is given by

$$0 < c\left(\mathbb{R}^N\right) := \inf_{u \in \mathcal{N}_\infty} J_\infty(u).$$

Lemma 3.3. *Let hypotheses (H1) and (H2) be satisfied. Then $c(\mathbb{R}^N)$ is achieved by a positive radially symmetric function.*

Proof. We define

$$f^+(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ f(t) & \text{if } t > 0 \end{cases}$$

with primitive $F^+(s) = \int_0^s f^+(t)\,\mathrm{d}t$. We set

$$J^+_\infty(u) := \int_{\mathbb{R}^N} \left(\frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p + \frac{\mu}{q}|\nabla u|^q - F^+(u) \right) \mathrm{d}x \quad \text{for all } u \in W^{1,p}_r(\mathbb{R}^N).$$

It is clear that (H2) remain valid for f^+ and F^+. Similar to the proof of Lemma 2.3, we can define

$$\hat{m} : W^{1,p}_r(\mathbb{R}^N) \setminus \{0\} \to \mathcal{N}_\infty, \quad w \mapsto \hat{m}(w) := t_w w,$$

where t_w is similar to the definition in the proof of Lemma 2.3. We set $m := \hat{m}|_{\mathcal{S}}$ and can show that m is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{N}_\infty$, where

$$\mathcal{S} = \left\{ w \in W^{1,p}_r(\mathbb{R}^N) \, : \, \|w\|_{1,p} = 1 \right\}.$$

Setting $\hat{\Psi}^+_\infty(w) := J^+_\infty(\hat{m}(w))$ we can show that $\hat{\Psi}^+_\infty$ satisfies the (PS)-condition on $\mathcal{S}$ as in Lemma 3.1(ii), since $W^{1,p}_r(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ is compact for all $\gamma \in (p, p^*)$. Therefore, it follows from Theorem 1 in Szulkin-Weth [25] that $\inf\limits_{\mathcal{S}} \hat{\Psi}^+_\infty$ is attained by a function $w \in W^{1,p}_r(\mathbb{R}^N)$. Just like Lemma 2.5 (iii), we are able to show that $\inf\limits_{\mathcal{S}} \hat{\Psi}^+_\infty = \inf\limits_{\mathcal{N}_\infty} J^+_\infty$, that is, $\inf\limits_{\mathcal{N}_\infty} J^+_\infty$ is attained by $m(w)$, which is obviously radially symmetric. By an argument similar to that in the proof of Theorem 1.4 of the first two authors [23], we can also prove that $m(w)$ is positive. ☐

We also need the auxiliary functional which is defined as in (1.2) replacing Ω_λ by $B_R := B_R(0)$ with $R > 0$, that is,

$$J_R(u) = \int_{B_R} \left(\frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p + \frac{\mu}{q}|\nabla u|^q - F(u) \right) \mathrm{d}x.$$

The corresponding Nehari manifold is denoted by

$$\mathcal{N}_R := \left\{ u \in W^{1,p}_0(B_R) \setminus \{0\} \, : \, \langle J'_R(u), u \rangle = 0 \right\}.$$

We write

$$c\left(B_R\right) := \inf_{u \in \mathcal{N}_R} J_R(u). \tag{3.1}$$

Then $c\left(B_R\right)$ is achieved by a positive radially symmetric function Ψ_R. Indeed, similar to the proof of Lemma 3.3, we can show that $c\left(B_R\right)$ is attained by a positive function $v \in W^{1,p}_0(B_R)$.

Let v^* be the Schwartz symmetrization of v, then we have that $v^* \in W_0^{1,p}(B_R)$ and

$$\int_{B_R} \left(\frac{1}{p}|\nabla v^*|^p + \frac{\mu}{q}|\nabla v^*|^q\right) \mathrm{d}x \leq \int_{B_R} \left(\frac{1}{p}|\nabla v|^p + \frac{\mu}{q}|\nabla v|^q\right) \mathrm{d}x,$$
$$\int_{B_R} \frac{1}{p}|v^*|^p \,\mathrm{d}x = \int_{B_R} \frac{1}{p}|v|^p \,\mathrm{d}x,$$
$$\int_{B_R} F\left(v^*\right) \mathrm{d}x = \int_{B_R} F(v) \,\mathrm{d}x$$

are satisfied.

Just as in the proof of Lemma 2.3, we can show that there exists a unique $t_{v^*} > 0$ such that $t_{v^*} v^* \in \mathcal{N}_R$. Moreover,

$$c\left(B_R\right) \leq J_R\left(t_{v^*} v^*\right) \leq J_R\left(t_{v^*} v\right) \leq \max_{t \geqslant 0} J_R(tv) = J_R(v) = c\left(B_R\right).$$

Setting $\Psi_R := t_{v^*} v^*$, then it has all the required properties. Furthermore, we can determine the asymptotic behavior of $c\left(B_R\right)$.

Lemma 3.4. *Let hypotheses* (H1) *and* (H2) *be satisfied and let* $c\left(B_R\right)$ *and* $c\left(\Omega_\lambda\right)$ *be defined as in* (3.1) *and* (2.6)*, respectively. Then it holds*

$$\lim_{R \to +\infty} c\left(B_R\right) = c\left(\mathbb{R}^N\right) \quad \text{and} \quad \lim_{\lambda \to +\infty} c\left(\Omega_\lambda\right) = c\left(\mathbb{R}^N\right).$$

Proof. We only prove the second equality, the other works very similarly.

We follow the ideas of Alves [2] who studied the p-Laplacian equation. To this end, fix $\tilde{\lambda} > 0$ and $R > 0$ such that $B_R \subseteq \Omega_{\tilde{\lambda}}$. Let $\eta_R\colon [0, +\infty) \to \mathbb{R}$ be a smooth, nonincreasing cut-off function such that

$$\eta_R(t) = 1 \quad \text{if } 0 \leq t \leq \frac{R}{2}, \quad \eta_R(t) = 0 \quad \text{if } t \geq R, \quad 0 \leq \eta_R \leq 1 \quad \text{and} \quad |\eta_R'(t)| \leq 2.$$

We write $w_R(x) = \eta_R(x) w(x)$, where $w \in \mathcal{N}_\infty$ such that $J_\infty(w) = c(\mathbb{R}^N)$. Let $t_R > 0$ be such that $t_R w_R \in \mathcal{N}_\lambda$. Then

$$c\left(\Omega_\lambda\right) \leq J_\lambda\left(t_R w_R\right) \quad \text{for all } \lambda > \tilde{\lambda}.$$

Passing to the limit as $\lambda \to +\infty$ we obtain

$$\limsup_{\lambda \to +\infty} c\left(\Omega_\lambda\right) \leq J_\infty\left(t_R w_R\right).$$

As in the proof of Lemma 2.3 we can show that $t_R \to 1$ as $R \to +\infty$. Then we have $J_\infty(t_R w_R) \to J_\infty(w) = c\left(\mathbb{R}^N\right)$ as $R \to +\infty$. Therefore,

$$\limsup_{\lambda \to +\infty} c\left(\Omega_\lambda\right) \leq c\left(\mathbb{R}^N\right). \tag{3.2}$$

On the other hand, from the definition of $c\left(\Omega_\lambda\right)$ and $c\left(\mathbb{R}^N\right)$ it follows that

$$c\left(\mathbb{R}^N\right) \leq c\left(\Omega_\lambda\right) \quad \text{for all } \lambda > 0,$$

which implies that

$$c\left(\mathbb{R}^N\right) \leq \liminf_{\lambda \to +\infty} c\left(\Omega_\lambda\right). \tag{3.3}$$

From (3.2) and (3.3) we get the assertion. □

4. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. In what follows, without any loss of generality, we shall assume that $0 \in \Omega$. Moreover, we choose $\tilde{R} \geq \operatorname{diam}(\Omega)$ and $\tilde{R} > R > 0$ such that $B_R(0) \subseteq \Omega \subseteq B_{\tilde{R}}(0)$ and the sets

$$\Omega_R^+ := \left\{x \in \mathbb{R}^N \,:\, \operatorname{dist}(x, \Omega) \leq R\right\} \quad \text{and} \quad \Omega_R^- := \left\{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega \cup \{0\}) \geq R\right\}$$

are homotopically equivalent to Ω. For $\lambda>0$, let $\Psi_{\lambda R}\in\mathcal{N}_{\lambda R}$ be given as in Section 3 satisfying $J_{\lambda R}(\Psi_{\lambda R})=c(B_{\lambda R})$. We define $\Phi_\lambda\colon \lambda\Omega_R^-\to\mathcal{N}_\lambda^\circ$ by

$$[\Phi_\lambda(\xi)](x)=\begin{cases} t_\lambda\left[\Psi_{\lambda R}(|x-\xi|)-\Psi_{\lambda R}(|x+\xi|)\right], & \text{if } x\in B_{\lambda R}(\xi),\\ 0, & \text{if } x\in\Omega_\lambda\setminus B_{\lambda R}(\xi),\end{cases}$$

where $t_\lambda>0$ is such that $\Phi_\lambda(\xi)\in\mathcal{N}_\lambda^\circ$. Note that

$$[\Phi_\lambda(\xi)](-x)=-[\Phi_\lambda(\xi)](x)\quad\text{and}\quad \Phi_\lambda(-\xi)=-\Phi_\lambda(\xi).$$

Hence $\Phi_\lambda(\xi)^\pm\in\mathcal{N}_\pm^\circ$.

Then we have the following lemma.

Lemma 4.1. *Let hypotheses (H1) and (H2) be satisfied. Then we have*

$$\lim_{\lambda\to+\infty}J_\lambda\left(\Phi_\lambda(\xi)^\pm\right)=c\left(\mathbb{R}^N\right)$$

uniformly in $\xi\in\lambda\Omega_R^-$.

Proof. For any $\xi\in\lambda\Omega_R^-$, by the definition of $\lambda\Omega_R^-$, we have $|\xi|\geq\lambda R$ and $|-\xi|\geq\lambda R$, and so $|\xi-(-\xi)|\geq 2\lambda R$. Following the same arguments as in the proofs of Lemmas 2.3 and 3.2 as well as Remark 2.6, it is easy to see that

$$\begin{aligned} c(\Omega_\lambda)\leq J_\lambda\left(\Phi_\lambda(\xi)^\pm\right)&=\begin{cases} J_\lambda\left(t_\lambda\Psi_{\lambda R}(|x-\xi|)\right)\\ J_\lambda\left(-t_\lambda\Psi_{\lambda R}(|x+\xi|)\right)\end{cases}\\ &=J_\lambda\left(t_\lambda\Psi_{\lambda R}(|x|)\right)\leq J_\lambda\left(\Psi_{\lambda R}(|x|)\right)=c(B_{\lambda R}).\end{aligned}$$

Here we have used translation invariance of the Lebesgue integral the in second equality. From Lemma 3.4 we then deduce that

$$\lim_{\lambda\to+\infty}c\left(B_{\lambda R}\right)=\lim_{\lambda\to+\infty}c\left(\Omega_\lambda\right)=c\left(\mathbb{R}^N\right)$$

Hence the assertion of the lemma follows. □

Given $\xi\in\lambda\Omega_R^-$, we set

$$h(\lambda):=\left|J_\lambda\left(\Phi_\lambda(\xi)^\pm\right)-c\left(\mathbb{R}^N\right)\right|.$$

From Lemma 4.1 we conclude that $h(\lambda)\to 0$ as $\lambda\to+\infty$. We define the sublevel set

$$\widetilde{\mathcal{N}_\pm^\circ}=\left\{u\in\mathcal{N}_\pm^\circ\,:\,J_\lambda(u)\leqslant c\left(\mathbb{R}^N\right)+h(\lambda)\right\}.$$

It is clear that $\Phi_\lambda(\xi)^\pm\in\widetilde{\mathcal{N}_\pm^\circ}$ which implies $\widetilde{\mathcal{N}_\lambda^\circ}\neq\emptyset$ for any $\lambda>0$.

For $u\in W^{1,p}(\mathbb{R}^N)$ with compact support in $B_{\tilde{R}}(0)$, we define the barycenter map

$$\begin{aligned} &\beta_+\colon W^{1,p}(\mathbb{R}^N)\setminus\{0\}\to\mathbb{R}^N,\quad \beta_+(u)=\frac{\displaystyle\int_{\mathbb{R}^N}x|u^+(x)|^p\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N}|u^+(x)|^p\,\mathrm{d}x},\\ &\beta_-\colon W^{1,p}(\mathbb{R}^N)\setminus\{0\}\to\mathbb{R}^N,\quad \beta_-(u)=\frac{\displaystyle\int_{\mathbb{R}^N}x|u^-(x)|^p\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N}|u^-(x)|^p\,\mathrm{d}x}.\end{aligned}\tag{4.1}$$

Proof of Theorem 1.1. From Lemmas 4.1 and 2.5 we know that

$$\lim_{\lambda\to+\infty}\hat{\Psi}\left(m^{-1}\left(\Phi_\lambda(\xi)^\pm\right)\right)=\lim_{\lambda\to+\infty}J_\lambda\left(\Phi_\lambda(\xi)^\pm\right)=c\left(\mathbb{R}^N\right)$$

uniformly in $\xi\in\lambda\Omega_R^-$. We set

$$\widetilde{\mathcal{S}_\pm^\circ}:=\left\{u\in\mathcal{S}_\pm^\circ\,:\,\hat{\Psi}(u)\leq c\left(\mathbb{R}^N\right)+h(\lambda)\right\},$$

where h is given in the definition of $\widetilde{\mathcal{N}^\circ_\pm}$. It is clear that $\widetilde{\mathcal{S}^\circ_\pm} \neq \emptyset$ since $m_\pm^{-1}(\Phi_\lambda(\xi)^\pm) \in \widetilde{\mathcal{S}^\circ_\pm}$. From Lemma 3.1 and Krasnosel'skii's genus theory, see for example Ambrosetti-Malchiodi [6, Theorem 10.9], it follows that $\hat{\Psi}$ has at least $\gamma(\widetilde{\mathcal{S}^\circ_\pm})$ pairs of critical points on $\widetilde{\mathcal{S}^\circ_\pm}$.

We claim that $\gamma(\widetilde{\mathcal{S}^\circ_\pm}) \geq 2\gamma(\Omega_\lambda \setminus \{0\})$. Indeed, suppose that $\gamma(\widetilde{\mathcal{S}^\circ_\pm}) = 2n$. For a set A, we denote $A^* = \{(x,-x) : x \in A\}$. From Theorem 3.9 of Rabinowitz [24] it follows that

$$\gamma(\widetilde{\mathcal{S}^\circ_\pm}) = \operatorname{cat}_{\left(W_0^{1,p}(\Omega_\lambda)\setminus\{0\}\right)^*} \widetilde{\mathcal{S}^\circ_\pm}^{\,*}.$$

Therefore, there exists a smallest positive integer n such that

$$\widetilde{\mathcal{S}^\circ_\pm}^{\,*} \subseteq \mathcal{D}^*_{\pm 1} \cup \mathcal{D}^*_{\pm 2} \cup \cdots \cup \mathcal{D}^*_{\pm n},$$

where $\mathcal{D}^*_{\pm i}$, $i = 1,2,\cdots,n$ are closed and contractible in $(W_0^{1,p}(\Omega_\lambda)\setminus\{0\})^*$, that is, there exist

$$h_i^* \in C\left([0,1]\times \mathcal{D}^*_{\pm i}, \left(W_0^{1,p}(\Omega_\lambda)\setminus\{0\}\right)^*\right) \qquad \text{for } i = 1,2,\cdots,n$$

such that

$$\begin{aligned}
&h_i^*(0,u^\pm) = (u^\pm,-u^\pm) \quad \text{for all } (u^\pm,-u^\pm) \in \mathcal{D}^*_{\pm i},\\
&h_i^*(1,u^\pm) = (\omega_i^\pm,-\omega_i^\pm) \in \left(W_0^{1,p}(\Omega_\lambda)\setminus\{0\}\right)^* \quad \text{for all } (u^\pm,-u^\pm) \in \mathcal{D}^*_{\pm i}.
\end{aligned}$$

Here we have used the fact that $-u^\pm(x) = u^\mp(-x) \in \mathcal{D}^*_{\pm i}$.

Let

$$\mathcal{D}_i = \left\{u^\pm \in W_0^{1,p}(\Omega_\lambda) : (u^\pm,-u^\pm) \in \mathcal{D}_i^*\right\}.$$

Then there exists a homotopy

$$h_i \in C\left([0,1]\times \mathcal{D}_i, \left(W_0^{1,p}(\Omega_\lambda)\setminus\{0\}\right)\right)$$

such that $h_i(0,\cdot) = \mathrm{id}$, $h_i(1,\cdot) = \omega_i^\pm$ or $-\omega_i^\pm$ and $h_i(t,u^\pm) = -h_i(t,-u^\pm)$.

We define $\Phi^*_\lambda = (\Phi^\pm_\lambda, -\Phi^\pm_\lambda)\colon \left(\lambda\Omega_R^-\right)^* \to \left(\mathcal{N}^\circ_\pm\right)^*$ by

$$\left[\Phi^*_\lambda(\xi,-\xi)\right](x) = \left(\left[\Phi^\pm_\lambda(\xi)\right](x), -\left[\Phi^\pm_\lambda(\xi)\right](x)\right) = \left(\left[\Phi_\lambda(\xi)^\pm\right](x), \left[\Phi_\lambda(-\xi)^\mp\right](x)\right).$$

Note that for any $(\xi,-\xi) \in \left(\lambda\Omega_R^-\right)^*$ we have

$$\beta_\pm\left(\Phi_\lambda(\xi)^\pm\right) = \xi \quad \text{and} \quad \beta_\mp\left(\Phi_\lambda(-\xi)^\mp\right) = -\xi,$$

that is,

$$\beta^*\left(\Phi_\lambda(\xi)^\pm, -\Phi_\lambda(\xi)^\pm\right) = \left(\beta_\pm\left(\Phi_\lambda(\xi)^\pm\right), \beta_\mp\left(\Phi_\lambda(-\xi)^\mp\right)\right) = (\xi,-\xi),$$

where $\beta^*(\cdot,\cdot) = (\beta_\pm(\cdot),\beta_\mp(\cdot))$ and $\beta_\pm$ is given in (4.1). We set

$$\mathcal{K}^*_{\pm i} = \left(\Phi^*_\lambda\right)^{-1}\left(m^*\left(\mathcal{D}^*_{\pm i}\right)\right),$$

where $m^*(\cdot,\cdot) = (m_\pm(\cdot),m_\pm(\cdot))$. It is clear that $\mathcal{K}^*_{\pm i}$ are closed subsets of $\left(\lambda\Omega_R^-\setminus\{0\}\right)^*$ and $\left(\lambda\Omega_R^-\setminus\{0\}\right)^* \subseteq \mathcal{K}^*_{\pm 1}\cup\cdots\cup\mathcal{K}^*_{\pm n}$. Moreover, for $i = 1,\ldots,n$, $\mathcal{K}^*_{\pm i}$ is contractible in $\left(\mathbb{R}^N\setminus\{0\}\right)^*$ by using the deformation $\mathfrak{h}_i\colon [0,1]\times\mathcal{K}^*_{\pm i} \to \left(\mathbb{R}^N\setminus\{0\}\right)^*$ defined by

$$\mathfrak{h}_i(t,x) = \left(\beta^*\circ h_i^*\right)\left(t, \left(m^*\right)^{-1}\left(\Phi^*_\lambda(\xi,-\xi)\right)\right).$$

From Lemma 4.1 and the definition of $\beta^\pm$ we conclude that

$$\begin{aligned}
&\mathfrak{h}_i \in C\left([0,1]\times\mathcal{K}^*_{\pm i}, \left(\mathbb{R}^N\setminus\{0\}\right)^*\right),\\
&\mathfrak{h}_i(0,x) = \left(\beta^*\circ h_i^*\right)\left(0,\left(m^*\right)^{-1}\left(\Phi^*_\lambda(\xi,-\xi)\right)\right) = (\xi,-\xi) \quad \text{for all } (\xi,-\xi)\in\mathcal{K}^*_{\pm i},\\
&\mathfrak{h}_i(1,x) = \left(\beta^*\circ h_i^*\right)\left(1,\left(m^*\right)^{-1}\left(\Phi^*_\lambda(\xi,-\xi)\right)\right)\\
&\qquad\quad = \beta^*\left(\omega_i^\pm,-\omega_i^\pm\right) = \left(\xi_i^0,-\xi_i^0\right) \in \left(\mathbb{R}^N\setminus\{0\}\right)^* \quad \text{for all } (\xi,-\xi)\in\mathcal{K}^*_{\pm i}.
\end{aligned}$$

Hence

$$\gamma\left(\Omega_\lambda \setminus \{0\}\right) = \mathrm{cat}_{(\mathbb{R}^N\setminus\{0\})^*}\left(\Omega_\lambda \setminus \{0\}\right)^* = \mathrm{cat}_{(\mathbb{R}^N\setminus\{0\})^*}\left(\lambda\Omega_R^- \setminus \{0\}\right)^* \leq n,$$

which implies that $\widetilde{\mathcal{S}_\pm^\circ}$ contains at least $2\gamma(\Omega_\lambda \setminus \{0\})$ pairs of critical points of $\hat{\Psi}$. Thus we conclude from Lemma 2.5 that there exist at least $2\gamma(\Omega_\lambda \setminus \{0\})$ pairs $(u^\pm, -u^\pm)$ of critical points of J_λ. It is clear that $u = u^+ + u^-$ is odd, and is also the critical point of J_λ, that is, problem (1.1) has at least $\gamma(\Omega_\lambda \setminus \{0\})$ pairs of odd solutions. □

Acknowledgment

W. Liu is supported by NNSF of China (No. 11961030). G. Dai is supported by NNSF of China (No. 11871129), the Fundamental Research Funds for the Central Universities (No. DUT17LK05) and Xinghai Youqing funds from Dalian University of Technology.

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