ANISOTROPIC AND ISOTROPIC IMPLICIT OBSTACLE PROBLEMS WITH NONLOCAL TERMS AND MULTIVALUED BOUNDARY CONDITIONS

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Abstract. In this paper we study an anisotropic implicit obstacle problem driven by the $(p(\cdot), q(\cdot))$-Laplacian and an isotropic implicit obstacle problem involving a nonlinear convection term (a reaction term depending on the gradient) which contain several interesting and challenging untreated problems. These two implicit obstacle problems have both highly nonlinear and nonlocal functions and three multivalued terms where two of them are appearing on the boundary and the other one is formulated in the domain. Under very general assumptions on the data, we develop general frameworks to examine the nonemptiness and compactness of the set of weak solutions to the problems under consideration. The proofs of our main results use the theory of nonsmooth analysis, Tychonoff's fixed point theorem for multivalued operators, the theory of pseudomonotone operators and variational approach.

1. Introduction

In this paper we study isotropic and anisotropic quasilinear implicit obstacle problems involving multivalued mappings and mixed boundary conditions. These classes of problems include several interesting special cases which have not been treated largely in the literature to date. Originally, the study of so-called obstacle problems is due the pioneering work by Stefan [41] in which the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water, was studied. In this direction we also mention the renowned contribution of Lions [23] who studied the equilibrium position of an elastic membrane which lies above a given obstacle and which turns out as the unique solution of the Dirichlet energy functional minimized on the closed convex set driven by the obstacle.

Let us formulate the two problems under consideration. To this end, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\Gamma := \partial\Omega$ such that Γ is divided into three mutually disjoint parts Γ_1, Γ_2, and Γ_3 where Γ_1 has positive Lebesgue measure. Note that Γ_2 and Γ_3 could be empty which means that Γ_1 could be the whole boundary $\Gamma_1 = \Gamma$. In this paper, we are interested in the study of two implicit obstacle problems. The first problem of this paper is formulated by the following anisotropic implicit obstacle problem given in the form

$$
\begin{aligned}
-a(u)\Delta_{p(\cdot)}u - b(u)\Delta_{q(\cdot)}u + g(x,u) &\in U_1(x,u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_n} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_n} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
L(u) &\leq J(u),
\end{aligned}
\tag{1.1}
$$

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where $p, q\colon \overline{\Omega} \to (1,+\infty)$ are continuous functions, $\Delta_{p(\cdot)}$ is the $p(\cdot)$-Laplace differential operator defined by

$$\Delta_{p(\cdot)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega),$$

and

$$\frac{\partial u}{\partial \nu_n} := \left(a(u)|\nabla u|^{p(x)-2}\nabla u + b(u)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nu, \tag{1.2}$$

with ν being the unit normal vector on Γ. Furthermore, $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\phi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is a convex function with respect to the second argument, $a\colon L^{p^*(\cdot)}(\Omega) \to (0,+\infty)$, $b\colon L^{p^*(\cdot)}(\Omega) \to [0,+\infty)$ are two continuous functions and $U_1\colon \Omega\times\mathbb{R} \to 2^{\mathbb{R}}$ as well as $U_2\colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ are two given multivalued functions. Also, $\partial_c\phi(x,u)$ is the convex subdifferential of $s \mapsto \phi(x,s)$, and $L, J\colon W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ are given functions defined on the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$, see Section 2 for its precise definition.

The second goal of this paper is the study of the following isotropic implicit obstacle problem involving a nonlinear convection function $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ of the form

$$\begin{aligned} -a(u)\Delta_p u - b(u)\Delta_q u + g(x,u) &\in U_1(x,u) + f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_1,\\ \frac{\partial u}{\partial \nu_n} &\in U_2(x,u) && \text{on } \Gamma_2,\\ -\frac{\partial u}{\partial \nu_n} &\in \partial_c\phi(x,u) && \text{on } \Gamma_3,\\ L(u) &\leq J(u), \end{aligned} \tag{1.3}$$

where $L, J\colon W^{1,p}(\Omega) \to \mathbb{R}$ are two given functions and $\frac{\partial u}{\partial \nu_n}$ is defined by

$$\frac{\partial u}{\partial \nu_n} := \left(a(u)|\nabla u|^{p-2}\nabla u + b(u)|\nabla u|^{q-2}\nabla u\right)\cdot \nu. \tag{1.4}$$

As mentioned above, problems (1.1) and (1.3) combine several interesting and challenging phenomena which have not been treated in the literature so far. To be more precise, these problems include

- a nonlinear, nonhomogeneous differential operator with different anisotropic/isotropic growth;
- two highly nonlinear nonlocal terms a and b, where the function b can be degenerate;
- mixed boundary conditions;
- multivalued mappings in which one of them is formulated by the subdifferential operator to a convex function;
- an implicit obstacle effect;
- a nonlinear convection term for the isotropic case.

The main goal of the paper is to develop general frameworks for determining the existence of a (weak) solution to the nonlinear implicit obstacle problems (1.1) and (1.3) via Tychonoff's fixed point theorem for multivalued operators, the theory of nonsmooth analysis and variational methods for pseudomonotone operators. In fact, to the best of our knowledge, this is the first work which combines a nonlinear anisotropic/isotropic partial differential operator along with two highly abstract nonlocal terms, an implicit obstacle constraint, a nonlinear convection term for the isotropic case, mixed boundary conditions and multivalued mixed terms which include a convex subdifferential operator and two abstract multivalued functions.

Such combination of an implicit obstacle effect with mixed boundary conditions along with multivalued mappings (which include as special case Clarke's generalized gradients, see Clarke [10]) arise in several engineering and economic models, such as Nash equilibrium problems with shared constraints and transport route optimization with feedback control. We refer to books of Panagiotopoulos [36, 37] and Naniewicz-Panagiotopoulos [35] for more models related to nonsmooth mechanical problems. In general, equations driven by the sum of two differential operators of different nature arise often in mathematical models of physical processes, see, for example, the works of Bahrouni-Rădulescu-Repovš [4] for transonic flow problems, Cherfils-Il'yasov [9] for reaction diffusion systems, Zhikov [49] for elasticity problems and Papageorgiou-Vetro-Vetro [39] for least energy problems. For implicit obstacle effects involving Clarke's generalized gradient or general multivalued mappings but without nonlocal term we refer to the papers of Alleche-Rădulescu [1], Aussel-Sultana-Vetrivel [3], Bonanno-Motreanu-Winkert [5], Liu et al. [25], Carl-Le-Winkert [8], Iannizzotto-Papageorgiou [21], Migórski-Khan-Zeng [30, 31], Liu-Migórski-Nguyen-Zeng [24], Zeng-Bai-Gasiński-Winkert [46, 47], Zeng-Rădulescu-Winkert [48] and the references therein. We also mention the recent monograph of Carl-Le [7] about multivalued variational inequalities and inclusions. For single-valued equations with convection term we refer to the works of Faraci-Motreanu-Puglisi [13], Faraci-Puglisi [14], Figueiredo-Madeira [15], Gasiński-Papageorgiou [18], Gasiński-Winkert [19], Liu-Motreanu-Zeng [27], Marano-Winkert [29] and Papageorgiou-Rădulescu-Repovš [38]. We also mention the overview articles of Rădulescu [42] about isotropic and anisotropic problems and of Mingione-Rădulescu [33] about recent developments for problems with nonstandard growth and nonuniform ellipticity.

Let us comment on some relevant special cases of problems (1.1) and (1.3). To the best of our knowledge, these problems have not been studied yet in the literature. We start with (1.1).

(i) Let $j_1\colon \Omega\times\mathbb{R}\to\mathbb{R}$ and $j_2\colon \Gamma_2\times\mathbb{R}\to\mathbb{R}$ be two functions which are measurable in the first argument and locally Lipschitz in the second one. Moreover, let $r_1,r_2\colon\mathbb{R}\to\mathbb{R}$ be two functions and denote by ∂j_i Clarke's generalized gradient of $j_i(x,\cdot)$ for $i=1,2$. If U_1 and U_2 are defined by $U_1(x,s)=r_1(s)\partial j_1(x,s)$ for a. a. $x\in\Omega$, $s\in\mathbb{R}$ and $U_2(x,s)=r_2(s)\partial j_2(x,s)$ for a. a. $x\in\Gamma_2$, $s\in\mathbb{R}$, then problem (1.1) becomes

$$\begin{aligned}
-a(u)\Delta_{p(\cdot)}u-b(u)\Delta_{q(\cdot)}u+g(x,u)&\in r_1(u)\partial j_1(x,u) &&\text{in } \Omega,\\
u&=0 &&\text{on } \Gamma_1,\\
\frac{\partial u}{\partial\nu_n}&\in r_2(u)\partial j_2(x,u) &&\text{on } \Gamma_2,\\
-\frac{\partial u}{\partial\nu_n}&\in\partial_c\phi(x,u) &&\text{on } \Gamma_3,\\
L(u)&\le J(u),
\end{aligned}\tag{1.5}$$

where $\frac{\partial u}{\partial\nu_n}$ is given in (1.2). We show in Theorem 3.13 that the solution set of (1.5) is nonempty and compact which follows from Theorem 3.4.

(ii) If $\Gamma_2=\emptyset$ and $\Gamma_3=\emptyset$, i.e., $\Gamma_1=\Gamma$, then problem (1.1) reduces to the following implicit obstacle inclusion problem with Dirichlet boundary condition

$$\begin{aligned}
-a(u)\Delta_{p(\cdot)}u-b(u)\Delta_{q(\cdot)}u+g(x,u)&\in U_1(x,u) &&\text{in } \Omega,\\
u&=0 &&\text{on } \Gamma,\\
L(u)&\le J(u),
\end{aligned}\tag{1.6}$$

where $\frac{\partial u}{\partial\nu_n}$ is given in (1.2). As a direct consequence, Corollary 3.12 guarantees the existence of a solution of (1.6).

(iii) Let $\Psi\colon\Omega\to\mathbb{R}$ be a given obstacle. When $J(u)\equiv 0$ and $L(u):=\displaystyle\int_\Omega (u(x)-\Psi(x))^+\,\mathrm{d}x$ for all $u\in W^{1,p(\cdot)}(\Omega)$, then our problem (1.1) can be rewritten to the following obstacle inclusion problem

$$\begin{aligned} -a(u)\Delta_{p(\cdot)}u - b(u)\Delta_{q(\cdot)}u + g(x,u) \in U_1(x,u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu_n} \in U_2(x,u) \qquad & \text{on } \Gamma_2, \\ -\frac{\partial u}{\partial \nu_n} \in \partial_c \phi(x,u) \qquad & \text{on } \Gamma_3, \\ u(x) \le \Psi(x) \qquad & \text{in } \Omega, \end{aligned} \tag{1.7}$$

where $\frac{\partial u}{\partial \nu_n}$ is given in (1.2). We can also suppose that $\Phi\colon \Gamma_a \to \mathbb{R}$ is a given obstacle on the boundary $\Gamma_a \subset \Gamma$ with Γ_a having positive Lebesgue measure. Then the last inequality in (1.7) is replaced by $u(x) \le \Phi(x)$ on Γ_a. The main results to problem (1.7) are given in Corollary 3.10.

(iv) Finally if $J(u) \equiv +\infty$ or $L(u) \equiv -\infty$ for all $u \in W^{1,p(\cdot)}(\Omega)$, then problem (1.1) turns into the following mixed boundary value problem without obstacle effect

$$\begin{aligned} -a(u)\Delta_{p(\cdot)}u - b(u)\Delta_{q(\cdot)}u + g(x,u) \in U_1(x,u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu_n} \in U_2(x,u) \qquad & \text{on } \Gamma_2, \\ -\frac{\partial u}{\partial \nu_n} \in \partial_c \phi(x,u) \qquad & \text{on } \Gamma_3, \end{aligned} \tag{1.8}$$

where $\frac{\partial u}{\partial \nu_n}$ is given in (1.2). We prove that there exists a weak solution of (1.8) and the solution set of (1.8) is compact, see Corollary 3.11.

Next, we mention some special cases of problem (1.3).

(a) If U_1 and U_2 are defined by $U_1(x,s) = r_1(s)\partial j_1(x,s)$ for a. a. $x \in \Omega$, $s \in \mathbb{R}$ and $U_2(x,s) = r_2(s)\partial j_2(x,s)$ for a. a. $x \in \Gamma_2$, $s \in \mathbb{R}$, where j_1, j_2, r_1, r_2 are given in problem (1.5), then problem (1.3) becomes the following implicit obstacle problem involving a nonlinear convection term and generalized Clarke's subgradients:

$$\begin{aligned} -a(u)\Delta_p u - b(u)\Delta_q u + g(x,u) \in r_1(u)\partial j_1(x,u) + f(x,u,\nabla u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu_n} \in r_2(u)\partial j_2(x,u) \qquad & \text{on } \Gamma_2, \\ -\frac{\partial u}{\partial \nu_n} \in \partial_c \phi(x,u) \qquad & \text{on } \Gamma_3, \\ L(u) \le J(u), \qquad & \end{aligned} \tag{1.9}$$

where $\frac{\partial u}{\partial \nu_n}$ is given in (1.4). We also obtain the nonemptiness and compactness of the solution set of problem (1.9), see Corollary 4.17. If f is independent of ∇u, then problem (1.9) can be seemed as a special case of problem (1.5).

(b) If $\Gamma_2 = \emptyset$ and $\Gamma_3 = \emptyset$, i.e., $\Gamma_1 = \Gamma$, problem (1.3) reduces to the following nonlinear implicit obstacle problem with nonlinear convection term and Dirichlet boundary condition:

$$\begin{aligned} -a(u)\Delta_p u - b(u)\Delta_q u + g(x,u) \in U_1(x,u) + f(x,u,\nabla u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma, \\ L(u) \le J(u). \qquad & \end{aligned} \tag{1.10}$$

In this case, we obtain Corollary 4.10 getting one weak solution to problem (1.10).

(c) If f is independent of ∇u, then problem (1.3) becomes the following problem:

$$\begin{aligned} -a(u)\Delta_p u - b(u)\Delta_q u + g(x,u) \in U_1(x,u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma, \\ L(u) \le J(u). \qquad & \end{aligned} \tag{1.11}$$

This is exactly the particular case of problem (1.6) if the exponents p, q are constants.

(d) Let $\Psi\colon \Omega \to \mathbb{R}$ be a given obstacle. When $J(u) \equiv 0$ and $L(u) := \int_\Omega (u(x) - \Psi(x))^+ \, dx$ for all $u \in W^{1,p}(\Omega)$, then problem (1.3) can be rewritten to the following obstacle inclusion problem with nonlinear convection term:

$$\begin{aligned} -a(u)\Delta_p u - b(u)\Delta_q u + g(x,u) \in U_1(x,u) + f(x,u,\nabla u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu_n} \in U_2(x,u) \qquad & \text{on } \Gamma_2, \\ -\frac{\partial u}{\partial \nu_n} \in \partial_c \phi(x,u) \qquad & \text{on } \Gamma_3, \\ u(x) \le \Psi(x) \qquad & \text{in } \Omega, \end{aligned} \tag{1.12}$$

where $\frac{\partial u}{\partial \nu_n}$ is given in (1.4). In the case a, b to be independent of $u \in W^{1,p}(\Omega)$, i.e., a, b are two nonnegative constants, problem (1.12) has been recently studied by Zeng-Bai-Gasiński [45].

(e) If $J(u) \equiv +\infty$ or $L(u) \equiv -\infty$ for all $u \in W^{1,p}(\Omega)$, then problem (1.3) turns into the following mixed boundary value problem with nonlinear convection term, but without obstacle effect:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \in U_1(x,u) + f(x,u,\nabla u) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu_n} \in U_2(x,u) \qquad & \text{on } \Gamma_2, \\ -\frac{\partial u}{\partial \nu_n} \in \partial_c \phi(x,u) \qquad & \text{on } \Gamma_3, \end{aligned} \tag{1.13}$$

where $\frac{\partial u}{\partial \nu_n}$ is given in (1.4).

The paper is organized as follows. Section 2 presents a detailed overview about variable exponent Lebesgue/Sobolev spaces, the eigenvalue problem of the p-Laplacian with Steklov boundary condition and we state some results from nonsmooth analysis, the properties of Clarke's generalized gradient and Tychonoff's fixed point theorem for multivalued operators which will be used in the next sections to establish the main results of this paper. In Section 3, in order to establish the solvability of the anisotropic implicit obstacle problem (1.1), we first introduce an auxiliary problem defined in (3.3) and apply an existence theorem for a class of mixed variational inequalities involving coercive and monotone operators to prove the existence and uniqueness of the auxiliary problem. Finally, we introduce two multivalued operators, which are proved to be strongly-weakly u.s.c. and apply Tychonoff's fixed point theorem for multivalued operators along with the theory of nonsmooth analysis to examine the nonemptiness and compactness of the solution set of problem (1.1). After that, in Section 4, we move our attention to prove the solvability of the implicit obstacle problem (1.3) with nonlinear convection term. Lastly, several special and interesting cases of our problem (1.3) are discussed and the corresponding and extended existence results are obtained at the end of the paper.

2. Preliminaries

In this section we present the main tools which are needed in the sequel. For this purpose, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$, where Γ is divided into three mutually disjoint parts Γ_1, Γ_2 and Γ_3 with Γ_1 having positive Lebesgue measure. For any fixed $r \in [1,\infty)$ and for any subset D of $\overline{\Omega}$ we denote the usual Lebesgue spaces by $L^r(D) := L^r(D;\mathbb{R})$ and $L^r(D;\mathbb{R}^N)$ equipped with the norm $\|\cdot\|_{r,D}$ given by

$$\|u\|_{r,D} := \left(\int_D |u|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \quad \text{for all } u \in L^r(D).$$

Moreover, we set $L^r(D)_+ := \{u \in L^r(D) \,:\, u(x) \geq 0 \text{ for a. a. } x \in D\}$. By $W^{1,r}(\Omega)$ we define the corresponding Sobolev space endowed with the norm $\|\cdot\|_{1,r,\Omega}$ given by

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

In the entire paper, the symbols "$\xrightarrow{w}$" and "$\to$" stand for the weak and the strong convergence, respectively. Moreover, the conjugate of $r>1$ is denoted by $r'>1$, e.g., $\frac{1}{r}+\frac{1}{r'}=1$. The critical exponents of $r>1$ in the domain and on the boundary, denoted by r^* and r_*, are defined by

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N,\\ +\infty & \text{if } r \geq N,\end{cases} \quad \text{and} \quad r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N,\\ +\infty & \text{if } r \geq N,\end{cases} \tag{2.1}$$

respectively. From Simon [44, formula (2.2)], we have the well-known inequality

$$\left(|x|^{r-2}x - |y|^{r-2}y\right)\cdot(x-y) \geq k(r)|x-y|^r \tag{2.2}$$

for $r \geq 2$ and for all $x,y \in \mathbb{R}^N$, where $k(r)$ is a positive constant.

The eigenvalue problem of the r-Laplacian ($r>1$) with Steklov boundary condition is given by

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u \qquad && \text{in } \Omega,\\ |u|^{r-2}u\cdot\nu &= \lambda|u|^{r-2}u && \text{on } \Gamma. \end{aligned} \tag{2.3}$$

We know that problem (2.3) has a smallest eigenvalue $\lambda_{1,r}^S > 0$ that is isolated and simple, see Lê [22]. Also, $\lambda_{1,r}^S > 0$ can be characterized by

$$\lambda_{1,r}^S = \inf_{u \in W^{1,r}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{r,\Omega}^r + \|u\|_{r,\Omega}^r}{\|u\|_{r,\Gamma}^r}. \tag{2.4}$$

In what follows, we denote by $u_{1,r}^S$ the first eigenfunction of problem (2.3) corresponding to the first eigenvalue $\lambda_{1,r}^S$. It is clear that $u_{1,r}^S \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, where $\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ stands for the interior of

$$C^1(\overline{\Omega})_+ := \{u \in C^1(\overline{\Omega}) \,:\, u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\},$$

that is

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega}) \,:\, u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

Without any loss of generality, we suppose that $\left\|u_{1,r}^S\right\|_{r,\Gamma} = 1$.

Next, we introduce the subset $C_+(\overline{\Omega})$ of $C(\overline{\Omega})$ defined by

$$C_+(\overline{\Omega}) := \left\{s \in C(\overline{\Omega}) \,:\, 1 < s(x) \text{ for all } x \in \overline{\Omega}\right\}.$$

For any $r \in C_+(\overline{\Omega})$, we define

$$r_- := \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_+ := \max_{x\in\overline{\Omega}} r(x).$$

Let $p \in C_+(\overline{\Omega})$. In what follows, we denote by $p' \in C_+(\overline{\Omega})$ the conjugate variable exponent to p, namely,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Also, we denote by s^* and s_* the critical Sobolev variable exponents to $s \in C_+(\overline{\Omega})$ in the domain and on the boundary, respectively, given by

$$s^*(x) = \begin{cases} \frac{Ns(x)}{N-s(x)} & \text{if } s(x) < N, \\ +\infty & \text{if } s(x) \geq N, \end{cases} \quad \text{for all } x \in \overline{\Omega}, \tag{2.5}$$

and

$$s_*(x) = \begin{cases} \frac{(N-1)s(x)}{N-s(x)} & \text{if } s(x) < N, \\ +\infty & \text{if } s(x) \geq N \end{cases} \quad \text{for all } x \in \overline{\Omega}, \tag{2.6}$$

respectively.

By $M(\Omega)$ we denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. For $r \in C_+(\overline{\Omega})$ the variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ is defined by

$$L^{r(\cdot)}(\Omega) := \left\{ u \in M(\Omega) \,:\, \int_\Omega |u|^{r(x)}\,\mathrm{d}x < +\infty \right\}.$$

It is well-known that $L^{r(\cdot)}(\Omega)$ equipped with the Luxemburg norm given by

$$\|u\|_{r(\cdot),\Omega} := \inf \left\{ \lambda > 0 \,:\, \int_\Omega \left(\frac{|u|}{\lambda} \right)^{r(x)} \mathrm{d}x \leq 1 \right\}$$

is a separable and reflexive Banach space, the dual space of $L^{r(\cdot)}(\Omega)$ is $L^{r'(\cdot)}(\Omega)$ and the following Hölder inequality holds:

$$\int_\Omega |uv|\,\mathrm{d}x \leq \left[\frac{1}{r_-} + \frac{1}{r'_-} \right] \|u\|_{r(\cdot),\Omega} \|v\|_{r'(\cdot),\Omega} \leq 2\|u\|_{r(\cdot),\Omega} \|v\|_{r'(\cdot),\Omega}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$. Moreover, if $r_1, r_2 \in C_+(\overline{\Omega})$ are such that $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega).$$

For any $r \in C_+(\overline{\Omega})$, we consider the modular function $\varrho_{r(\cdot),\Omega}\colon L^{r(\cdot)}(\Omega) \to \mathbb{R}_+ := [0, +\infty)$ given by

$$\varrho_{r(\cdot),\Omega}(u) := \int_\Omega |u|^{r(x)}\,\mathrm{d}x \quad \text{for all } u \in L^{r(\cdot)}(\Omega). \tag{2.7}$$

The following proposition states some important relations between the norm of $L^{r(\cdot)}(\Omega)$ and the modular function $\varrho_{r(\cdot),\Omega}$ defined in (2.7).

Proposition 2.1. *If $r \in C_+(\overline{\Omega})$ and $u \in L^{r(\cdot)}(\Omega)$, then we have the following assertions:*

(i) $\|u\|_{r(\cdot),\Omega} = \lambda \iff \varrho_{r(\cdot),\Omega}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$*;*
(ii) $\|u\|_{r(\cdot),\Omega} < 1$ *(resp.* $= 1, > 1$*)* $\iff \varrho_{r(\cdot),\Omega}(u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) $\|u\|_{r(\cdot),\Omega} < 1 \implies \|u\|_{r(\cdot),\Omega}^{r_+} \leq \varrho_{r(\cdot),\Omega}(u) \leq \|u\|_{r(\cdot),\Omega}^{r_-}$*;*
(iv) $\|u\|_{r(\cdot),\Omega} > 1 \implies \|u\|_{r(\cdot),\Omega}^{r_-} \leq \varrho_{r(\cdot),\Omega}(u) \leq \|u\|_{r(\cdot),\Omega}^{r_+}$*;*
(v) $\|u\|_{r(\cdot),\Omega} \to 0 \iff \varrho_{r(\cdot),\Omega}(u) \to 0$*;*
(vi) $\|u\|_{r(\cdot),\Omega} \to +\infty \iff \varrho_{r(\cdot),\Omega}(u) \to +\infty$.

Let D be a nonempty subset of $\overline{\Omega}$. In what follows, we denote by $\|\cdot\|_{r(\cdot),D}$ the norm of the variable exponent Lebesgue space $L^{r(\cdot)}(D)$. We set $\varrho_{r(\cdot),D}(u) = \displaystyle\int_D |u|^{r(x)}\,\mathrm{d}x$ for $u \in L^{r(\cdot)}(D)$.

Further, for $r \in C_+(\overline{\Omega})$, we denote by $W^{1,r(\cdot)}(\Omega)$ the variable exponent Sobolev space given in

$$W^{1,r(\cdot)}(\Omega) := \left\{ u \in L^{r(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{r(\cdot)}(\Omega) \right\},$$

which is equipped with the norm

$$\|u\|_{1,r(\cdot),\Omega} := \|u\|_{r(\cdot),\Omega} + \|\nabla u\|_{r(\cdot),\Omega} \quad \text{for all } u \in W^{1,r(\cdot)}(\Omega),$$

to be a separable and reflexive Banach space, where $\|\nabla u\|_{r(\cdot),\Omega} := \||\nabla u|\|_{r(\cdot),\Omega}$. Moreover we define

$$W_0^{1,r(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(\cdot),\Omega}}.$$

with norm $\|\cdot\|_{1,r(\cdot),\Omega}$. From Poincaré's inequality, we know that we can endow the space $W_0^{1,r(\cdot)}(\Omega)$ with the equivalent norm

$$\|u\|_{1,r(\cdot),0,\Omega} = \|\nabla u\|_{r(\cdot),\Omega} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

Additionally, we introduce a closed subset V of $W^{1,p(\cdot)}(\Omega)$ given by

$$V := \left\{ u \in W^{1,p(\cdot)}(\Omega) \,:\, u = 0 \text{ for a. a. } x \in \Gamma_1 \right\}.$$

It is clear that V equipped with the norm $V \ni u \mapsto \|u\|_{1,r(\cdot),\Omega} \in \mathbb{R}$ becomes a reflexive Banach space.

Employing Proposition 2.1, we also have the following proposition.

Proposition 2.2. *Let $r \in C_+(\overline{\Omega})$ and $\iota_{r(\cdot),\Omega}\colon W^{1,r(\cdot)}(\Omega) \to \mathbb{R}_+ := [0,+\infty)$ be the modular function given by*

$$\iota_{r(\cdot),\Omega}(u) := \int_\Omega |\nabla u|^{r(x)}\,\mathrm{d}x + \int_\Omega |u|^{r(x)}\,\mathrm{d}x \quad \textit{for all } u \in W^{1,r(\cdot)}(\Omega).$$

If $u \in W^{1,r(\cdot)}(\Omega)$, then we have the following assertions:

(i) $\|u\|_{1,r(\cdot),\Omega} = \lambda \iff \iota_{r(\cdot),\Omega}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$*;*
(ii) $\|u\|_{1,r(\cdot),\Omega} < 1$ *(resp.* $= 1, > 1$*)* $\iff \iota_{r(\cdot),\Omega}(u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) $\|u\|_{1,r(\cdot),\Omega} < 1 \implies \|u\|^{r_+}_{1,r(\cdot),\Omega} \le \iota_{r(\cdot),\Omega}(u) \le \|u\|^{r_-}_{1,r(\cdot),\Omega}$*;*
(iv) $\|u\|_{1,r(\cdot),\Omega} > 1 \implies \|u\|^{r_-}_{1,r(\cdot),\Omega} \le \iota_{r(\cdot),\Omega}(u) \le \|u\|^{r_+}_{1,r(\cdot),\Omega}$*;*
(v) $\|u\|_{1,r(\cdot),\Omega} \to 0 \iff \iota_{r(\cdot),\Omega}(u) \to 0$*;*
(vi) $\|u\|_{1,r(\cdot),\Omega} \to +\infty \iff \iota_{r(\cdot),\Omega}(u) \to +\infty$*.*

In the sequel, we denote by $C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ the set of all functions $r\colon \overline{\Omega} \to \mathbb{R}$ that are log-Hölder continuous, namely, there is a constant $C > 0$ satisfying

$$|r(x) - r(y)| \le \frac{C}{|\log|x-y||} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } |x-y| < \frac{1}{2}.$$

The following propositions give several important embeddings results, its detailed proof can be founded in Diening-Harjulehto-Hästö-Ružička [11, Corollary 8.3.2] and Fan [12, Propositions 2.1 and 2.2].

Proposition 2.3.

(i) *If $r \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega}) \cap C_+(\overline{\Omega})$ and $s \in C(\overline{\Omega})$ is such that*

$$1 \le s(x) \le r^*(x) \quad \textit{for all } x \in \overline{\Omega},$$

then the embedding

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$$

is continuous.

(ii) *If $s \in C_+(\overline{\Omega})$ is such that*

$$1 \le s(x) < r^*(x) \quad \textit{for all } x \in \overline{\Omega},$$

then the embedding

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$$

is compact.

Proposition 2.4.

(i) *If $r \in C_+(\overline{\Omega}) \cap W^{1,\varsigma}(\Omega)$ for some $\varsigma > N$ and $s \in C(\overline{\Omega})$ is such that*

$$1 \le s(x) \le r_*(x) \quad \textit{for all } x \in \overline{\Omega},$$

then the embedding

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\partial\Omega)$$

is continuous.

(ii) *If $s \in C_+(\overline{\Omega})$ is such that*

$$1 \le s(x) < r_*(x) \quad \textit{for all } x \in \overline{\Omega},$$

then the embedding

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\partial\Omega)$$

is compact.

Remark 2.5. *The embeddings in Propositions 2.3 and 2.4 remain valid if we replace the space $W^{1,r(\cdot)}(\Omega)$ by V.*

Next, we introduce the nonlinear operator $F\colon V \to V^*$ given by

$$\langle F(u), v\rangle := \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega |u|^{p(x)-2} uv \,\mathrm{d}x, \tag{2.8}$$

for $u, v \in V$ with $\langle \cdot,\cdot\rangle$ being the duality pairing between V and its dual space V^*. Arguing as in the proof of Proposition 2.5 of Gasiński-Papageorgiou [17] or Rădulescu-Repovš [43, p. 40], we have the following result which states the main properties of $F\colon V \to V^*$.

Proposition 2.6. *The operator F defined by (2.8) is bounded, continuous, monotone (hence maximal monotone) and of type (S_+), that is,*

$$u_n \xrightarrow{w} u \quad \textit{in } V \quad \textit{and} \quad \limsup_{n\to\infty} \langle F(u_n), u_n - u\rangle \le 0,$$

imply $u_n \to u$ in V.

In the last part of this section we are going to recall some results from nonsmooth analysis and multivalued analysis. First, we recall some definitions and properties of semicontinuous multivalued operators.

Definition 2.7. *Let Y and Z be topological spaces, let $D \subset Y$ be a nonempty set, and let $G\colon Y \to 2^Z$ be a multivalued map.*

(i) *The map G is called upper semicontinuous (u.s.c. for short) at $y \in Y$, if for each open set $O \subset Z$ such that $G(y) \subset O$, there exists a neighborhood $N(y)$ of y satisfying $G(N(y)) := \cup_{z\in N(y)} G(z) \subset O$. If it holds for each $y \in D$, then G is called to be upper semicontinuous in D.*

(ii) *The map G is closed at $y \in Y$, if for every sequence $\{(y_n, z_n)\}_{n\in\mathbb{N}} \subset \operatorname{Gr}(G)$ satisfying $(y_n, z_n) \to (y, z)$ in $Y \times Z$, it holds $(y, z) \in \operatorname{Gr}(G)$, where $\operatorname{Gr}(G)$ is the graph of G defined by*

$$\operatorname{Gr}(G) := \{(y, z) \in Y \times Z \,|\, z \in G(y)\}.$$

If it holds for each $y \in Y$, then G is called to be closed or G has a closed graph.

The next proposition gives equivalent characterizations of multivalued functions to be upper semicontinuous.

Proposition 2.8. *Let $F\colon X \to 2^Y$ with X and Y being topological spaces. The following statements are equivalent:*

(i) *F is upper semicontinuous.*
(ii) *For each closed set $C \subset Y$, $F^-(C) := \{x \in X \,|\, F(x) \cap C \neq \emptyset\}$ is closed in X.*
(iii) *For each open set $O \subset Y$, $F^+(O) := \{x \in X \,|\, F(x) \subset O\}$ is open in X.*

In the following, let E be real Banach space with norm $\|\cdot\|_E$. A function $\varphi\colon E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be proper, convex and lower semicontinuous, if the following conditions are fulfilled:

- $D(\varphi) := \{u \in E : \varphi(u) < +\infty\} \neq \emptyset$;
- for any $u, v \in E$ and $t \in (0, 1)$, it holds $\varphi(tu + (1-t)v) \le t\varphi(u) + (1-t)\varphi(v)$;
- $\liminf_{n\to\infty} \varphi(u_n) \ge \varphi(u)$ where the sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ is such that $u_n \to u$ in E as $n \to \infty$ for some $u \in E$.

Let φ be a convex mapping. An element $x^* \in E^*$ is said to be a subgradient of φ at $u \in E$ if

$$\langle x^*, v - u\rangle \le \varphi(v) - \varphi(u) \tag{2.9}$$

holds for all $v \in E$. The set of all elements $x^* \in E^*$ which satisfies (2.9) is called the convex subdifferential of φ at u and is denoted by $\partial_c\varphi(u)$.

Moreover, a function $j\colon E \to \mathbb{R}$ is said to be locally Lipschitz at $x \in E$ if there is a neighborhood $O(x)$ of x and a constant $L_x > 0$ such that

$$|j(y) - j(z)| \le L_x\|y - z\|_E \quad \text{for all } y, z \in O(x).$$

We denote by

$$j^\circ(x; y) := \limsup_{z\to x,\, \lambda\downarrow 0} \frac{j(z + \lambda y) - j(z)}{\lambda},$$

the generalized directional derivative of j at the point x in the direction y and $\partial j\colon E \to 2^{E^*}$ given by

$$\partial j(x) := \{\, \xi \in E^* : j^\circ(x; y) \ge \langle \xi, y\rangle_{E^*\times E} \text{ for all } y \in E\} \quad \text{for all } x \in E$$

is the generalized gradient of j at x in the sense of Clarke.

The next proposition summarizes the properties of generalized gradients and generalized directional derivatives of a locally Lipschitz function. We refer to Migórski-Ochal-Sofonea [32, Proposition 3.23] for its proof.

Proposition 2.9. *Let $j\colon E \to \mathbb{R}$ be locally Lipschitz with Lipschitz constant $L_x > 0$ at $x \in E$. Then we have the following:*

(i) *The function $y \mapsto j^\circ(x; y)$ is positively homogeneous, subadditive, and satisfies*

$$|j^\circ(x; y)| \le L_x\|y\|_E \quad \text{for all } y \in E.$$

(ii) *The function $(x, y) \mapsto j^\circ(x; y)$ is upper semicontinuous.*
(iii) *For each $x \in E$, $\partial j(x)$ is a nonempty, convex, and weak* compact subset of E^* with $\|\xi\|_{E^*} \le L_x$ for all $\xi \in \partial j(x)$.*
(iv) *$j^\circ(x; y) = \max\{\langle \xi, y\rangle_{E^*\times E} \,|\, \xi \in \partial j(x)\}$ for all $y \in E$.*

(v) *The multivalued function $E \ni x \mapsto \partial j(x) \subset E^*$ is upper semicontinuous from E into the subsets of E^* with weak* topology.*

Finally, we recall Tychonoff's fixed point theorem for multivalued operators. The proof of this result can be found in Granas-Dugundji [20, Theorem 8.6].

Theorem 2.10. *Let D be a bounded, closed and convex subset of a reflexive Banach space E, and $\Lambda\colon D \to 2^D$ be a multivalued map such that*

(i) *Λ has bounded, closed and convex values,*
(ii) *Λ is weakly-weakly u.s.c.*

Then Λ has a fixed point in D.

3. Anisotropic implicit obstacle problems

The main objective of this section is to develop a generalized framework for examining the existence of weak solutions to the nonlinear implicit obstacle inclusion problem with multivalued boundary conditions and nonlocal terms given by (1.1). Our method is based on the theory of nonsmooth analysis, convex analysis, Tychonoff's fixed point theorem for multivalued operators and variational approach.

We start by imposing the precise assumptions on the data of problem (1.1).

H(0): $p, q \in C_+(\overline{\Omega})$ are such that

$$q(x) < p(x) \quad \text{for all } x \in \overline{\Omega}.$$

H(1): $a\colon L^{p^*(\cdot)}(\Omega) \to (0, +\infty)$ and $b\colon L^{p^*(\cdot)}(\Omega) \to [0, +\infty)$ are such that

(i) a is weakly continuous in V, i.e., if $\{u_n\}_{n\in\mathbb{N}} \subset V \subset L^{p^*(\cdot)}(\Omega)$ is such that $u_n \xrightarrow{w} u$ in V, then it holds

$$a(u) = \lim_{n\to\infty} a(u_n),$$

and there exists a constant $c_a > 0$ satisfying

$$a(u) \geq c_a \text{ for all } u \in V,$$

where p^* is the critical exponent of p in the domain Ω given in (2.5);

(ii) b is a weakly continuous in V.

H(g): The function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) the function $x \mapsto g(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$;
(ii) the function $s \mapsto g(x, s)$ is continuous for a. a. $x \in \Omega$;
(iii) there exist a constant $\alpha_g > 0$ and a function $\beta_g \in L^{\delta_0'(\cdot)}(\Omega)_+$ such that

$$|g(x, s)| \leq \beta_g(x) + \alpha_g |s|^{\delta_0(x)-1}$$

for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $\delta_0 \in C_+(\overline{\Omega})$ is such that

$$\delta_0(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega};$$

(iv) there exist a constant $a_g > 0$ and a function $b_g \in L^1(\Omega)$ such that

$$g(x, s)s \geq a_g |s|^{\varsigma(x)} - b_g(x)$$

for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $\varsigma \in C_+(\overline{\Omega})$ is such that

$$p(x) < \varsigma(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega};$$

(v) the function $s \mapsto g(x, s)$ is nondecreasing for a. a. $x \in \Omega$, i.e.,

$$(g(x, s_1) - g(x, s_2))(s_1 - s_2) \geq 0$$

for all $s_1, s_2 \in \mathbb{R}$ and for a. a. $x \in \Omega$.

H(U_1): The multivalued function $U_1\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that

(i) $U_1(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. a. $x \in \Omega$ and all $s \in \mathbb{R}$;
(ii) $x \mapsto U_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$;
(iii) $s \mapsto U_1(x,s)$ is u.s.c. for a. a. $x \in \Omega$;
(iv) there exist a function $\alpha_{U_1} \in L^{\delta_1'(\cdot)}(\Omega)_+$ and a constant $a_{U_1} \geq 0$ such that
$$|\eta| \leq \alpha_{U_1}(x) + a_{U_1}|s|^{\delta_1(x)-1}$$
for all $\eta \in U_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $\delta_1 \in C_+(\overline{\Omega})$ is such that
$$\delta_1(x) < p(x) \quad \text{for all } x \in \overline{\Omega}.$$

H(U_2): The multivalued function $U_2 \colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that
(i) $U_2(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. a. $x \in \Gamma_2$ and all $s \in \mathbb{R}$;
(ii) $x \mapsto U_2(x,s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$;
(iii) $s \mapsto U_2(x,s)$ is u.s.c. for a. a. $x \in \Gamma_2$;
(iv) there exist a function $\alpha_{U_2} \in L^{\delta_2'(\cdot)}(\Gamma_2)_+$ and a constant $a_{U_2} > 0$ such that
$$|\xi| \leq \alpha_{U_2}(x) + a_{U_2}|s|^{\delta_2(x)-1}$$
for all $\xi \in U_2(x,s)$, for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$, where $\delta_2 \in C_+(\overline{\Omega})$ is such that
$$\delta_2(x) < p(x) \quad \text{for all } x \in \overline{\Omega}.$$

H(ϕ): The function $\phi \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that
(i) $x \mapsto \phi(x,r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$;
(ii) $r \mapsto \phi(x,r)$ is convex and l.s.c. for a. a. $x \in \Gamma_3$;
(iii) for each function $u \in L^{p_*(\cdot)}(\Gamma_3)$ the function $x \mapsto \phi(x,u(x))$ belongs to $L^1(\Gamma_3)$, where p_* is the critical exponent of p on the boundary Γ given in (2.6).

H(L): $L \colon V \to \mathbb{R}$ is positively homogeneous and subadditive such that
$$L(u) \leq \limsup_{n\to\infty} L(u_n), \tag{3.1}$$
whenever $\{u_n\}_{n\in\mathbb{N}} \subset V$ is such that $u_n \xrightarrow{w} u$ in V for some $u \in V$.

H(J): $J \colon V \to (0,+\infty)$ is weakly continuous, that is, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$ such that $u_n \xrightarrow{w} u$ for some $u \in V$, we have
$$J(u_n) \to J(u).$$

Remark 3.1. *From hypotheses* H(L)*, we can observe that on the one hand, the homogeneity and subadditivity of* L *guarantee the convexity of* L *and on the other hand, if* $L \colon V \to \mathbb{R}$ *is weak lower semicontinuous, then inequality* (3.1) *holds automatically.*

Example 3.2. *Given a constant* $c_a > 0$*, the functions*
$$a(u) = c_a + \int_\Omega |u|^\tau \,\mathrm{d}x \quad \textit{and} \quad b(u) = \prod_{i=1}^k \left| \int_\Omega |u|^{\tau_i}\,\mathrm{d}x - \pi_i \right|,$$
satisfy hypotheses H(g)*, where* $\tau, \tau_1, \ldots, \tau_k \in [1, p_-^*)$ *and* $\pi_1, \ldots, \pi_k \in [0,+\infty)$*. Observe that the function* b *given above is finite degenerate.*

Let $c_g > 0$ *and* $\varsigma_0, \varsigma \in C_+(\overline{\Omega})$ *and* $\beta_g \in L^{\varsigma'(\cdot)}(\Omega)$ *be such that*
$$\varsigma_0(x) \leq p(x) < \varsigma(x) < p^*(x) \quad \textit{for all } x \in \overline{\Omega}.$$
Then, the following function satisfies hypotheses H(g)
$$g(x,s) = \begin{cases} c_g|s|^{\varsigma_0(x)-2}s + \beta_g(x) & \textit{if } |s| \leq 1, \\ c_g|s|^{\varsigma(x)-2}s + \beta_g(x) & \textit{if } |s| > 1, \end{cases} \quad \textit{for a. a. } x \in \Omega.$$

Let $\omega \in L^\infty(\Gamma_3)_+$*. Then, the function* $\phi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ *fulfills assumption* H(ϕ)

$$\phi(x,s) = \begin{cases} \omega(x)|s| & \text{if } |s| \le 1, \\ \omega(x)|s|^{\varsigma_2(x)} & \text{if } |s| > 1, \end{cases} \quad \text{for a. a. } x \in \Gamma_3.$$

In order to formulate the implicit obstacle effect to a suitable variational constraint, we consider the multivalued map $K\colon V \to 2^V$ defined by

$$K(u) := \{v \in V \;:\; L(v) \le J(u)\} \tag{3.2}$$

for all $u \in V$.

Next, we state the definition of a weak solution of problem (1.1).

Definition 3.3. *A function* $u \in V$ *is said to be a weak solution of problem* (1.1)*, if* $u \in K(u)$ *and there exist functions* $\eta \in L^{\delta_1'(\cdot)}(\Omega)$*,* $\xi \in L^{\delta_2'(\cdot)}(\Gamma_2)$ *such that* $\eta(x) \in U_1(x,u(x))$ *for a. a.* $x \in \Omega$*,* $\xi(x) \in U_2(x,u(x))$ *for a. a.* $x \in \Gamma_2$ *and the inequality*

$$\begin{aligned} & a(u)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(v-u)\,\mathrm{d}x + b(u)\int_\Omega |\nabla u|^{q(x)-2}\nabla u \cdot \nabla(v-u)\,\mathrm{d}x \\ & \quad + \int_\Omega g(x,u)(v-u)\,\mathrm{d}x + \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma \\ & \ge \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)(v-u)\,\mathrm{d}\Gamma \end{aligned}$$

is satisfied for all $v \in K(u)$*, where the multivalued function* $K\colon V \to 2^V$ *is defined by* (3.2)*.*

The main result in this section is stated by the following theorem.

Theorem 3.4. *Assume that* H(0), H(1), H(g), H(U_1), H(U_2), H(ϕ), H(L) *and* H(J) *are satisfied. Then, the solution set of problem* (1.1)*, denoted by* Υ*, is nonempty and compact in* V*.*

In order to prove Theorem 3.4, we need the following important auxiliary result which delivers several significant properties for the multivalued mapping $K\colon V \to 2^V$. More precisely, this lemma reveals an essential characteristic that K is Mosco continuous (see Mosco [34], i.e., K is sequentially weakly-weakly closed and sequentially weakly-strongly l.s.c.). The detailed proof of this lemma can be found in Lemma 3.3 of Zeng-Rădulescu-Winkert [48].

Lemma 3.5. *Let* $J\colon V \to (0,+\infty)$ *and* $L\colon V \to \mathbb{R}$ *be two functions such that* H(L) *and* H(J) *are satisfied. Then, the following statements hold:*

(i) *for each* $u \in V$*,* $K(u)$ *is closed and convex in* V *such that* $0 \in K(u)$*;*
(ii) *the graph* $\mathrm{Gr}(K)$ *of* K *is sequentially closed in* $V_w \times V_w$*, that is,* K *is sequentially closed from* V *with the weak topology into the subsets of* V *with the weak topology;*
(iii) *if* $\{u_n\}_{n\in\mathbb{N}} \subset V$ *is a sequence such that*

$$u_n \xrightarrow{w} u \quad \text{in } V$$

for some $u \in V$*, then for each* $v \in K(u)$ *there exists a sequence* $\{v_n\}_{n\in\mathbb{N}} \subset V$ *such that*

$$v_n \in K(u_n) \quad \text{and} \quad v_n \to v \quad \text{in } V.$$

Note that problem (1.1) has several interesting and complicated characterizations, such as, highly abstract nonlocal functions (which could be specialized to a nonlinear Kirchhoff type condition (see for example, in [2], the authors combined the effects of a nonlocal Kirchhoff coefficient and a double phase operator with a singular term and a critical Sobolev nonlinearity in which the proof of main result is based on a suitable minimization argument on the Nehari manifold; the work [16] investigates the effects of an indefinite Kirchhoff type function on the geometry of an elliptic problem, by adopting an approximation process based on the Galerkin method), multivalued terms (which can be seemed as feedback control effect from the control point of view), and also nonsmooth boundary conditions. This leads to tremendous difficulties from various perspectives. For example, we are not able to use directly variational methods, topological techniques and the theory of set-valued analysis for determining the existence of a weak solution. In order to bypass those difficulties, we consider the following auxiliary problem: for given functions $(w,\eta,\xi)\in V\times X^*\times Y^*$, find a function $u\colon\Omega\to\mathbb{R}$ such that

$$\begin{aligned}
-a(w)\Delta_{p(\cdot)}u-b(w)\Delta_{q(\cdot)}u+g(x,u)&=\eta(x) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma_1,\\
\frac{\partial u(x)}{\partial\nu_w}&=\xi(x) && \text{on } \Gamma_2,\\
-\frac{\partial u(x)}{\partial\nu_w}&\in\partial_c\phi(x,u) && \text{on } \Gamma_3,\\
L(u)&\le J(w),
\end{aligned}\tag{3.3}$$

where $X:=L^{\delta_1(\cdot)}(\Omega)$, $Y:=L^{\delta_2(\cdot)}(\Gamma_2)$, and X^* and Y^* are the dual spaces of X and Y (i.e., $X^*:=L^{\delta_1'(\cdot)}(\Omega)$ and $Y^*:=L^{\delta_2'(\cdot)}(\Gamma_2)$), respectively, and

$$\frac{\partial u}{\partial\nu_w}:=\Big(a(w)|\nabla u|^{p(x)-2}\nabla u+b(w)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nu.$$

Note that problem (3.3) is an anisotropic obstacle problem with mixed boundary conditions.

From Definition 3.3, it is not difficult to see that a function $u\in V$ is a weak solution of problem (3.3), if the following holds: $u\in K(w)$ and

$$\begin{aligned}
&a(w)\int_\Omega|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x+b(w)\int_\Omega|\nabla u|^{q(x)-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x\\
&\quad+\int_\Omega g(x,u)(v-u)\,\mathrm{d}x+\int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma\\
&\ge\int_\Omega\eta(x)(v-u)\,\mathrm{d}x+\int_{\Gamma_2}\xi(x)(v-u)\,\mathrm{d}\Gamma
\end{aligned}$$

for all $v\in K(w)$.

The following lemma examines the existence and uniqueness of problem (3.3).

Lemma 3.6. *Suppose that* H(0), H(g) *and* H(ϕ) *are fulfilled. Then, for each fixed* $(w,\eta,\xi)\in V\times X^*\times Y^*$, *problem* (3.3) *has a unique solution.*

Proof. Recall that $V\hookrightarrow X$, $V\hookrightarrow Y$ and $V\hookrightarrow L^{\delta_0(\cdot)}(\Omega)$ are continuous embeddings. We introduce the nonlinear operator $\mathcal{F}\colon V\to V^*$ given by

$$\begin{aligned}
\langle\mathcal{F}(u),v\rangle:=&\,a(w)\int_\Omega|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x+b(w)\int_\Omega|\nabla u|^{q(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x\\
&+\int_\Omega g(x,u)v\,\mathrm{d}x-\int_\Omega\eta(x)v\,\mathrm{d}x-\int_{\Gamma_2}\xi(x)v\,\mathrm{d}\Gamma
\end{aligned}$$

for all $u,v\in V$. By virtue of hypotheses H(0) and H(g), we can see that $\mathcal{F}\colon V\to V^*$ is a continuous, bounded and strictly monotone operator. Furthermore, let us consider the function $\varphi\colon V\to\mathbb{R}$ defined by

$$\varphi(u):=\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma\quad\text{for all } u\in V,$$

which is well-defined due to hypothesis H(ϕ)(iii). Applying standard arguments, it is not difficult to prove that φ is a proper, convex and l.s.c. function in V. In fact, it is convex and continuous, because the effective domain of φ contains V.

Utilizing the notation above, it is obvious that u is a weak solution of problem (3.3), if and only if it solves the following mixed variational inequality problem: find $u\in K(w)$ such that

$$\langle\mathcal{F}(u),v-u\rangle+\varphi(v)-\varphi(u)\ge0\tag{3.4}$$

for all $v \in K(w)$. Moreover, using hypotheses H(0) and H(g)(iv), we obtain

$$\begin{aligned}
&\langle \mathcal{F}u, u\rangle \\
&\geq a(w)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + b(w)\int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x + \int_\Omega a_g|u|^{\varsigma(x)} - b_g(x)\,\mathrm{d}x \\
&\quad - \int_\Omega \eta(x)u\,\mathrm{d}x - \int_{\Gamma_2}\xi(x)u\,\mathrm{d}\Gamma \\
&\geq c_a\varrho_{p(\cdot),\Omega}(\nabla u) + b(w)\varrho_{q(\cdot),\Omega}(\nabla u) - \left(\|\eta\|_{V^*} + \|\xi\|_{V^*}\right)\|u\|_V - \|b_g\|_{1,\Omega} + a_g\varrho_{\varsigma(\cdot),\Omega}(u),
\end{aligned}$$

where $c_a > 0$ is given in hypotheses $H(1)$. Keeping in mind that $p(x) < \varsigma(x)$ for all $x \in \overline{\Omega}$, it follows from Young's inequality that

$$a_g\varrho_{\varsigma(\cdot),\Omega}(u) \geq c_a\varrho_{p(\cdot),\Omega}(u) - m_0$$

for some $m_0 > 0$. Taking the last two inequalities into account, we have

$$\begin{aligned}
&\langle \mathcal{F}u, u\rangle \\
&\geq c_a\varrho_{p(\cdot),\Omega}(\nabla u) + b(w)\varrho_{q(\cdot),\Omega}(\nabla u) - \left(\|\eta\|_{V^*} + \|\xi\|_{V^*}\right)\|u\|_V - \|b_g\|_{1,\Omega} + c_a\varrho_{p(\cdot),\Omega}(u) - m_0 \\
&\geq c_a\left(\varrho_{p(\cdot),\Omega}(\nabla u) + \varrho_{p(\cdot),\Omega}(u)\right) - \left(\|\eta\|_{V^*} + \|\xi\|_{V^*}\right)\|u\|_V - \|b_g\|_{1,\Omega} - m_0 \\
&\geq c_a\min\left\{\|u\|_V^{p_-}, \|u\|_V^{p_+}\right\} - \left(\|\eta\|_{V^*} + \|\xi\|_{V^*}\right)\|u\|_V - \|b_g\|_{1,\Omega} - m_0,
\end{aligned}$$

where the last inequality is obtained by using Proposition 2.2 (iii) and (iv). This means that $\mathcal{F}$ is a coercive operator.

Therefore, all conditions of Theorem 3.2 of Liu-Migórski-Zeng [26] are satisfied. Using this theorem, we conclude that inequality (3.4) has at least one solution. On the other hand, the strict monotonicity of $\mathcal{F}$ implies that this solution is unique. This completes the proof. ☐

In particular, if $J(w) = +\infty$ for all $w \in V$, problem (3.3) reduces to the following nonlinear anisotropic mixed boundary problem involving a convex subdifferential term: find $u \in V$ such that

$$\begin{aligned}
-a(w)\Delta_{p(\cdot)}u - b(w)\Delta_{q(\cdot)}u + g(x,u) &= \eta(x) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_w} &= \xi(x) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_w} &\in \partial_c\phi(x,u) && \text{on } \Gamma_3.
\end{aligned} \tag{3.5}$$

In this special case, we have the following result.

Corollary 3.7. *Suppose that* H(0), H(g) *and* H(ϕ) *are fulfilled. Then, problem* (3.5) *has a unique solution.*

Lemma 3.6 permits us to consider the solution mapping $\mathcal{S}\colon V \times X^* \times Y^* \to V$ of problem (3.3) defined by

$$\mathcal{S}(w,\eta,\xi) := u_{(w,\eta,\xi)} \quad \text{for all } (w,\eta,\xi) \in V \times X^* \times Y^*,$$

where $u_{(w,\eta,\xi)}$ is the unique solution of problem (3.3) corresponding to $(w,\eta,\xi) \in V \times X^* \times Y^*$. The following lemma shows that the solution mapping $\mathcal{S}$ is a completely continuous operator, that is, if $\{(w_n,\eta_n,\xi_n)\}_{n\in\mathbb{N}} \subset V \times X^* \times Y^*$ and $(u,\eta,\xi) \in V \times X^* \times Y^*$ satisfy $(w_n,\eta_n,\xi_n) \xrightarrow{w} (w,\eta,\xi)$ in $V \times X^* \times Y^*$, then we have $\mathcal{S}(w_n,\eta_n,\xi_n) \to \mathcal{S}(w,\eta,\xi)$ in V.

Lemma 3.8. *Assume that* H(0), H(1), H(g), H(ϕ), H(L) *and* H(J) *are satisfied. Then, the solution map* $\mathcal{S}\colon V \times X^* \times Y^* \to V$ *of problem* (3.3) *is completely continuous.*

Proof. Let $\{(w_n, \eta_n, \xi_n)\}_{n\in\mathbb{N}} \subset V \times X^* \times Y^*$, $\{u_n\}_{n\in\mathbb{N}} \subset V$ be sequences and $(w, \eta, \xi) \in V \times X^* \times Y^*$ such that

$$(w_n, \eta_n, \xi_n) \xrightarrow{w} (w, \eta, \xi) \quad \text{in } V \times X^* \times Y^*$$

and $u_n = \mathcal{S}(w_n, \eta_n, \xi_n)$ for each $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$, the function $u_n \in K(w_n)$ is the unique solution of the following inequality

$$\begin{aligned}
&a(w_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(v-u_n)\,\mathrm{d}x + b(w_n)\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\cdot\nabla(v-u_n)\,\mathrm{d}x\\
&\quad+\int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma+\int_\Omega g(x,u_n)(v-u_n)\,\mathrm{d}x\\
&\geq \int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x+\int_{\Gamma_2}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma \quad \text{for all } v\in K(w_n).
\end{aligned}\tag{3.6}$$

Claim 1: The solution sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in V.

If the sequence $\{u_n\}_{n\in\mathbb{N}}$ is unbounded in V, then, passing to a subsequence if necessary, we may suppose that

$$\|u_n\|_V \to +\infty \quad \text{as } n \to \infty. \tag{3.7}$$

Note that $0 \in K(w_n)$ for each $n \in \mathbb{N}$ (see Lemma 3.5(i)), we can take $v = 0$ in inequality (3.6) in order to obtain

$$\begin{aligned}
&a(w_n)\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x+b(w_n)\int_\Omega |\nabla u_n|^{q(x)}\,\mathrm{d}x+\int_\Omega g(x,u_n)u_n\,\mathrm{d}x+\int_{\Gamma_3}\phi(x,u_n(x))\,\mathrm{d}\Gamma\\
&\leq \int_\Omega \eta_n(x)u_n(x)\,\mathrm{d}x+\int_{\Gamma_2}\xi_n(x)u_n(x)\,\mathrm{d}\Gamma+\int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma\\
&\leq \|\phi(\cdot,0)\|_{1,\Gamma_3}+\|\eta_n\|_{V^*}\|u_n\|_V+\|\xi_n\|_{V^*}\|u_n\|_V.
\end{aligned}\tag{3.8}$$

Condition H(g)(iv) and Young's inequality imply that

$$\begin{aligned}
\int_\Omega g(x,u_n)u_n\,\mathrm{d}x &\geq \int_\Omega a_g|u_n|^{\varsigma(x)}-b_g(x)\,\mathrm{d}x\\
&= a_g\varrho_{\varsigma(\cdot),\Omega}(u_n)-\|b_g\|_{1,\Omega}\geq c_a\varrho_{p(\cdot),\Omega}(u_n)-m_1-\|b_g\|_{1,\Omega}
\end{aligned}\tag{3.9}$$

for some $m_1 > 0$ which is independent of n. Recall that $v \mapsto \varphi(v) = \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma$ is a proper, convex and l.s.c. function. Thus, from Brézis [6, Proposition 1.10], we are able to find two positive constants $\alpha_\varphi, \beta_\varphi \geq 0$ such that

$$\varphi(v) \geq -\alpha_\varphi\|v\|_V - \beta_\varphi \tag{3.10}$$

for all $v \in V$. Taking into account (3.8), (3.9) and (3.10) and using hypothesis H(1) leads to

$$\begin{aligned}
0 &\geq c_a\varrho_{p(\cdot),\Omega}(\nabla u_n)+b(w_n)\varrho_{q(\cdot),\Omega}(\nabla u_n)+c_a\varrho_{p(\cdot),\Omega}(u_n)-m_1-\|b_g\|_{1,\Omega}-\alpha_\varphi\|u_n\|_V-\beta_\varphi\\
&\quad-\|\phi(\cdot,0)\|_{1,\Gamma_3}-\left(\|\eta_n\|_{V^*}+\|\xi_n\|_{V^*}\right)\|u_n\|_V\\
&\geq c_a\left(\varrho_{p(\cdot),\Omega}(\nabla u_n)+\varrho_{p(\cdot),\Omega}(u_n)\right)-m_1-\|b_g\|_{1,\Omega}-\alpha_\varphi\|u_n\|_V-\beta_\varphi-\|\phi(\cdot,0)\|_{1,\Gamma_3}\\
&\quad-\left(\|\eta_n\|_{V^*}+\|\xi_n\|_{V^*}\right)\|u_n\|_V\\
&\geq c_a\min\left\{\|u_n\|_V^{p_-},\|u_n\|_V^{p_+}\right\}-m_1-\|b_g\|_{1,\Omega}-\alpha_\varphi\|u_n\|_V-\beta_\varphi-\|\phi(\cdot,0)\|_{1,\Gamma_3}\\
&\quad-\left(\|\eta_n\|_{V^*}+\|\xi_n\|_{V^*}\right)\|u_n\|_V.
\end{aligned}$$

Because $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are bounded in X^* and Y^*, respectively, and the embeddings of V into X and of V into Y are continuous, we know that $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are bounded in V^*. Passing to the lower limit as $n \to \infty$ in the above inequalities and then using (3.7), it leads to a contradiction. Therefore, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in V. The claim follows.

Using this claim, without any loss of generality, we are able to find a function $u \in V$ satisfying

$$u_n \xrightarrow{w} u \quad \text{in } V.$$

Claim 2: The sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in V.

Recall that the graph of K, $\mathrm{Gr}(K)$, is sequentially closed in $V_w \times V_w$ (see Lemma 3.5(ii)). So, it follows from the convergence $(u_n, w_n) \xrightarrow{w} (u, w)$ in $V \times V$ and $\{(u_n, w_n)\}_{n\in\mathbb{N}} \subset \mathrm{Gr}(K)$, that u belongs to $K(w)$, that is, $u \in K(w)$. By means of Lemma 3.5(iii), it permits us to find a sequence $\{y_n\}_{n\in\mathbb{N}} \subset V$ such that $y_n \in K(w_n)$ for each $n \in \mathbb{N}$ and

$$y_n \to u \quad \text{in } V.$$

Taking $v = y_n$ in (3.6) one has

$$\begin{aligned}
&a(w_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x + b(w_n)\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x\\
&\le \int_{\Gamma_3}\phi(x,y_n)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \int_\Omega g(x,u_n)(y_n-u_n)\,\mathrm{d}x\\
&\quad - \int_\Omega \eta_n(x)(y_n-u_n)\,\mathrm{d}x - \int_{\Gamma_2}\xi_n(x)(y_n-u_n)\,\mathrm{d}\Gamma.
\end{aligned}$$

Passing to the upper limit as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\Big[a(w_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x\\
&\quad + b(w_n)\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x\Big]\\
&\le \limsup_{n\to\infty}\Big[\int_{\Gamma_3}\phi(x,y_n)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \int_\Omega g(x,u_n)(y_n-u_n)\,\mathrm{d}x\\
&\quad - \int_\Omega \eta_n(x)(y_n-u_n)\,\mathrm{d}x - \int_{\Gamma_2}\xi_n(x)(y_n-u_n)\,\mathrm{d}\Gamma\Big]\\
&\le \limsup_{n\to\infty}\int_{\Gamma_3}\phi(x,y_n)\,\mathrm{d}\Gamma - \liminf_{n\to\infty}\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \limsup_{n\to\infty}\int_\Omega g(x,u_n)(y_n-u_n)\,\mathrm{d}x\\
&\quad - \liminf_{n\to\infty}\int_\Omega \eta_n(x)(y_n-u_n)\,\mathrm{d}x - \liminf_{n\to\infty}\int_{\Gamma_2}\xi_n(x)(y_n-u_n)\,\mathrm{d}\Gamma.
\end{aligned} \tag{3.11}$$

Keeping in mind that V is embedded compactly into $L^{\delta_0(\cdot)}(\Omega)$ (resp. X and Y), we have

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega g(x,u_n)(y_n-u_n)\,\mathrm{d}x &= 0,\\
\lim_{n\to\infty}\int_\Omega \eta_n(x)(y_n-u_n)\,\mathrm{d}x &= 0,\\
\liminf_{n\to\infty}\int_{\Gamma_2}\xi_n(x)(y_n-u_n)\,\mathrm{d}\Gamma &= 0,
\end{aligned} \tag{3.12}$$

where we have used the boundedness of $\{\eta_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset Y^*$ as well as hypotheses H(g). Hypotheses H(ϕ) indicates that $s \mapsto \phi(x,s)$ is continuous for a. a. $x \in \Gamma_3$. Employing Fatou's lemma and the convergence $(u_n, y_n) \to (u, u)$ in $Y \times Y$ implies

$$\limsup_{n\to\infty}\int_{\Gamma_3}\phi(x,y_n)\,\mathrm{d}\Gamma - \liminf_{n\to\infty}\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma \le 0. \tag{3.13}$$

Note that a and b are continuous. Applying Hölder's inequality we get

$$\begin{aligned}
&\liminf_{n\to\infty} b(w_n)\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x\\
&\geq \liminf_{n\to\infty} b(w_n)\int_\Omega |\nabla y_n|^{q(x)-2}\nabla y_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x=0
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} a(w_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x\\
&=\limsup_{n\to\infty}\left[a(w_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right]\\
&\geq \limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right]\\
&\quad -\limsup_{n\to\infty}|a(w_n)-a(w)|\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x\right|\\
&\geq \limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-y_n)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right]\\
&\quad -\limsup_{n\to\infty}2k_0|a(w_n)-a(w)|\|\nabla u_n\|_{p(\cdot),\Omega}\|\nabla(y_n-u_n)\|_{p(\cdot),\Omega}\\
&\geq \limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right.\\
&\quad \left.+a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u-y_n)\,\mathrm{d}x\right]\\
&\geq \limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right]\\
&\quad +\liminf_{n\to\infty}a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u-y_n)\,\mathrm{d}x\\
&\geq \limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right]
\end{aligned}\tag{3.15}$$

for some $k_0>0$ which is independent of n, where we have used the compactness of the embedding of V into $L^{p(\cdot)}(\Omega)$ and the equality

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x=0.$$

Let us consider the bifunction $A\colon V\times V\to V^*$ defined by

$$\langle A(w,u),v\rangle := a(w)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x\quad\text{for all } w,u,v\in V.$$

Inserting (3.12), (3.13), (3.14) and (3.15) into (3.11) yields

$$\begin{aligned}
&\limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,\mathrm{d}x\right]\\
&=\limsup_{n\to\infty}\langle A(w,u_n),u_n-u\rangle\leq 0.
\end{aligned}$$

The latter combined with the (S_+)-property of $A(w,\cdot)$ (see Proposition 2.6) implies that $u_n\to u$ in V. Therefore, the claim is proved.

Claim 3: The function u is the unique solution of problem (3.3) corresponding to $(w,\eta,\xi)\in V\times X^*\times Y^*$, that is, $u=\mathcal{S}(w,\eta,\xi)$.

Let $z\in K(w)$ be arbitrary. We use Lemma 3.5(iii) to find a sequence $\{z_n\}_{n\in\mathbb{N}}\subset V$ satisfying

$$z_n\in K(w_n)\quad\text{and}\quad z_n\to z\quad\text{in } V.$$

Choosing $v = z_n$ in (3.6) and passing to the upper limit as $n \to \infty$, we obtain

$$
\begin{aligned}
& a(w) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(z-u)\,\mathrm{d}x + b(w) \int_\Omega |\nabla u|^{q(x)-2} \nabla u \cdot \nabla(z-u)\,\mathrm{d}x \\
& \quad + \int_{\Gamma_3} \phi(x,z)\,\mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x,u)\,\mathrm{d}\Gamma + \int_\Omega g(x,u)(z-u)\,\mathrm{d}x \\
& \geq \limsup_{n\to\infty} a(w_n) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(z_n-u_n)\,\mathrm{d}x \\
& \quad + \limsup_{n\to\infty} b(w_n) \int_\Omega |\nabla u_n|^{q(x)-2} \nabla u_n \cdot \nabla(z_n-u_n)\,\mathrm{d}x + \limsup_{n\to\infty} \int_{\Gamma_3} \phi(x,z_n)\,\mathrm{d}\Gamma \\
& \quad - \liminf_{n\to\infty} \int_{\Gamma_3} \phi(x,u_n)\,\mathrm{d}\Gamma + \limsup_{n\to\infty} \int_\Omega g(x,u_n)(z_n-u_n)\,\mathrm{d}x \\
& \geq \limsup_{n\to\infty} \int_\Omega \eta_n(x)(z_n-u_n)\,\mathrm{d}x + \limsup_{n\to\infty} \int_{\Gamma_2} \xi_n(x)(z_n-u_n)\,\mathrm{d}\Gamma \\
& = \int_\Omega \eta(x)(z-u)\,\mathrm{d}x + \int_{\Gamma_2} \xi(x)(z-u)\,\mathrm{d}\Gamma.
\end{aligned}
$$

Since $z \in K(w)$ is arbitrary, we can apply Lemma 3.6 and have that u is the unique solution of problem (3.3) corresponding to (w,η,ξ), that is, $u = \mathcal{S}(w,\eta,\xi)$.

Because each convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$ converges to the same limit u, we know that the whole sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in V. This means that $\mathcal{S}\colon V\times X^*\times Y^* \to V$ is completely continuous. $\square$

In what follows, we write $i\colon V \to X$ and $\gamma\colon V \to Y$ for the embedding operators of V to X and the trace operator from V into Y, respectively. It is obvious that i and γ are linear, bounded and compact. Also, by $i^*\colon X^* \to V^*$ and $\gamma^*\colon Y^* \to V^*$ we denote the dual operators of i and γ, respectively. Moreover, let us consider two multivalued mappings $\mathcal{U}_1\colon X \to 2^{X^*}$ and $\mathcal{U}_2\colon Y \to 2^{Y^*}$ given by

$$\mathcal{U}_1(u) := \big\{\eta \in X^* \,:\, \eta(x) \in U_1(x,u(x)) \text{ a.a. in } \Omega\big\}, \tag{3.16}$$

$$\mathcal{U}_2(v) := \big\{\xi \in Y^* \,:\, \xi(x) \in U_2(x,v(x)) \text{ a.a. on } \Gamma_2\big\}, \tag{3.17}$$

for all $(u,v) \in X\times Y$, respectively. The following lemma indicates that $\mathcal{U}_1$ and $\mathcal{U}_2$ are well-defined and strongly-weakly u.s.c.

Lemma 3.9. *Let* $\mathrm{H}(U_1)$ *and* $\mathrm{H}(U_2)$ *be satisfied. Then, the following statements hold:*

(i) *$\mathcal{U}_1$ and $\mathcal{U}_2$ are well-defined and for each $u \in X$ and for each $v \in Y$, the sets $\mathcal{U}_1(u)$ and $\mathcal{U}_2(v)$ are bounded, closed and convex in X^* and Y^*, respectively;*

(ii) *$\mathcal{U}_1$ and $\mathcal{U}_2$ are strongly-weakly u.s.c., i.e., $\mathcal{U}_1$ is u.s.c. from X with the strong topology to the subsets of X^* with the weak topology, and $\mathcal{U}_2$ is u.s.c. from Y with the strong topology to the subsets of Y^* with the weak topology.*

Proof. (i) Note that U_1 and U_2 satisfy an upper Carathéodory condition, that is, $\Omega \ni x \mapsto U_1(x,s) \subset \mathbb{R}$ and $\Gamma_2 \ni x \mapsto U_2(x,s) \subset \mathbb{R}$ are measurable and $\mathbb{R} \ni s \mapsto U_1(x,s) \subset \mathbb{R}$ and $\mathbb{R} \ni s \mapsto U_2(x,s) \subset \mathbb{R}$ are u.s.c. Employing Theorem 1.3.4 of Kamenskii-Obukhovskii-Zecca [28], we can see that for each $(u,v) \in X \times Y$, the functions $\Omega \ni x \mapsto U_1(x,u(x)) \subset \mathbb{R}$ and $\Gamma_2 \ni x \mapsto U_2(x,v(x)) \subset \mathbb{R}$ are both measurable in Ω and on Γ_2, respectively. This allows us to invoke the Yankov-von Neumann-Aumann selection theorem (see e.g. Papageorgiou-Winkert [40, Theorem 2.7.25]) which implies that there are two measurable functions $\eta\colon \Omega \to \mathbb{R}$ and $\xi\colon \Gamma_2 \to \mathbb{R}$ satisfying

$$\eta(x) \in U_1(x,u(x)) \quad \text{for a.a. } x \in \Omega \quad \text{and} \quad \xi(x) \in U_2(x,v(x)) \quad \text{for a.a. } x \in \Gamma_2.$$

From hypotheses H(U_1)(iv) and H(U_2)(iv) we have that

$$\begin{aligned}\varrho_{\delta_1'(\cdot),\Omega}(\eta)&=\int_\Omega|\eta(x)|^{\delta_1'(x)}\,\mathrm{d}x\le\int_\Omega\left(\alpha_{U_1}(x)+a_{U_1}|u(x)|^{\delta_1(x)-1}\right)^{\delta_1'(x)}\,\mathrm{d}x\\&\le m_2\int_\Omega\left(\alpha_{U_1}(x)^{\delta_1'(x)}+|u(x)|^{\delta_1(x)}\right)\,\mathrm{d}x\\&=m_2\left(\varrho_{\delta_1'(\cdot),\Omega}(\alpha_{U_1})+\varrho_{\delta_1(\cdot),\Omega}(u)\right)\\&<+\infty,\end{aligned}\tag{3.18}$$

for some $m_2>0$, and

$$\begin{aligned}\varrho_{\delta_2'(\cdot),\Gamma_2}(\xi)&=\int_{\Gamma_2}|\xi(x)|^{\delta_2'(x)}\,\mathrm{d}\Gamma\le\int_{\Gamma_2}\left(\alpha_{U_2}(x)+a_{U_2}|s|^{\delta_2(x)-1}\right)^{\delta_2'(x)}\,\mathrm{d}\Gamma\\&\le m_3\int_{\Gamma_2}\left(\alpha_{U_2}(x)^{\delta_2'(x)}+|u(x)|^{\delta_2(x)}\right)\,\mathrm{d}\Gamma\\&=m_3\left(\varrho_{\delta_2'(\cdot),\Gamma_2}(\alpha_{U_2})+\varrho_{\delta_2(\cdot),\Gamma_2}(u)\right)\\&<+\infty,\end{aligned}\tag{3.19}$$

for some $m_3>0$, where we have used the elementary inequality $(s+t)^r\le 2^{r-1}(s^r+t^r)$ for all $s,t\ge 0$ and $r\ge 1$ as well as the continuity of δ_1 and δ_2. The latter together with Proposition 2.1(vi) implies that $\eta\in X^*$ and $\xi\in Y^*$. Thus, the multivalued mappings $\mathcal{U}_1$ and $\mathcal{U}_2$ are well-defined and for each $(u,v)\in X\times Y$, the sets $\mathcal{U}_1(u)$ and $\mathcal{U}_2(v)$ are bounded in X^* and Y^*, respectively. Recall that U_1 and U_2 have closed and convex values. So we can use standard arguments to show that for each $(u,v)\in X\times Y$ the sets $\mathcal{U}_1(u)$ and $\mathcal{U}_2(v)$ are closed and convex in X^* and Y^*, respectively.

(ii) We only prove that $\mathcal{U}_1$ is u.s.c., the upper semicontinuity of $\mathcal{U}_2$ can be shown in a similar way. It follows from Proposition 2.8 that it is sufficient to show that for each weakly closed set D of X^*, the set $\mathcal{U}_1^-(D)$ is closed in X. Let $\{u_n\}_{n\in\mathbb{N}}\subset U^-(D)$ be such that $u_n\to u$ in X for some $u\in X$. Due to the continuity of the embedding $V\hookrightarrow L^1(\Omega)$, by passing to a subsequence if necessary, we may assume that

$$u_n(x)\to u(x)\quad\text{as } n\to\infty\quad\text{for a. a. } x\in\Omega.\tag{3.20}$$

Let $\{\eta_n\}_{n\in\mathbb{N}}\subset X^*$ be a sequence such that $\eta_n\in\mathcal{U}_1(u_n)\cap D$ for each $n\in\mathbb{N}$. By virtue of (3.18), we infer that sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in X^*. Because X^* is reflexive, we may assume that

$$\eta_n\xrightarrow{w}\eta\quad\text{in } X^*$$

for some $\eta\in D$ owing to the weak closedness of D. Our objective is to prove that $\eta\in\mathcal{U}_1(u)$, namely, $\eta(x)\in\mathcal{U}_1(x,u(x))$ for a. a. $x\in\Omega$.

Employing Mazur's theorem, we are able to find a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of convex combinations of $\{\eta_n\}_{n\in\mathbb{N}}$ such that

$$\zeta_n\to\eta\quad\text{in } L^{\delta_1'(\cdot)}(\Omega)\quad\text{and}\quad\zeta_n(x)\to\eta(x)\quad\text{for a. a. } x\in\Omega\quad\text{as}\quad n\to\infty.\tag{3.21}$$

The convexity of U_1 guarantees that $\zeta_n(x)\in U_1(x,u_n(x))$ for a. a. $x\in\Omega$. Applying the convergences in (3.20) and (3.21) along with the upper semicontinuity of U_1 (see hypothesis H(U_1)(iii)), we get that $\eta(x)\in U_1(x,u(x))$ for a. a. $x\in\Omega$. This means that $\eta\in\mathcal{U}_1(u)\cap D$. Hence, $u\in\mathcal{U}_1^-(D)$. Therefore, we can apply Proposition 2.8 to conclude that $\mathcal{U}_1$ is strongly-weakly u.s.c. This completes the proof. ☐

Using the results above, we are now in a position to provide the detailed proof of Theorem 3.4.

Proof of Theorem 3.4. First, we prove the following claims.

Claim 4: The solution set Υ of problem (1.1) is bounded, if Υ is nonempty.

Let $u \in V$ be a weak solution of problem (1.1). Then, there exist functions $(\eta, \xi) \in X^* \times Y^*$ with $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$ and $\xi(x) \in U_2(x, u(x))$ for a. a. $x \in \Gamma_2$ such that

$$\begin{aligned}
&a(u) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(v-u)\,\mathrm{d}x + b(u) \int_\Omega |\nabla u|^{q(x)-2} \nabla u \cdot \nabla(v-u)\,\mathrm{d}x\\
&\quad + \int_\Omega g(x,u)(v-u)\,\mathrm{d}x + \int_{\Gamma_3} \phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x,u)\,\mathrm{d}\Gamma\\
&\geq \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_2} \xi(x)(v-u)\,\mathrm{d}\Gamma
\end{aligned}$$

for all $v \in K(u)$. Since $0 \in K(u)$ we take $v = 0$ in the above inequality to obtain

$$\begin{aligned}
&a(u) \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + b(u) \int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x + \int_\Omega g(x,u)u\,\mathrm{d}x\\
&\leq \int_{\Gamma_3} \phi(x,0)\,\mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x,u)\,\mathrm{d}\Gamma + \int_\Omega \eta(x)u\,\mathrm{d}x + \int_{\Gamma_2} \xi(x)u\,\mathrm{d}\Gamma.
\end{aligned} \tag{3.22}$$

It follows from hypotheses H(U_1)(iv) and H(U_2)(iv) that

$$\begin{aligned}
\int_\Omega \eta(x)u(x)\,\mathrm{d}x &\leq \int_\Omega |\eta(x)||u(x)|\,\mathrm{d}x\\
&\leq \int_\Omega \left(\alpha_{U_1}(x) + a_{U_1}|u(x)|^{\delta_1(x)-1}\right)|u(x)|\,\mathrm{d}x\\
&\leq a_{U_1}\varrho_{\delta_1(\cdot),\Omega}(u) + 2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}\|u\|_{\delta_1(\cdot),\Omega},
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
\int_{\Gamma_2} \xi(x)u(x)\,\mathrm{d}\Gamma &\leq \int_{\Gamma_2} |\xi(x)||u(x)|\,\mathrm{d}\Gamma\\
&\leq \int_{\Gamma_2} \left(\alpha_{U_2}(x) + a_{U_2}|u(x)|^{\delta_2(x)-1}\right)|u(x)|\,\mathrm{d}\Gamma\\
&\leq a_{U_2}\varrho_{\delta_2(\cdot),\Gamma_2}(u) + 2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Omega}\|u\|_{\delta_2(\cdot),\Gamma_2}.
\end{aligned} \tag{3.24}$$

Since the embeddings of V into X and of V into Y are continuous, we are able to find two constants $C_X, C_Y > 0$ such that

$$\|u\|_{\delta_1(\cdot),\Omega} \leq C_X\|u\|_V \quad \text{and} \quad \|u\|_{\delta_2(\cdot),\Gamma_2} \leq C_Y\|u\|_V \quad \text{for all } u \in V. \tag{3.25}$$

Keeping in mind that $\varsigma(x) > p(x)$ for all $x \in \Omega$, using hypothesis H(g)(iv), we have

$$\int_\Omega g(x,u)u\,\mathrm{d}x \geq \int_\Omega a_g|u|^{\varsigma(x)} - b_g(x)\,\mathrm{d}x = a_g\varrho_{\varsigma(\cdot),\Omega}(u) - \|b_g\|_{1,\Omega}. \tag{3.26}$$

Putting (3.23), (3.24), (3.25) and (3.26) into (3.22), we have

$$\begin{aligned}
&c_a\varrho_{p(\cdot),\Omega}(\nabla u) + a_g\varrho_{\varsigma(\cdot),\Omega}(u) - \|b_g\|_{1,\Omega} - \alpha_\varphi\|u\|_V\\
&\leq a_{U_2}\varrho_{\delta_2(\cdot),\Gamma_2}(u) + 2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Omega}\|u\|_{\delta_2(\cdot),\Gamma_2} + a_{U_1}\varrho_{\delta_1(\cdot),\Omega}(u) + 2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}\|u\|_{\delta_1(\cdot),\Omega} + \beta_\varphi,
\end{aligned}$$

where we have used inequality (3.10). Employing Propositions 2.1(iii), (iv) and 2.2(iii) and (iv) we get

$$
\begin{aligned}
& c_a \min\left\{\|u\|_V^{p_-}, \|u\|_V^{p_+}\right\} - c_a \min\left\{\|u\|_{p(\cdot),\Omega}^{p_-}, \|u\|_{p(\cdot),\Omega}^{p_+}\right\} + a_g \min\left\{\|u\|_{\varsigma(\cdot),\Omega}^{\varsigma_-}, \|u\|_{\varsigma(\cdot),\Omega}^{\varsigma_+}\right\} \\
&\quad - \alpha_\varphi \|u\|_V \\
& \leq c_a \left(\varrho_{p(\cdot),\Omega}(\nabla u) + \varrho_{p(\cdot),\Omega}(u)\right) - c_a \varrho_{p(\cdot),\Omega}(u) + a_g \varrho_{\varsigma(\cdot),\Omega}(u) - \alpha_\varphi \|u\|_V \\
& \leq a_{U_2} \varrho_{\delta_2(\cdot),\Gamma_2}(u) + 2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Omega} \|u\|_{\delta_2(\cdot),\Gamma_2} + a_{U_1} \varrho_{\delta_1(\cdot),\Omega}(u) + 2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega} \|u\|_{\delta_1(\cdot),\Omega} \\
&\quad + \beta_\varphi + \|b_g\|_{1,\Omega} \\
& \leq a_{U_2} \max\left\{\|u\|_{\delta_2(\cdot),\Gamma_2}^{\delta_{2-}}, \|u\|_{\delta_2(\cdot),\Gamma_2}^{\delta_{2+}}\right\} + a_{U_1} \max\left\{\|u\|_{\delta_1(\cdot),\Omega}^{\delta_{1-}}, \|u\|_{\delta_1(\cdot),\Omega}^{\delta_{1+}}\right\} \\
&\quad + 2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Omega} \|u\|_{\delta_2(\cdot),\Gamma_2} + 2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega} \|u\|_{\delta_1(\cdot),\Omega} + \beta_\varphi + \|b_g\|_{1,\Omega} \\
& \leq a_{U_2} \max\left\{C_Y^{\delta_{2-}} \|u\|_V^{\delta_{2-}}, C_Y^{\delta_{2+}} \|u\|_V^{\delta_{2+}}\right\} + a_{U_1} \max\left\{C_X^{\delta_{1-}} \|u\|_V^{\delta_{1-}}, C_X^{\delta_{1+}} \|u\|_{\delta_1(\cdot),\Omega}^{\delta_{1+}}\right\} \\
&\quad + 2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Omega} \|u\|_{\delta_2(\cdot),\Gamma_2} + 2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega} \|u\|_{\delta_1(\cdot),\Omega} + \beta_\varphi + \|b_g\|_{1,\Omega}.
\end{aligned}
\tag{3.27}
$$

Recall that $\varsigma_- > p_- > \delta_{1-}$ and $p_- > \delta_{2-}$. From the estimates above, it is not difficult to prove that there exists a constant $m_4 > 0$ such that

$$\|u\|_V \leq m_4 \quad \text{for all } u \in \Upsilon.$$

Thus, the claim is verified.

Claim 5: There exists a constant $M^* > 0$ such that

$$\mathcal{S}(\overline{B_V(0, M^*)}, \mathcal{U}_1(i\overline{B_V(0, M^*)}), \mathcal{U}_2(\gamma\overline{B_V(0, M^*)})) \subset \overline{B_V(0, M^*)}, \tag{3.28}$$

where $\overline{B_V(0, M^*)} := \{u \in V : \|u\|_V \leq M^*\}$.

Arguing by contradiction, suppose that there is no such constant M^* such that the inclusion holds. Then for each $n > 0$ there exist $w_n, z_n, y_n \in \overline{B_V(0, n)}$ and $(\eta_n, \xi_n) \in X^* \times Y^*$ with $\eta_n \in \mathcal{U}_1(iz_n)$ and $\xi_n \in \mathcal{U}_2(\gamma y_n)$ such that

$$u_n = \mathcal{S}(w_n, \eta_n, \xi_n) \quad \text{and} \quad \|u_n\|_V > n.$$

Hence, for every $n > 0$, we have

$$
\begin{aligned}
& a(w_n) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(v - u_n) \,\mathrm{d}x + b(w_n) \int_\Omega |\nabla u_n|^{q(x)-2} \nabla u_n \cdot \nabla(v - u_n) \,\mathrm{d}x \\
&\quad + \int_{\Gamma_3} \phi(x, v) \,\mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x, u_n) \,\mathrm{d}\Gamma + \int_\Omega g(x, u_n)(v - u_n) \,\mathrm{d}x \\
& \geq \int_\Omega \eta_n(x)(v - u_n) \,\mathrm{d}x + \int_{\Gamma_2} \xi_n(x)(v - u_n) \,\mathrm{d}\Gamma
\end{aligned}
$$

for all $v \in K(w_n)$. Taking $v = 0$ in the above inequality gives

$$
\begin{aligned}
& a(w_n) \int_\Omega |\nabla u_n|^{p(x)} \,\mathrm{d}x + b(w_n) \int_\Omega |\nabla u_n|^{q(x)} \,\mathrm{d}x \\
&\quad + \int_\Omega g(x, u_n) u_n \,\mathrm{d}x + \int_{\Gamma_3} \phi(x, u_n(x)) \,\mathrm{d}\Gamma \\
& \leq \int_\Omega \eta_n(x) u_n(x) \,\mathrm{d}x + \int_{\Gamma_2} \xi_n(x) u_n(x) \,\mathrm{d}\Gamma + \int_{\Gamma_3} \phi(x, 0) \,\mathrm{d}\Gamma.
\end{aligned}
\tag{3.29}
$$

From hypotheses H(U_1)(iv) and H(U_2)(iv), we have

$$\begin{aligned}\int_\Omega \eta_n(x)u_n(x)\,\mathrm{d}x &\le \int_\Omega |\eta_n(x)||u_n(x)|\,\mathrm{d}x\\ &\le \int_\Omega \left(\alpha_{U_1}(x)+a_{U_1}|z_n(x)|^{\delta_1(x)-1}\right)|u_n(x)|\,\mathrm{d}x\\ &\le m_5\left(\varrho_{\delta_1(\cdot),\Omega}(z_n)+\varrho_{\delta_1(\cdot),\Omega}(u_n)\right)+2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}\|u_n\|_{\delta_1(\cdot),\Omega},\end{aligned} \tag{3.30}$$

for some $m_5>0$, and

$$\begin{aligned}\int_{\Gamma_2} \xi_n(x)u_n(x)\,\mathrm{d}x &\le \int_{\Gamma_2} |\xi_n(x)||u_n(x)|\,\mathrm{d}x\\ &\le \int_{\Gamma_2} \left(\alpha_{U_2}(x)+a_{U_2}|y_n(x)|^{\delta_2(x)-1}\right)|u_n(x)|\,\mathrm{d}x\\ &\le m_6\left(\varrho_{\delta_2(\cdot),\Gamma_2}(y_n)+\varrho_{\delta_2(\cdot),\Gamma_2}(u_n)\right)+2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Gamma_2}\|u_n\|_{\delta_2(\cdot),\Gamma_2},\end{aligned} \tag{3.31}$$

for some $m_6>0$, where we have used Young's inequality and the continuity of δ_1 and δ_2. Putting $u=u_n$ into (3.26) leads to

$$\int_\Omega g(x,u_n)u_n\,\mathrm{d}x \ge a_g\varrho_{\varsigma(\cdot),\Omega}(u_n)-\|b_g\|_{1,\Omega}. \tag{3.32}$$

Inserting (3.30), (3.31), (3.32) into (3.29), we obtain

$$\begin{aligned}&c_a\left(\varrho_{p(\cdot),\Omega}(\nabla u_n)+\varrho_{p(\cdot),\Omega}(u_n)\right)-c_a\varrho_{p(\cdot),\Omega}(u_n)+a_g\varrho_{\varsigma(\cdot),\Omega}(u_n)-\alpha_\varphi\|u_n\|_V\\ &\le m_6\left(\varrho_{\delta_2(\cdot),\Gamma_2}(y_n)+\varrho_{\delta_2(\cdot),\Gamma_2}(u_n)\right)+2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Gamma_2}\|u_n\|_{\delta_2(\cdot),\Gamma_2}+\beta_\varphi+\|\phi(\cdot,0)\|_{1,\Gamma_3}+\|b_g\|_{1,\Omega}\\ &\quad+m_5\left(\varrho_{\delta_1(\cdot),\Omega}(z_n)+\varrho_{\delta_1(\cdot),\Omega}(u_n)\right)+2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}\|u_n\|_{\delta_1(\cdot),\Omega}.\end{aligned}$$

Hence,

$$\begin{aligned}&c_a\min\left\{\|u_n\|_V^{p_-},\|u_n\|_V^{p_+}\right\}-c_a\left\{\|u_n\|_{p(\cdot),\Omega}^{p_-},\|u_n\|_{p(\cdot),\Omega}^{p_+}\right\}\\ &\quad+a_g\left\{\|u_n\|_{\varsigma(\cdot),\Omega}^{\varsigma_-},\|u_n\|_{\varsigma(\cdot),\Omega}^{\varsigma_+}\right\}-\alpha_\varphi\|u_n\|_V\\ &\le m_6\left(\max\left\{\|y_n\|_{\delta_2(\cdot),\Gamma_2}^{\delta_{2-}},\|y_n\|_{\delta_2(\cdot),\Gamma_2}^{\delta_{2+}}\right\}+\max\left\{\|u_n\|_{\delta_2(\cdot),\Gamma_2}^{\delta_{2-}},\|u_n\|_{\delta_2(\cdot),\Gamma_2}^{\delta_{2+}}\right\}\right)\\ &\quad+2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Gamma_2}\|u_n\|_{\delta_2(\cdot),\Gamma_2}+\beta_\varphi+\|\phi(\cdot,0)\|_{1,\Gamma_3}+2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}\|u_n\|_{\delta_1(\cdot),\Omega}\\ &\quad+m_5\left(\max\left\{\|z_n\|_{\delta_1(\cdot),\Omega}^{\delta_{1-}},\|z_n\|_{\delta_1(\cdot),\Omega}^{\delta_{1+}}\right\}+\max\left\{\|u_n\|_{\delta_1(\cdot),\Omega}^{\delta_{1-}},\|u_n\|_{\delta_1(\cdot),\Omega}^{\delta_{1+}}\right\}\right)\\ &\le m_6\left(\max\left\{C_Y^{\delta_{2-}}\|y_n\|_V^{\delta_{2-}},C_Y^{\delta_{2+}}\|y_n\|_V^{\delta_{2+}}\right\}+\max\left\{C_Y^{\delta_{2-}}\|u_n\|_V^{\delta_{2-}},C_Y^{\delta_{2+}}\|u_n\|_V^{\delta_{2+}}\right\}\right)\\ &\quad+2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Gamma_2}C_Y\|u_n\|_V+\beta_\varphi+\|\phi(\cdot,0)\|_{1,\Gamma_3}+2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}C_X\|u_n\|_V\\ &\quad+m_5\left(\max\left\{C_X^{\delta_{1-}}\|z_n\|_V^{\delta_{1-}},C_X^{\delta_{1+}}\|z_n\|_V^{\delta_{1+}}\right\}+\max\left\{C_X^{\delta_{1-}}\|u_n\|_V^{\delta_{1-}},C_X^{\delta_{1+}}\|u_n\|_V^{\delta_{1+}}\right\}\right)\\ &\le m_6\left(\max\left\{C_Y^{\delta_{2-}}\|u_n\|_V^{\delta_{2-}},C_Y^{\delta_{2+}}\|u_n\|_V^{\delta_{2+}}\right\}+\max\left\{C_Y^{\delta_{2-}}\|u_n\|_V^{\delta_{2-}},C_Y^{\delta_{2+}}\|u_n\|_V^{\delta_{2+}}\right\}\right)\\ &\quad+2\|\alpha_{U_2}\|_{\delta_2'(\cdot),\Gamma_2}C_Y\|u_n\|_V+\beta_\varphi+\|\phi(\cdot,0)\|_{1,\Gamma_3}+2\|\alpha_{U_1}\|_{\delta_1'(\cdot),\Omega}C_X\|u_n\|_V\\ &+m_5\left(\max\left\{C_X^{\delta_{1-}}\|u_n\|_V^{\delta_{1-}},C_X^{\delta_{1+}}\|u_n\|_V^{\delta_{1+}}\right\}+\max\left\{C_X^{\delta_{1-}}\|u_n\|_V^{\delta_{1-}},C_X^{\delta_{1+}}\|u_n\|_V^{\delta_{1+}}\right\}\right).\end{aligned}$$

Because of $\varsigma_->p_->\delta_{1-}$ and $p_->\delta_{2-}$, passing to the upper limit as $n\to\infty$ in the above inequalities, we get a contradiction. Hence there exists a constant $M^*>0$ such that (3.28) is fulfilled.

As mentioned before, the main tool in the proof of the existence of a solution to problem (1.1) is the Tychonoff's fixed point theorem for multivalued operators, see Theorem 2.10. For this purpose, let us consider the multivalued mapping $\Lambda\colon V\times X^*\times Y^*\to 2^{V\times X^*\times Y^*}$ defined by

$$\Lambda(u,\eta,\xi):=(\mathcal{S}(u,\eta,\xi),\mathcal{U}_1(iu),\mathcal{U}_2(\gamma u)),$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are given in (3.16) and (3.17). Observe that if (u,η,ξ) is a fixed point of Λ, then we have $u = S(u,\eta,\xi)$ and $(\eta,\xi)\in\mathcal{U}_1(iu)\times\mathcal{U}_2(\gamma u)$. It is obvious from the definitions of $\mathcal{S}$, $\mathcal{U}_1$ and $\mathcal{U}_2$ that u is also a weak solution of problem (1.1). Therefore, we are going to examine the validity of the conditions of Theorem 2.10. Invoking Lemmas 3.6 and 3.9, we can see that for each $(w,\eta,\xi)\in V\times X^*\times Y^*$, the set $\Lambda(w,\eta,\xi)$ is a nonempty, bounded, closed and convex subset of $V\times X^*\times Y^*$.

Employing hypotheses H(U_1)(iv) and H(U_2)(iv), it is not difficult to prove that $\mathcal{U}_1\colon X\to 2^{X^*}$ and $\mathcal{U}_2\colon Y\to 2^{Y^*}$ are two bounded operators (see (3.18) and (3.19)), and there exist two constants $M_1>0$ and $M_2>0$ satisfying

$$\|\mathcal{U}_1(i\overline{B_V(0,M^*)})\|_{X^*}\le M_1\quad\text{ and }\quad \|\mathcal{U}_2(\gamma\overline{B_V(0,M^*)})\|_{Y^*}\le M_2.$$

Additionally, we introduce a bounded, closed and convex subset D of $V\times X^*\times Y^*$ defined by

$$D=\left\{(u,\eta,\xi)\in V\times X^*\times Y^* \,:\, \|u\|_V\le M^*,\ \|\eta\|_{X^*}\le M_1 \text{ and } \|\xi\|_{Y^*}\le M_2\right\}.$$

From this and (3.28) we know that Λ maps D into itself.

Next, we are going to prove that the multivalued mapping Λ is weakly-weakly u.s.c. For any weakly closed set E in $V\times X^*\times Y^*$ such that $\Lambda^-(E)\neq\emptyset$, let $\{(w_n,\eta_n,\xi_n)\}_{n\in\mathbb{N}}\subset\Lambda^-(E)$ be such that $(w_n,\eta_n,\xi_n)\xrightarrow{w}(w,\eta,\xi)$ in $V\times X^*\times Y^*$ for some $(w,\eta,\xi)\in V\times X^*\times Y^*$. Our goal is to show that $(w,\eta,\xi)\in\Lambda^-(E)$, namely, there exists $(u,\delta,\sigma)\in\Lambda(w,\eta,\xi)\cap E$. Indeed, for each $n\in\mathbb{N}$, we are able to find $(u_n,\delta_n,\sigma_n)\in\Lambda(w_n,\eta_n,\xi_n)\cap E$, so, $u_n=\mathcal{S}(w_n,\eta_n,\xi_n)$, $\delta_n\in\mathcal{U}_1(iw_n)$ and $\sigma_n\in\mathcal{U}_2(\gamma w_n)$. From (3.18) and (3.19), one has that the sequences $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ are bounded in X^* and Y^*, respectively. Passing to a subsequence if necessary, we may assume that

$$\delta_n\xrightarrow{w}\delta\quad\text{in } X^*\quad\text{and}\quad\sigma_n\xrightarrow{w}\sigma\quad\text{in } Y^*$$

for some $(\delta,\sigma)\in X^*\times Y^*$. Recall that $\mathcal{S}$ is completely continuous. So, it holds $u_n=\mathcal{S}(w_n,\eta_n,\xi_n)\to\mathcal{S}(w,\eta,\xi):=u$ in V. Note that i and γ are both compact. Hence $iw_n\to iw$ in X and $\gamma w_n\to\gamma w$ in Y. Since $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is strongly-weakly u.s.c. and has nonempty, bounded, closed and convex values, it follows from Theorem 1.1.4 of Kamenskii-Obukhovskii-Zecca [28] that $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is strongly-weakly closed. The latter combined with the convergences above implies that $\delta\in\mathcal{U}_1(iw)$ and $\sigma\in\mathcal{U}_2(\gamma w)$, namely, $(u,\delta,\sigma)\in\Lambda(w,\eta,\xi)\cap E$, because of the weak closedness of E. Therefore, we conclude that Λ is weakly-weakly u.s.c.

Therefore, all conditions of Theorem 2.10 are satisfied. Using this theorem, we conclude that Λ has at least a fixed point, say $(u^*,\eta^*,\xi^*)\in V\times X^*\times Y^*$. Hence, $u^*\in V$ is a weak solution of problem (1.1).

Next, let us prove the compactness of the solution set Υ. As proved before, we can see that the solution set Υ of problem (1.1) is bounded in V. By the definitions of a weak solution (see Definition 3.3) and of Λ, there exist $(\eta,\xi)\in X^*\times Y^*$ such that $u=\mathcal{S}(u,\eta,\xi)$, $\eta\in\mathcal{U}_1(iu)$ and $\xi\in\mathcal{U}_2(\gamma u)$, that is, $(u,\eta,\xi)\in\Lambda(u,\eta,\xi)$. Let $\{u_n\}_{n\in\mathbb{N}}$ be any sequence of solutions to problem (1.1). Then, there are two sequences $\{\eta_n\}_{n\in\mathbb{N}}\subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}}\subset Y^*$ such that $\eta_n\in\mathcal{U}_1(iu_n)$, $\xi_n\in\mathcal{U}_2(\gamma u_n)$ and $u_n=\mathcal{S}(u_n,\eta_n,\xi_n)$ for all $n\in\mathbb{N}$. From the boundedness of Υ we may assume that

$$u_n\xrightarrow{w}u\quad\text{in } V$$

for some $u\in V$. This together with the estimates (3.18) and (3.19) deduces that $\{\eta_n\}_{n\in\mathbb{N}}\subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}}\subset Y^*$ are both bounded. So, passing to a subsequence if necessary, we suppose that

$$\eta_n\xrightarrow{w}\eta\quad\text{in } X^*\quad\text{and}\quad\xi_n\xrightarrow{w}\xi\quad\text{in } Y^*$$

for some $\eta\in\mathcal{U}_1(iu)$ and $\xi\in\mathcal{U}_2(\gamma u)$, owing to the compactness of i and γ as well as the strongly-weakly closedness of $\mathcal{U}_1$ and $\mathcal{U}_2$. Using the complete continuity of $\mathcal{S}$, we conclude that

$$u_n=\mathcal{S}(u_n,\eta_n,\xi_n)\to\mathcal{S}(u,\eta,\xi)=u.$$

This means that u is a solution to problem (1.1). Consequently, the solution set Υ of problem (1.1) is compact. $\square$

We end this section by considering some particular cases of problem (1.1).

Let $\Psi\colon \Omega \to (0,+\infty)$. If $J(u) \equiv 0$ and

$$L(u) = \int_\Omega (u(x) - \Psi(x))^+ \,\mathrm{d}x \quad \text{for all } u \in V,$$

then problem (1.1) becomes the anisotropic obstacle problem (1.7) with mixed boundary conditions. A careful observation gives the following corollary.

Corollary 3.10. *Assume that* H(0), H(1), H(g), H(U_1), H(U_2) *and* H(ϕ) *are satisfied. Then, the solution set of problem* (1.7) *is nonempty and compact in* V.

If $J(u) \equiv +\infty$ for all $u \in V$, then problem (1.1) becomes the non-obstacle mixed boundary value problem (1.8). In this situation, we obtain the following corollary.

Corollary 3.11. *Assume that* H(0), H(1), H(g), H(U_1), H(U_2) *and* H(ϕ) *are satisfied. Then, the solution set of problem* (1.8) *is nonempty and compact in* V.

In addition, if $\Gamma_2 = \emptyset$ and $\Gamma_3 = \emptyset$, i.e., $\Gamma_1 = \Gamma$, then problem (1.1) reduces to problem (1.6). Using Theorem 3.4, we have the following corollary.

Corollary 3.12. *Assume that* H(0), H(1), H(U_1) *and* H(ϕ) *are satisfied. Then, the solution set of problem* (1.6) *with* $g \equiv 0$ *is nonempty and compact in* V.

Let us now consider problem (1.5) and suppose the following assumptions:

H(j_1): The functions $j_1\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $r_1\colon \mathbb{R} \to \mathbb{R}$ are such that

(i) $x \mapsto j_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$ with $x \mapsto j_1(x,0)$ belonging to $L^1(\Omega)$;

(ii) for a. a. $x \in \Omega$, $s \mapsto j_1(x,s)$ is locally Lipschitz continuous and the function $r_1\colon \mathbb{R} \to \mathbb{R}$ is continuous;

(iii) there exist a function $\alpha_{j_1} \in L^{\delta_1'(\cdot)}(\Omega)_+$ and a constant $a_{j_1} \geq 0$ such that

$$|r_1(s)\eta| \leq \alpha_{j_1}(x) + a_{j_1}|s|^{\delta_1(x)-1}$$

for all $\eta \in \partial j_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $\delta_1 \in C_+(\overline{\Omega})$ is such that

$$\delta_1(x) < p(x) \quad \text{for all } x \in \overline{\Omega}.$$

H(j_2): The functions $j_2\colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ and $r_2\colon \mathbb{R} \to \mathbb{R}$ are such that

(i) $x \mapsto j_2(x,s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$ with $x \mapsto j_2(x,0)$ belonging to $L^1(\Gamma_2)$;

(ii) for a. a. $x \in \Gamma_2$, $s \mapsto j_2(x,s)$ is locally Lipschitz continuous and the function $r_2\colon \mathbb{R} \to \mathbb{R}$ is continuous;

(iii) there exist a function $\alpha_{j_2} \in L^{\delta_2'(\cdot)}(\Gamma_2)_+$ and a constant $a_{j_2} \geq 0$ such that

$$|r_2(s)\xi| \leq \alpha_{j_2}(x) + a_{j_2}|s|^{\delta_2(x)-1}$$

for all $\xi \in \partial j_2(x,s)$, for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$, where $\delta_2 \in C_+(\overline{\Omega})$ is such that

$$\delta_2(x) < p(x) \quad \text{for all } x \in \overline{\Omega}.$$

If U_1 and U_2 are given by $U_1(x,s) = r_1(s)\partial j_1(x,s)$ for a. a. $x \in \Omega$, for $s \in \mathbb{R}$ and $U_2(x,s) = r_2(s)\partial j_2(x,s)$ for a. a. $x \in \Gamma_2$, for $s \in \mathbb{R}$, problem (1.1) becomes the implicit obstacle problem (1.3) with generalized subgradient term in the sense of Clarke. We have the following result.

Theorem 3.13. *Assume that* H(0), H(1), H(g), H(ϕ), H(L), H(J), H(j_1) *and* H(j_2) *are satisfied. Then, the solution set of problem* (1.5) *is nonempty and compact in* V.

Proof. It is obvious that the conclusion is a direct consequence of Theorem 3.4. So, we have to verify that the functions U_1 and U_2, defined by $U_1(x,s) = r_1(s)\partial j_1(x,s)$ for a. a. $x \in \Omega$, for $s \in \mathbb{R}$ and $U_2(x,s) = r_2(s)\partial j_2(x,s)$ for a. a. $x \in \Gamma_2$, for $s \in \mathbb{R}$, fulfill hypotheses $\mathrm{H}(U_1)$ and $\mathrm{H}(U_2)$, respectively.

It follows from Proposition 2.9 that for a. a. $x \in \Omega$ (resp. for a. a. $x \in \Gamma_2$) and all $s \in \mathbb{R}$ the set $U_1(x,s)$ (resp. $U_2(x,s)$) is nonempty, bounded, closed and convex in $\mathbb{R}$, namely, condition $\mathrm{H}(U_1)$(i) (resp. $\mathrm{H}(U_2)$) is satisfied. Hypotheses $\mathrm{H}(j_1)$(i) and $\mathrm{H}(j_2)$(i) indicate that for all $s \in \mathbb{R}$, the functions $x \mapsto U_1(x,s) = r_1(s)\ \partial j_1(x,s)$ and $x \mapsto U_2(x,s) = r_2(s)\partial j_2(x,s)$ are measurable in Ω and on Γ_2, respectively. This means that $\mathrm{H}(U_1)$(ii) and $\mathrm{H}(U_2)$(ii) hold.

We claim that $s \mapsto r_1(s)\partial j_1(x,s)$ is u.s.c. From Proposition 2.8, it is sufficient to show that $(r_1(\cdot)\partial j_1(x,\cdot))^-(D)$ is closed for each closed set $D \subset \mathbb{R}$. Let $\{s_n\}_{n\in\mathbb{N}} \subset (r_1(\cdot)\partial j_1(x,\cdot))^-(D)$ be such that $s_n \to s$. Then, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ satisfying $\eta_n \in r_1(s_n)\partial j_1(x,s_n) \cap D$ for each $n \in \mathbb{N}$. We are able to find a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\eta_n = r_1(s_n)\xi_n$ and $\xi_n \in \partial j_1(x,s_n)$ for all $n \in \mathbb{N}$ and for a. a. $x \in \Omega$. Recall that $s_n \to s$, we can apply Proposition 2.9(iii) and (v) to conclude that $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Hence, we may assume that $\xi_n \to \xi$ in $\mathbb{R}$ for some $\xi \in D$, because of the closedness of D. But, the closedness of ∂j_1 (see Proposition 2.9(v)) admits that $\xi \in \partial j_1(x,s)$. This combined with the continuity of r_1 deduces that $\eta_n = r_1(s_n)\xi_n \to r_1(s)\xi \in r_1(s)\partial j_1(x,s)$. This implies that $s \in (r_1(\cdot)\partial j_1(x,\cdot))^-(D)$, that is, $(r_1(\cdot)\partial j_1(x,\cdot))^-(D)$ is closed. Applying Proposition 2.8 we see that $s \mapsto r_1(s)\partial j_1(x,s)$ is u.s.c. Using the same arguments as before, we can also show that $s \mapsto r_2(s)\partial j_2(x,s)$ is u.s.c. Therefore, $\mathrm{H}(U_1)$(iii) and $\mathrm{H}(U_2)$(iii) are verified.

Finally, hypotheses $\mathrm{H}(U_1)$(iv) and $\mathrm{H}(U_2)$(iv) are consequences of the assumptions $\mathrm{H}(j_1)$(iii) and $\mathrm{H}(j_2)$(iii). Consequently, we apply Theorem 3.4 to obtain the desired conclusion. ☐

In particular, when p, q are constants such that $1 < q < p$, then problem (1.1) reduces to the following isotropic implicit obstacle problem:

$$\begin{aligned}
-a(u)\Delta_p u - b(u)\Delta_q u + g(x,u) &\in U_1(x,u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_n} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_n} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
L(u) &\le J(u),
\end{aligned} \tag{3.33}$$

where Δ_p is the well-known p-Laplace operator, i.e.,

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Then, we have the following corollary.

Corollary 3.14. *Assume that* $\mathrm{H}(1)$, $\mathrm{H}(g)$, $\mathrm{H}(U_1)$, $\mathrm{H}(U_2)$, $\mathrm{H}(\phi)$, $\mathrm{H}(L)$ *and* $\mathrm{H}(J)$ *are satisfied such that the exponents* $p, q, \delta_0, \delta_1, \delta_2$ *are constants. Then, the solution set of problem* (3.33) *is nonempty and weakly compact in* V.

4. Isotropic implicit obstacle problems with nonlinear convection terms

In this section, we are going to move our attention to study the implicit obstacle problem (1.3) which involves a nonlinear convection function, two nonlocal terms and three multivalued mappings where two of them are formulated on the boundary and the other one is defined in the domain. If the exponents p, q are constants in problem (1.1), then problem (1.3) is a generalization of problem (1.1). The goal of this section is to establish the existence of a weak solution to problem (1.3) under more general assumptions.

We suppose the following assumptions on the functions g, U_1 and U_2.

H(g'): The function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is such that H(g)(i), (ii), (iii) are satisfied, $x \mapsto g(x,0)$ belongs to $L^{p'}(\Omega)$ and there exists a constant $m_g > 0$ such that

$$(g(x,s) - g(x,t))(s-t) \geq m_g |s-t|^p$$

for all $s,t \in \mathbb{R}$ and for a. a. $x \in \Omega$.

H(U_1'): The multivalued function $U_1\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that H(U_1)(i), (ii), (iii) are satisfied and there exist a function $\alpha_{U_1} \in L^{p'}(\Omega)_+$ and a constant $a_{U_1} \geq 0$ such that

$$|\eta| \leq \alpha_{U_1}(x) + a_{U_1}|s|^{p-1} \tag{4.1}$$

for all $\eta \in U_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

H(U_2'): The multivalued function $U_2\colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that H(U_2)(i), (ii), (iii) are satisfied and there exist a function $\alpha_{U_2} \in L^{p'}(\Gamma_2)_+$ and a constant $a_{U_2} > 0$ such that

$$|\xi| \leq \alpha_{U_2}(x) + a_{U_2}|s|^{p-1} \tag{4.2}$$

for all $\xi \in U_2(x,s)$, for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$.

For the convection term we suppose the following conditions.

H(f): $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) there exist $a_f, b_f \geq 0$ and a function $\alpha_f \in L^{\frac{q_1}{q_1-1}}(\Omega)_+$ satisfying

$$|f(x,s,\xi)| \leq a_f |\xi|^{\frac{p(q_1-1)}{q_1}} + b_f |s|^{q_1-1} + \alpha_f(x)$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $1 < q_1 < p^*$ and p^* is the critical exponents to p in the domain (see (2.1) with $r = p$);

(ii) there exist $c_f, d_f \geq 0$ and a function $\beta_f \in L^1(\Omega)_+$ such that

$$f(x,s,\xi)s \leq c_f|\xi|^p + d_f|s|^p + \beta_f(x)$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$;

(iii) there exist $e_f, h_f \geq 0$ such that

$$(f(x,s,\xi) - f(x,t,\xi))(s-t) \leq e_f |s-t|^p$$
$$|f(x,s,\xi_1) - f(x,s,\xi_2)| \leq h_f |\xi_1 - \xi_2|^{p-1}$$

for a. a. $x \in \Omega$, for all $s,t \in \mathbb{R}$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

H(2): The inequalities

$$\begin{aligned} c_a &> a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + c_f, \\ k(p)c_a &> h_f \hat{\lambda}^{\frac{1}{p}} \\ m_g &> \max\{a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1}, e_f\} \end{aligned}$$

hold, where $k(p)$ is given in (2.2), $\lambda_{1,p}^S$ is the first eigenvalue of the p-Laplacian with Steklov boundary condition (see (2.3) and (2.4)) and $\hat{\lambda} > 0$ is the smallest constant such that

$$\|u\|_{p,\Omega} \leq \hat{\lambda}\|\nabla u\|_{p,\Omega} \quad \text{for all } u \in W^{1,p}(\Omega). \tag{4.3}$$

Remark 4.1. *Observe that hypotheses* H(U_1') *and* H(U_2') *are weaker than* H(U_1) *and* H(U_2) *in case if* δ_1, δ_2 *are constants. Indeed, if* δ_1 *(resp.* δ_2*) is a constant and* $\varepsilon > 0$ *is arbitrary, then from* H(U_1)(iv) *(resp.* H(U_1)(iv)*), there exists a constant* $l(\varepsilon) > 0$ *such that*

$$|\eta| \leq \alpha_{U_1}(x) + a_{U_1}|s|^{\delta_1 - 1} \leq \alpha_{U_1}(x) + l(\varepsilon) + \varepsilon|s|^{p-1}$$

for all $\eta \in U_1(x,s)$*, for a. a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$*, where we have used Young's inequality and the fact that* $1 < \delta_1 < p$*. Then, the inequality* (4.1) *(resp.* (4.2)*) is valid. Therefore,* H(U_1') *(resp.* H(U_2')*) holds.*

Example 4.2. *The following functions satisfy hypotheses* H(g) *and* H(f)

$$g(x,s) = \zeta(x) + \kappa_0 s,$$
$$f(x,s,\xi) = \sum_{i=1}^N \zeta_i \xi_i - \kappa_1 s + \omega(x)$$

for a. a. $x \in \Omega$, *for all* $s \in \mathbb{R}$ *and for all* $\xi \in \mathbb{R}$, *where* $p = q_1 = 2$, $\omega \in L^2(\Omega)$, $\kappa_0 > 0$ *and* $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N$ *is a given vector.*

Now, let V be the closed subspace of $W^{1,p}(\Omega)$ defined by

$$V := \left\{ u \in W^{1,p}(\Omega) \,:\, u = 0 \text{ on } \Gamma_1 \right\}.$$

As in Section 3, the multivalued mapping K is defined as in (3.2). In what follows, if we refer to the conditions mentioned in Section 3, then it should be regarded as that the conditions hold in the constant exponents setting. For example, if we assume that H(ϕ) holds, then condition H(ϕ)(iii) is valid in the following sense: for each function $u \in L^{p_*}(\Gamma_3)$ the function $x \mapsto \phi(x, u(x))$ belongs to $L^1(\Gamma_3)$, where p_* is the critical exponent of p on the boundary Γ (see (2.1) with $r = p$).

Next, we give the definition of a weak solution.

Definition 4.3. *We say that a function* $u \in V$ *is a weak solution of problem* (1.3) *if* $u \in K(u)$ *and there exist functions* $\eta \in L^{p'}(\Omega)$, $\xi \in L^{p'}(\Gamma_2)$ *such that* $\eta(x) \in U_1(x, u(x))$ *for a. a.* $x \in \Omega$, $\xi(x) \in U_2(x, u(x))$ *for a. a.* $x \in \Gamma_2$ *and the inequality*

$$\begin{aligned}
&a(u)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(v-u)\,\mathrm{d}x + b(u)\int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla(v-u)\,\mathrm{d}x \\
&\quad + \int_\Omega g(x,u)(v-u)\,\mathrm{d}x + \int_{\Gamma_3} \phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x,u)\,\mathrm{d}\Gamma \\
&\geq \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_2} \xi(x)(v-u)\,\mathrm{d}\Gamma + \int_\Omega f(x,u,\nabla u)(v-u)\,\mathrm{d}x
\end{aligned} \tag{4.4}$$

holds for all $v \in K(u)$.

Let $(w, \eta, \xi) \in V \times X^* \times Y^*$ be arbitrary fixed, where $X = L^p(\Omega)$, $Y = L^p(\Gamma_2)$, $X^* = L^{p'}(\Omega)$ and $Y^* = L^{p'}(\Gamma_2)$. In order to solve problem (1.3), we first consider the following auxiliary obstacle problem with dependence on the gradient

$$\begin{aligned}
-a(w)\Delta_p u - b(w)\Delta_q u + g(x,u) &= \eta(x) + f(x,u,\nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u(x)}{\partial \nu_w} &= \xi(x) && \text{on } \Gamma_2,\\
-\frac{\partial u(x)}{\partial \nu_w} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
L(u) &\leq J(w),
\end{aligned} \tag{4.5}$$

where $\frac{\partial u(x)}{\partial \nu_w}$ is defined by

$$\frac{\partial u}{\partial \nu_w} := \left(a(w)|\nabla u|^{p-2}\nabla u + b(w)|\nabla u|^{q-2}\nabla u \right) \cdot \nu.$$

The next lemma shows that problem (4.5) has a unique solution.

Lemma 4.4. *Let* $p \geq 2$ *and* $1 < q < p$. *Suppose that* H(1), H(g'), H(ϕ), H(f), H(L) *and* H(J) *are fulfilled. Then, for each fixed* $(w, \eta, \xi) \in V \times X^* \times Y^*$, *problem* (4.5) *has a unique solution.*

Proof. The existence result is a direct consequence of Theorem 3.4 of Zeng-Bai-Gasiński [45]. It remains to verify the uniqueness of problem (4.5).

Let $u_1, u_2 \in V$ be two weak solutions of problem (4.5). Then, for each $i = 1, 2$, we have $u_i \in K(w)$ and

$$\begin{aligned}
&a(w)\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \cdot \nabla(v-u_i)\,\mathrm{d}x + b(w)\int_\Omega |\nabla u_i|^{q-2}\nabla u_i \cdot \nabla(v-u_i)\,\mathrm{d}x\\
&\quad + \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_i)\,\mathrm{d}\Gamma + \int_\Omega g(x,u_i)(v-u_i)\,\mathrm{d}x\\
&\geq \int_\Omega \eta(x)(v-u_i)\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)(v-u_i)\,\mathrm{d}\Gamma + \int_\Omega f(x,u_i,\nabla u_i)(v-u_i)\,\mathrm{d}x
\end{aligned}$$

for all $v \in K(w)$. Inserting $v = u_2$ and $v = u_1$ in the above inequalities with $i = 1$ and $i = 2$, respectively, we sum up the resulting inequalities to obtain

$$\begin{aligned}
&a(w)\int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\
&\quad + b(w)\int_\Omega \left(|\nabla u_1|^{q-2}\nabla u_1 - |\nabla u_2|^{q-2}\nabla u_2\right)\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\
&\quad + \int_\Omega \left(g(x,u_1)-g(x,u_2)\right)(u_1-u_2)\,\mathrm{d}x\\
&\leq \int_\Omega (f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_2))(u_1-u_2)\,\mathrm{d}x\\
&= \int_\Omega (f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_1))(u_1-u_2)\,\mathrm{d}x\\
&\quad + \int_\Omega (f(x,u_2,\nabla u_1)-f(x,u_2,\nabla u_2))(u_1-u_2)\,\mathrm{d}x.
\end{aligned}$$

Taking (2.2), H(g') and H(f)(iii) into account implies

$$\begin{aligned}
&k(p)c_a\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p + m_g\|u_1-u_2\|_{p,\Omega}^p\\
&\leq \int_\Omega e_f|u_1-u_2|^p\,\mathrm{d}x + \int_\Omega h_f|\nabla u_1-\nabla u_2|^{p-1}|u_1-u_2|\,\mathrm{d}x.
\end{aligned}$$

Applying Hölder's inequality and (4.3) gives

$$\begin{aligned}
&k(p)c_a\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p + m_g\|u_1-u_2\|_{p,\Omega}^p\\
&\leq e_f\|u_1-u_2\|_{p,\Omega}^p + h_f\|\nabla u_1-\nabla u_2\|_{p,\Omega}^{p-1}\|u_1-u_2\|_{p,\Omega}\\
&\leq e_f\|u_1-u_2\|_{p,\Omega}^p + h_f\hat{\lambda}^{\frac{1}{p}}\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p.
\end{aligned}$$

Hence,

$$\left(k(p)c_a - h_f\hat{\lambda}^{\frac{1}{p}}\right)\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p + (m_g-e_f)\,\|u_1-u_2\|_{p,\Omega}^p \leq 0.$$

By assumption, we know that $h_f\hat{\lambda}^{\frac{1}{p}} < c_a k(p)$ and $m_g > e_f$, thus $u_1 = u_2$. Therefore, for each $(w,\eta,\xi) \in V \times X^* \times Y^*$, problem (4.5) has a unique weak solution $u \in V$. $\square$

Let $\mathcal{S}\colon V \times X^* \times Y^* \to V$ be the solution mapping of problem (4.5) defined by

$$\mathcal{S}(w,\eta,\xi) = u_{w,\eta,\xi} \quad \text{for all } (w,\eta,\xi) \in V \times X^* \times Y^*,$$

where $u_{w,\eta,\xi}$ is the unique solution of problem (4.5) corresponding to $(w,\eta,\xi) \in V \times X^* \times Y^*$, see Lemma 4.4.

Next, we can prove that $\mathcal{S}$ is a completely continuous operator.

Lemma 4.5. *Let $p \geq 2$ and $1 < q < p$. Assume that* H(1), H(g'), H(ϕ), H(f), H(L) *and* H(J) *are satisfied. Then, the solution map $\mathcal{S}\colon V \times X^* \times Y^* \to V$ of problem* (4.5) *is completely continuous.*

Proof. Let $\{(w_n, \eta_n, \xi_n)\}_{n\in\mathbb{N}} \subset V \times X^* \times Y^*$ and $(w, \eta, \xi) \in V \times X^* \times Y^*$ be such that

$$(w_n, \eta_n, \xi_n) \xrightarrow{w} (w, \eta, \xi) \quad \text{in } V \times X^* \times Y^*.$$

Then, for any $n \in \mathbb{N}$, we have $u_n \in K(w_n)$ and

$$\begin{aligned}
&a(w_n)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(v-u_n)\,\mathrm{d}x + b(w_n)\int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(v-u_n)\,\mathrm{d}x\\
&\quad + \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \int_\Omega g(x,u_n)(v-u_n)\,\mathrm{d}x\\
&\geq \int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x + \int_{\Gamma_2}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma + \int_\Omega f(x,u_n,\nabla u_n)(v-u_n)\,\mathrm{d}x
\end{aligned} \tag{4.6}$$

for all $v \in K(w_n)$. Using hypotheses H(f)(ii) and $H(g')$, we have

$$\begin{aligned}
\int_\Omega f(x,u_n,\nabla u_n)u_n(x)\,\mathrm{d}x &\leq \int_\Omega c_f|\nabla u_n(x)|^p + d_f|u_n(x)|^p + \beta_f(x)\,\mathrm{d}x\\
&= c_f\|\nabla u_n\|_{p,\Omega}^p + d_f\|u_n\|_{p,\Omega}^p + \|\beta_f\|_{1,\Omega},
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\int_\Omega g(x,u_n)u_n(x)\,\mathrm{d}x\\
&= \int_\Omega \left(g(x,u_n) - g(x,0)\right)u_n(x)\,\mathrm{d}x + \int_\Omega g(x,0)u_n(x)\,\mathrm{d}x\\
&\geq \int_\Omega m_g|u_n(x)|^p\,\mathrm{d}x - \|g(\cdot,0)\|_{p',\Omega}\|u_n\|_{p,\Omega} = m_g\|u_n\|_{p,\Omega}^p - \|g(\cdot,0)\|_{p',\Omega}\|u_n\|_{p,\Omega}.
\end{aligned} \tag{4.8}$$

Putting $v = 0$ in (4.6) and using the inequalities (3.10), (4.6), (4.7) and (4.8), we get

$$\begin{aligned}
&\min\left\{(c_a - c_f), (m_g - d_f)\right\}\|u_n\|_V^p - \|g(\cdot,0)\|_{p',\Omega}\|u_n\|_V + \|\beta_f\|_{1,\Omega} - \alpha_\varphi\|u_n\|_V\\
&\leq (c_a - c_f)\|\nabla u_n\|_{p,\Omega}^p + (m_g - d_f)\|u_n\|_{p,\Omega}^p - \|g(\cdot,0)\|_{p',\Omega}\|u_n\|_{p,\Omega} + \|\beta_f\|_{1,\Omega} - \alpha_\varphi\|u_n\|_V\\
&\leq \|\phi(\cdot,0)\|_{1,\Gamma_3} + \|\eta_n\|_{V^*}\|u_n\|_V + \|\xi_n\|_{V^*}\|u_n\|_V + \beta_\varphi.
\end{aligned}$$

From the inequalities $c_a > c_f$ and $m_g > d_f$, it is not difficult to see that sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } V$$

for some $u \in K(w)$ due to Lemma 3.5(ii). Again from Lemma 3.5(ii), we are able to find a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n \in K(w_n)$ satisfying $y_n \to u$ in V. Condition H(f)(i) reveals that the sequence $\{f(\cdot,u_n,\nabla u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{q_1'}(\Omega)$ and since $q_1 < p^*$ we have

$$\lim_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)(y_n-u_n)\,\mathrm{d}x = 0. \tag{4.9}$$

Inserting $v = y_n$ in (4.6) and passing to the upper limit as $n \to \infty$ for the resulting inequality gives

$$\begin{aligned}
&\limsup_{n\to\infty}\left[a(w_n)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - y_n)\,\mathrm{d}x + b(w_n)\int_\Omega |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(u_n-y_n)\,\mathrm{d}x\right]\\
&\leq \limsup_{n\to\infty}\Bigg[\int_{\Gamma_3}\phi(x,y_n)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \int_\Omega g(x,u_n)(y_n-u_n)\,\mathrm{d}x\\
&\quad - \int_\Omega \eta_n(x)(y_n-u_n)\,\mathrm{d}x - \int_{\Gamma_2}\xi_n(x)(y_n-u_n)\,\mathrm{d}\Gamma - \int_\Omega f(x,u_n,\nabla u_n)(y_n-u_n)\Bigg].
\end{aligned}$$

Applying (3.12), (3.13), (3.14), (4.9) and the arguments of the proof of inequality (3.15) leads to

$$\limsup_{n\to\infty}\left[a(w)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-u)\,\mathrm{d}x+\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x\right]\le 0.$$

Therefore, it holds that $u_n\to u$ in V.

For any fixed $z\in K(w)$, we apply Lemma 3.5(iii) to find a sequence $\{z_n\}_{n\in\mathbb{N}}\subset V$ such that $z_n\in K(w_n)$ and $z_n\to z$ in V. We take $v=z_n$ in (4.6) and pass to the upper limit as $n\to\infty$ for the resulting inequality to obtain that

$$\begin{aligned}
&a(w)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(z-u)\,\mathrm{d}x+b(w)\int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla(z-u)\,\mathrm{d}x\\
&\quad+\int_{\Gamma_3}\phi(x,z)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma+\int_\Omega g(x,u)(z-u)\,\mathrm{d}x\\
&\ge\int_\Omega \eta(x)(z-u)\,\mathrm{d}x+\int_{\Gamma_2}\xi(x)(z-u)\,\mathrm{d}\Gamma+\int_\Omega f(x,u,\nabla u)(z-u)\,\mathrm{d}x.
\end{aligned}$$

Because $z\in K(w)$ is arbitrary, we conclude that u is the unique solution of problem (4.5) corresponding to $(w,\eta,\xi)\in V\times X^*\times Y^*$. Consequently, it holds $u_n=\mathcal{S}(w_n,\eta_n,\xi_n)\to\mathcal{S}(w,\eta,\xi)=u$ in V, namely, $\mathcal{S}$ is completely continuous. $\square$

Furthermore, we introduce the following multivalued mappings $\mathcal{U}_1\colon X\to 2^{X^*}$ and $\mathcal{U}_2\colon Y\to 2^{Y^*}$ given by

$$\begin{aligned}
\mathcal{U}_1(u)&:=\left\{\eta\in X^*\,:\,\eta(x)\in U_1(x,u(x))\text{ a. a. in }\Omega\right\},\\
\mathcal{U}_2(v)&:=\left\{\xi\in Y^*\,:\,\xi(x)\in U_2(x,v(x))\text{ a. a. on }\Gamma_2\right\},
\end{aligned}$$

for all $(u,v)\in X\times Y$, respectively. As before, by $i\colon V\to X$ and $\gamma\colon V\to Y$, we denote the embedding operator of $V\hookrightarrow X$ and the trace operator from $V\hookrightarrow Y$, respectively. It is clear that both are linear, bounded and compact. Then, their dual operators $i^*\colon X^*\to V^*$ and $\gamma^*\colon Y^*\to V^*$ are linear, bounded and compact as well. The following lemma is a direct consequence of Lemma 3.8.

Lemma 4.6. *Let* $\mathrm{H}(U_1')$ *and* $\mathrm{H}(U_2')$ *be satisfied. Then, the following statements hold:*

- (i) $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are well-defined and for each* $u\in X$ *and* $v\in Y$*, the sets* $\mathcal{U}_1(u)$ *and* $\mathcal{U}_2(v)$ *are bounded, closed and convex in* X^* *and* Y^**, respectively;*
- (ii) $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are strongly-weakly u.s.c., i.e.,* $\mathcal{U}_1$ *is u.s.c. from* X *with the strong topology to the subsets of* X^* *with the weak topology, and* $\mathcal{U}_2$ *is u.s.c. from* Y *with the strong topology to the subsets of* Y^* *with the weak topology.*

We are now in a position to give the following existence theorem to problem (1.3).

Theorem 4.7. *Let* $2\le p$ *and* $1<q<p$*. Assume that* $\mathrm{H}(1)$, $\mathrm{H}(2)$, $\mathrm{H}(f)$, $\mathrm{H}(g')$, $\mathrm{H}(U_1')$, $\mathrm{H}(U_2')$, $\mathrm{H}(\phi)$, $\mathrm{H}(L)$ *and* $\mathrm{H}(J)$ *are satisfied. Then, the solution set of problem* (1.3) *is nonempty and compact in* V*.*

Proof. From the proof of Theorem 3.4, it is sufficient to prove that the solution set of problem (1.3) is bounded and that the inclusion

$$\mathcal{S}(\overline{B_V(0,M^*)},\ \mathcal{U}_1(i\overline{B_V(0,M^*)}),\ \mathcal{U}_2(\gamma\overline{B_V(0,M^*)}))\subset\overline{B_V(0,M^*)}\tag{4.10}$$

is satisfied for some $M^*>0$.

We only examine the boundedness of Υ. The validity of (4.10) can be obtained by employing the same arguments to the boundedness of Υ and the techniques applied in the proof of Claim 5 in Theorem 3.4.

For any $u \in \Upsilon$, we are able to find functions $\eta \in \mathcal{U}_1(iu)$ and $\xi \in \mathcal{U}_2(\gamma u)$ such that inequality (4.4) holds. Using hypotheses H(g') and H(f)(ii) yields

$$\begin{aligned}\int_\Omega g(x,u)u(x)\,\mathrm{d}x &= \int_\Omega \left(g(x,u)-g(x,0)\right)u(x)\,\mathrm{d}x + \int_\Omega g(x,0)u(x)\,\mathrm{d}x\\ &\geq m_g\|u\|_{p,\Omega}^p - \|g(\cdot,0)\|_{p',\Omega}\|u\|_{p,\Omega}\end{aligned} \tag{4.11}$$

and

$$\int_\Omega f(x,u,\nabla u)u(x)\,\mathrm{d}x \leq c_f\|\nabla u\|_{p,\Omega}^p + d_f\|u\|_{p,\Omega}^p + \|\beta_f\|_{1,\Omega}. \tag{4.12}$$

By means of H(U_1') and H(U_2'), we have

$$\begin{aligned}\int_\Omega \eta(x)u(x)\,\mathrm{d}x &\leq \int_\Omega |\eta(x)||u(x)|\,\mathrm{d}x\\ &\leq \int_\Omega \left(\alpha_{U_1}(x) + a_{U_1}|u(x)|^{p-1}\right)|u(x)|\,\mathrm{d}x\\ &\leq a_{U_1}\|u\|_{p,\Omega}^p + \|\alpha_{U_1}\|_{p',\Omega}\|u\|_{p,\Omega},\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}&\int_{\Gamma_2} \xi(x)u(x)\,\mathrm{d}\Gamma \leq \int_{\Gamma_2}|\xi(x)||u(x)|\,\mathrm{d}\Gamma\\ &\leq \int_{\Gamma_2}\left(\alpha_{U_2}(x) + a_{U_2}|u(x)|^{\delta_2(x)-1}\right)|u(x)|\,\mathrm{d}\Gamma\\ &\leq a_{U_2}\|u\|_{\Gamma_2,p}^p + \|\alpha_{U_2}\|_{p',\Gamma_2}\|u\|_{p,\Gamma_2}\\ &\leq a_{U_2}\left(\lambda_{1,p}^S\right)^{-1}\left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right) + \|\alpha_{U_2}\|_{p',\Gamma_2}\left(\lambda_{1,p}^S\right)^{-\frac{1}{p}}\left(\|\nabla u\|_{p,\Omega} + \|u\|_{p,\Omega}\right),\end{aligned} \tag{4.14}$$

where we have used the elementary inequality $(s+t)^r \leq s^r + t^r$ for all $s,t>0$ with $0<r<1$ and the inequality

$$\|u\|_{p,\Gamma_2}^p \leq \left(\lambda_{1,p}^S\right)^{-1}\left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right) \quad \text{for all } u \in W^{1,p}(\Omega),$$

which comes from the eigenvalue problem of the p-Laplacian with Steklov boundary condition (see (2.3) and (2.4)).

Taking (4.11), (4.12), (4.13) and (4.14) into account, we have the following estimate

$$\begin{aligned}&a(u)\int_\Omega |\nabla u|^p\,\mathrm{d}x + b(u)\int_\Omega |\nabla u|^q\,\mathrm{d}x + \int_\Omega g(x,u)u\,\mathrm{d}x - \int_\Omega \eta(x)u(x)\,\mathrm{d}x\\ &\quad - \int_{\Gamma_2}\xi(x)u(x)\,\mathrm{d}\Gamma - \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x\\ &\geq \left(c_a - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - c_f\right)\|\nabla u\|_{p,\Omega}^p + \left(m_g - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - d_f - a_{U_1}\right)\|u\|_{p,\Omega}^p\\ &\quad - \|\alpha_{U_2}\|_{p',\Gamma_2}\left(\lambda_{1,p}^S\right)^{-\frac{1}{p}}\|u\|_V - \|\alpha_{U_1}\|_{p',\Omega}\|u\|_{p,\Omega} - \|g(\cdot,0)\|_{p',\Omega}\|u\|_{p,\Omega} - \|\beta_f\|_{1,\Omega}.\end{aligned}$$

Arguing as in the proof of (3.27), we can use the estimates above and hypotheses H(2) to conclude that Υ is bounded.

Subsequently, we can invoke the same arguments as in the proof of Theorem 3.4 to conclude that the solution set of problem (1.3) is nonempty and compact in V. □

Let us consider some special cases to problem (1.3).

If $J(u) \equiv 0$ and

$$L(u) = \int_\Omega (u(x) - \Psi(x))^+\,\mathrm{d}x \quad \text{for all } u \in V,$$

then problem (1.1) becomes the obstacle problem (1.12) with mixed boundary conditions, where $\Psi\colon \Omega \to (0,+\infty)$ is a given obstacle function. A careful observation gives the following corollary.

Corollary 4.8. *Let $1 < q < p$. Assume that* H(1), H(f)(i), (ii), H(g'), H(U_1'), H(U_2') *and* H(ϕ) *are satisfied. If, in addition, the following inequalities hold*

$$c_a > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + c_f \quad \text{and} \quad m_g > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1},$$

then the solution set of problem (1.12) *is nonempty and compact in* V.

If $J(u) \equiv +\infty$ or $L(u) \equiv -\infty$ for all $u \in V$, then problem (1.3) becomes the non-obstacle elliptic inclusion problem (1.13) involving a monotone and a nonmonotone multivalued boundary conditions, respectively. Hence, we have the following corollary.

Corollary 4.9. *Let $1 < q < p$. Assume that* H(1), H(f)(i), (ii), H(g'), H(U_1'), H(U_2') *and* H(ϕ) *are satisfied. If, in addition, the following inequalities hold*

$$c_a > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + c_f \quad \text{and} \quad m_g > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1},$$

then the solution set of problem (1.13) *is nonempty and compact in* V.

In addition, if $\Gamma_2 = \emptyset$ and $\Gamma_3 = \emptyset$, i.e., $\Gamma_1 = \Gamma$, then problem (1.3) reduces to implicit obstacle problem (1.10) with Dirichlet boundary condition. Using Theorem 4.7, we obtain the following corollary.

Corollary 4.10. *Assume that* H(1), H(f)(i), (ii), H(U_1) *and* H(ϕ) *are satisfied. If, in addition, the following inequalities hold*

$$c_a > c_f \quad \text{and} \quad m_g > d_f + a_{U_1},$$

then the solution set of problem (1.10) *with $g \equiv 0$ is nonempty and compact in* V.

It should be mentioned that hypotheses H(1) in problem (1.13) can be relaxed to the following weaker conditions.

H($1'$): $a\colon V \to (0, +\infty)$ and $b\colon L^{p^*}(\Omega) \to [0, +\infty)$ are such that $a(u) = l_a(u) + k_a(u)$ for all $u \in V$ and b is a continuous function, where $l_a\colon V \to [c_a, +\infty)$ is weakly continuous with some $c_a > 0$ and $k_a\colon V \to [0, +\infty)$ is continuous.

Obviously, we do not require in H($1'$) that a and b are weakly continuous on V. This extends enormously the scope of applications to our results. A concrete example to hypotheses H($1'$) is the following functions

$$a(u) = c_a + e^{-\int_\Omega |\nabla u|^\tau \,\mathrm{d}x} \text{ and } b(u) = \|u\|_{p^*,\Omega} \quad \text{for all } u \in V,$$

where $1 \le \tau \le p$.

We have the following result for (1.13) by using H($1'$) instead of H(1).

Theorem 4.11. *Let $1 < q < p$. Assume that* H($1'$), H(f)(i), (ii), H(g'), H(U_1'), H(U_2') *and* H(ϕ) *are satisfied. If, in addition, the following inequalities hold*

$$c_a > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + c_f \quad \text{and} \quad m_g > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1},$$

then the solution set of problem (1.13) *is nonempty and compact in* V.

Proof. Let $B\colon V \times V \to V^*$, $F\colon V \to V^*$ and $G\colon V \to V^*$ be the functions defined by

$$\begin{aligned}
\langle B(u,u), v\rangle &:= b(u) \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla v \,\mathrm{d}x,\\
\langle Fu, v\rangle &:= \int_\Omega f(x,u,\nabla u) v \,\mathrm{d}x,\\
\langle G(u), v\rangle &:= \int_\Omega g(x,u) v \,\mathrm{d}x,
\end{aligned}$$

for all $u, v \in V$. Then, using standard arguments, it is not difficult to see that $u \in V$ is a solution of problem (1.13) if and only if it solves the following inclusion problem:

$$\mathcal{G}(u) + \partial_c \varphi(u) \ni 0 \quad \text{in } V^*,$$

where the multivalued mapping $\mathcal{G}\colon V \to 2^{V^*}$ is defined by

$$\mathcal{G}(u) = A(u,u) + B(u,u) + G(u) - F(u) - i^*\mathcal{U}_1(u) - \gamma^*\mathcal{U}_2(u) \tag{4.15}$$

for all $u \in V$. From the proof of Theorem 3.4 of Zeng-Bai-Gasiński [45], we can see that the continuity of a and b plays a significant role to verify the pseudomonotonicity of $\mathcal{G}$. More precisely, it directly effects the validity of the condition that

- if $\{u_n\}_{n\in\mathbb{N}} \subset V$ with $u_n \xrightarrow{w} u$ in V and $u_n^* \in \mathcal{G}(u_n)$ are such that
$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \le 0, \tag{4.16}$$
then to each element $v \in V$, there exists $u^*(v) \in \mathcal{G}(u)$ with
$$\langle u^*(v), u - v\rangle \le \liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle. \tag{4.17}$$

Let $\{u_n\}_{n\in\mathbb{N}} \subset V$ and $\{u_n^*\}_{n\in\mathbb{N}} \subset V^*$ be sequences such that $u_n^* \in \mathcal{G}(u_n)$ and suppose inequality (4.16) holds. Then, there exist sequences $\{\eta_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset Y^*$ satisfying

$$u_n^* = A(u_n,u_n) + B(u_n,u_n) + G(u_n) - F(u_n) - i^*\eta_n - \gamma^*\xi_n \quad \text{for all } n \in \mathbb{N}.$$

Using hypotheses H(U_1') and H(U_2'), we know that the sequences $\{\eta_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset Y^*$ are both bounded. Passing to a subsequence if necessary, we may assume that

$$\eta_n \xrightarrow{w} \eta \quad \text{in } X^* \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } Y^* \tag{4.18}$$

for some $(\eta,\xi) \in X^* \times Y^*$. Besides, hypotheses H($f$) reveal that the sequence $\{F(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{q_1'}(\Omega)$. Then, we use the compactness of i and γ as well as of the embedding from V into $L^{q_1}(\Omega)$ to obtain

$$\begin{aligned}
0 &\ge \limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle\\
&\ge \limsup_{n\to\infty} \langle A(u_n,u_n), u_n - u\rangle + \liminf_{n\to\infty} \langle B(u_n,u_n), u_n - u\rangle + \liminf_{n\to\infty} \langle G(u_n), u_n - u\rangle\\
&\quad - \limsup_{n\to\infty} \langle F(u_n), u_n - u\rangle_{L^{q_1'}(\Omega)\times L^{q_1}(\Omega)} - \limsup_{n\to\infty} \langle \eta_n, u_n - u\rangle_{L^{p'}(\Omega)\times L^{p}(\Omega)}\\
&\quad - \limsup_{n\to\infty} \langle \xi_n, u_n - u\rangle_{L^{p'}(\Gamma_2)\times L^{p}(\Gamma_2)}\\
&\ge \limsup_{n\to\infty} \langle A(u_n,u_n), u_n - u\rangle + \liminf_{n\to\infty} \langle B(u_n,u), u_n - u\rangle + \liminf_{n\to\infty} \langle G(u), u_n - u\rangle\\
&\ge \limsup_{n\to\infty} \langle A(u_n,u_n), u_n - u\rangle,
\end{aligned}$$

where we have used the monotonicity of $u \mapsto B(v,u)$ and $u \mapsto G(u)$. Hence, we have

$$\begin{aligned}
0 &\ge \limsup_{n\to\infty} \langle A(u_n,u_n), u_n - u\rangle\\
&= \limsup_{n\to\infty} \left((l_a(u_n) + k_a(u_n)) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x \right)\\
&\ge \limsup_{n\to\infty} l_a(u_n) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x\\
&\quad + \liminf_{n\to\infty} k_a(u_n) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x\\
&\ge \limsup_{n\to\infty} l_a(u_n) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x\\
&\quad + \liminf_{n\to\infty} k_a(u_n) \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla (u_n - u)\, \mathrm{d}x\\
&\ge \limsup_{n\to\infty} l_a(u) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x\\
&\quad - \limsup_{n\to\infty} |l_a(u_n) - l_a(u)| \left| \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x \right|
\end{aligned}$$

$$\geq \limsup_{n\to\infty} l_a(u) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (u_n - u)\, \mathrm{d}x.$$

This implies that $u_n \to u$ in V.

Recall that $\mathcal{U}_1$ and $\mathcal{U}_2$ are strongly-weakly closed. Therefore, from (4.18) it follows that $\eta \in \mathcal{U}_1(u)$ and $\xi \in \mathcal{U}_2(u)$. For any $v \in V$, we have, due to the continuity of a, b, F and G, that

$$\lim_{n\to\infty} \langle u_n^*, u_n - v\rangle = \langle A(u,u) + B(u,u) + G(u) - F(u) - i^*\eta - \gamma^*\xi, u - v\rangle.$$

The latter combined with the fact that $\eta \in \mathcal{U}_1(iu)$ and $\xi \in \mathcal{U}_2(\gamma u)$ implies that $u^* = A(u,u) + B(u,u) + G(u) - F(u) - i^*\eta - \gamma^*\xi \in \mathcal{G}(u)$. Therefore, we conclude that (4.17) holds.

Arguing as in the proof of Theorem 3.4 of Zeng-Bai-Gasiński [45], we can prove that the solution set of problem (1.13) is nonempty. Invoking the same arguments as in the proof of Theorem 3.4, we conclude that the solution set of problem (1.13) is compact. ☐

Furthermore, we suppose that the function k_a in hypotheses H($1'$) satisfies the following condition:

$$k_a(u) \to \infty \quad \text{as } u \in V \text{ with } \|\nabla u\|_{p,\Omega} \to \infty. \tag{4.19}$$

Then inequality $c_a > a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + c_f$ can be dropped and the domain of l_a can be replaced by $(0,+\infty)$.

Theorem 4.12. *Let $1 < q < p$. Assume that* H($1'$) *with* $l_a\colon (0,+\infty) \to (0,+\infty)$, H($f$)(i), (ii), H($g'$), H($U_1'$), H($U_2'$) *and* H($\phi$) *are satisfied. If, in addition,* (4.19) *and the inequality*

$$m_g > a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1}$$

hold, then the solution set of problem (1.13) *is nonempty and compact in V.*

Proof. We will see that the inequality $c_a > a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + c_f$ plays an important role in order to prove that the operator $\mathcal{G}\colon V \to 2^{V^*}$ defined in (4.15) is coercive in the sense that

$$\lim_{n\to\infty} \frac{\langle \mathcal{G}u_n, u_n\rangle}{\|u_n\|_V} = +\infty, \tag{4.20}$$

whenever the sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$ is such that $\|u_n\|_V \to +\infty$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset V$ be such that $\|u_n\|_V \to +\infty$. Then, from (4.11), (4.12), (4.13) and (4.14), we have

$$\begin{aligned}
&\langle \mathcal{G}(u_n), u_n\rangle \\
&\geq \left(a(u_n) - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - c_f\right)\|\nabla u_n\|_{p,\Omega}^p + \left(m_g - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - d_f - a_{U_1}\right)\|u_n\|_{p,\Omega}^p \\
&\quad - \|\alpha_{U_2}\|_{p',\Gamma_2}\left(\lambda_{1,p}^S\right)^{-\frac{1}{p}}\|u_n\|_V - \|\alpha_{U_1}\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|g(\cdot,0)\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|\beta_f\|_{1,\Omega}.
\end{aligned} \tag{4.21}$$

Since $\|u_n\|_V = \|u_n\|_{p,\Omega} + \|\nabla u_n\|_{p,\Omega} \to +\infty$, one of the following cases can occur:

(a) $\|u_n\|_{p,\Omega} \to +\infty$ and $\{\|\nabla u_n\|_{p,\Omega}\}_{n\in\mathbb{N}}$ is bounded;
(b) $\|\nabla u_n\|_{p,\Omega} \to \infty$ and $\{\|u_n\|_{p,\Omega}\}_{n\in\mathbb{N}}$ is bounded;
(c) $\|u_n\|_{p,\Omega} \to +\infty$ and $\|\nabla u_n\|_{p,\Omega} \to \infty$.

Let us discuss the cases above separately. If case (a) holds, then we have

$$\liminf_{n\to\infty} \frac{\left(a(u_n) - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - c_f\right)\|\nabla u_n\|_{p,\Omega}^p}{\|u_n\|_V} = 0,$$

and

$$\liminf_{n\to\infty} \frac{\left(m_g - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - d_f - a_{U_1}\right)\|u_n\|_{p,\Omega}^p}{\|u_n\|_V} = +\infty.$$

This shows that (4.20) is valid. If (b) occurs, then from (4.19) we are able to find $n_0 \in \mathbb{N}$ such that

$$a(u_n) - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - c_f > 0 \quad \text{for all } n \geq n_0$$

and

$$\lim_{n\to\infty} \frac{\left(a(u_n) - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - c_f\right)\|\nabla u_n\|_{p,\Omega}^p + \left(m_g - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - d_f - a_{U_1}\right)\|u_n\|_{p,\Omega}^p}{\|u_n\|_V} + \infty.$$

Hence, also in this case we have (4.20). Finally, if case (c) takes place, then we have

$$a(u_n) - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - c_f \geq m_g - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - d_f - a_{U_1} > 0$$

for all $n \geq n_1$, for some $n_1 \in \mathbb{N}$, and

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\langle \mathcal{G}(u_n), u_n\rangle}{\|u_n\|_V} \\
&\geq \lim_{n\to\infty} \frac{\left(m_g - a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} - d_f - a_{U_1}\right)\left(\|\nabla u_n\|_{p,\Omega}^p + \|u_n\|_{p,\Omega}^p\right)}{\|u_n\|} \\
&- \limsup_{n\to\infty} \frac{\|\alpha_{U_2}\|_{p',\Gamma_2}\left(\lambda_{1,p}^S\right)^{-\frac{1}{p}}\|u_n\|_V + \|\alpha_{U_1}\|_{p',\Omega}\|u_n\|_{p,\Omega} + \|g(\cdot,0)\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|\beta_f\|_{1,\Omega}}{\|u_n\|_V} \\
&= +\infty.
\end{aligned}$$

Thus, (4.20) is verified. Therefore, we have shown that $\mathcal{G}$ is coercive.

Employing the same arguments as in the proof of Theorem 4.11, we can conclude that the solution set of problem (1.13) is nonempty and compact in V. □

Example 4.13. *The following functions satisfy hypotheses* H(1′) *and* (4.19)*:*

$$a(u) = c_0 + \int_\Omega |\nabla u|^p \,\mathrm{d}x \quad \text{and} \quad a(u) = e^{\int_\Omega |u|^p \,\mathrm{d}x} + \int_\Omega |\nabla u|^p \,\mathrm{d}x$$

for all $u \in V$ *and for some* $c_0 > 0$.

Similarly, if $J(u) \equiv 0$ and

$$L(u) = \int_\Omega (u(x) - \Psi(x))^+ \,\mathrm{d}x \quad \text{for all } u \in V \tag{4.22}$$

we also have the following theorem concerning problem (1.12).

Theorem 4.14. *Let* $1 < q < p$. *Assume that* H(1′), H(f)(i), (ii), H(g'), H(U_1'), H(U_2') *and* H(ϕ) *are satisfied. If, in addition,* (4.19) *and the inequality*

$$m_g > a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1},$$

hold, then the solution set of problem (1.12) *is nonempty and compact in* V.

Additionally, if $g \equiv 0$ and $\Gamma_2 = \Gamma_3 = \emptyset$, i.e., $\Gamma_1 = \Gamma$, and $J(u) \equiv 0$ and L as in (4.22) (resp. $J(u) \equiv +\infty$ for all $u \in V$), then problem (1.3) reduces to the following elliptic obstacle inclusion problem with Dirichlet boundary and nonlinear convection (resp. elliptic non-obstacle inclusion problem with Dirichlet boundary and nonlinear convection):

$$\begin{aligned}
-a(u)\Delta_p u - b(u)\Delta_q u &\in U_1(x,u) + f(x,u,\nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma,\\
u(x) &\leq \Psi(x) && \text{in } \Omega,
\end{aligned} \tag{4.23}$$

resp.,

$$\begin{aligned}
-a(u)\Delta_p u - b(u)\Delta_q u &\in U_1(x,u) + f(x,u,\nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma.
\end{aligned} \tag{4.24}$$

Now, we can remove the inequality

$$m_g > a_{U_2} \left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{U_1}.$$

For problems (4.23) and (4.24) we have the following results.

Theorem 4.15. *Let $1 < q < p$. Assume that* H($1'$), H(f)(i), (ii), *and* H(U_1') *are satisfied. If, in addition,* (4.19) *holds, then the solution set of problem* (4.23) *is nonempty and compact in V.*

Proof. Since $\Gamma_1 = \Gamma$, we see that $V = W_0^{1,p}(\Omega)$ and $\|u\|_V = \|\nabla u\|_{p,\Omega}$ for all $u \in V$. From the proof of Theorem 4.12, it is sufficient to examine that $\mathcal{G}$ is coercive in the sense of (4.20).

Let $\{u_n\}_{n\in\mathbb{N}} \subset V$ be such that $\|u_n\|_V \to +\infty$. Then, we have

$$\langle \mathcal{G}(u_n), u_n\rangle \geq (a(u_n) - c_f)\, \|\nabla u_n\|_{p,\Omega}^p - \|\alpha_{U_1}\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|\beta_f\|_{1,\Omega}.$$

Applying (4.19), there exists $n_2 \in \mathbb{N}$ such that $a(u_n) - c_f \geq 1$ for all $n \geq n_2$. Passing to the limit as $n \to \infty$ in the last inequality, we conclude that (4.20) holds, that is, $\mathcal{G}$ is coercive.

Arguing as in the proof of Theorem 4.11, we infer that the solution set of problem (1.13) is nonempty and compact in V. ☐

A similar result holds for problem (4.24).

Theorem 4.16. *Let $1 < q < p$. Assume that* H($1'$), H(f)(i), (ii), *and* H(U_1') *are satisfied. If, in addition,* (4.19) *holds, then the solution set of problem* (4.24) *is nonempty and compact in V.*

Next, we consider the problems (1.9) and (3.33). For this purpose, we assume the following conditions.

H(j_1'): The functions $j_1\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $r_1\colon \mathbb{R} \to \mathbb{R}$ are such that
- (i) $x \mapsto j_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$ with $x \mapsto j_1(x,0)$ belonging to $L^1(\Omega)$;
- (ii) $s \mapsto j_1(x,s)$ is locally Lipschitz continuous for a. a. $x \in \Omega$ and the function $r_1\colon \mathbb{R} \to \mathbb{R}$ is continuous;
- (iii) there exist a function $\alpha_{j_1} \in L^{p'}(\Omega)_+$ and a constant $a_{j_1} \geq 0$ such that
$$|r_1(s)\eta| \leq \alpha_{j_1}(x) + a_{j_1}|s|^{p-1}$$
for all $\eta \in \partial j_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

H(j_2'): The functions $j_2\colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ and $r_2\colon \mathbb{R} \to \mathbb{R}$ are such that
- (i) $x \mapsto j_2(x,s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$ with $x \mapsto j_2(x,0)$ belonging to $L^1(\Gamma_2)$;
- (ii) $s \mapsto j_2(x,s)$ is locally Lipschitz continuous for a. a. $x \in \Gamma_2$ and the function $r_2\colon \mathbb{R} \to \mathbb{R}$ is continuous;
- (iii) there exist a function $\alpha_{j_2} \in L^{p'}(\Gamma_2)_+$ and a constant $a_{j_2} \geq 0$ such that
$$|r_2(s)\xi| \leq \alpha_{j_2}(x) + a_{j_2}|s|^{p-1}$$
for all $\xi \in \partial j_2(x,s)$, for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$.

From the proofs of Theorems 3.13 and 4.7, we obtain the following result.

Corollary 4.17. *Let $2 \leq p$ and $1 < q < p$. Assume that* H(1), H(f), H(g'), H(j_1'), H(j_2') *and* H(ϕ) *are satisfied. If, in addition, the inequalities*

$$c_a > a_{j_2} \left(\lambda_{1,p}^S\right)^{-1} + c_f, \quad k(p)c_a > h_f \left(\hat{\lambda}\right)^{\frac{1}{p}} \quad \text{and} \quad m_g > \max\{a_{j_2} \left(\lambda_{1,p}^S\right)^{-1} + d_f + a_{j_1}, e_f\}$$

hold, then the solution set of problem (1.9) *is nonempty and compact in V.*

More particularly, when $f \equiv 0$, then problem (1.3) reduces to problem (3.33). In some sense, the following corollary extends the one in Corollary 3.14.

Corollary 4.18. *Let $1 < q < p$. Assume that* H(1), H(g'), H(U_1'), H(U_2'), H(ϕ), H(L) *and* H(J) *are satisfied. If, in addition, the inequalities*

$$c_a > a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} \quad \text{and} \quad m_g > a_{U_2}\left(\lambda_{1,p}^S\right)^{-1} + a_{U_1}$$

hold, then the solution set of problem (3.33) *is nonempty and compact in* V.

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