THE FUČÍK SPECTRUM FOR THE NEGATIVE p-LAPLACIAN WITH DIFFERENT BOUNDARY CONDITIONS

DUMITRU MOTREANU AND PATRICK WINKERT

Dedicated to Professor Themistocles M. Rassias on the occasion of his 60th birthday.

Abstract. This chapter represents a survey on the Fučík spectrum of the negative p-Laplacian with different boundary conditions (Dirichlet, Neumann, Steklov, and Robin). The close relationship between the Fučík spectrum and the ordinary spectrum is briefly discussed. It is also pointed out that for every boundary condition there exists a first nontrivial curve $\mathcal{C}$ in the Fučík spectrum which has important properties such as Lipschitz continuity, being decreasing and a certain asymptotic behavior depending on the boundary condition. As a consequence, one obtains a variational characterization of the second eigenvalue λ_2 of the negative p-Laplacian with the corresponding boundary condition. The applicability of the abstract results is illustrated to elliptic boundary value problems with jumping nonlinearities.

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^N$, let T be a selfadjoint linear operator on $L^2(\Omega)$ with compact resolvent and eigenvalues

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots .$$

The so-called Fučík spectrum[1] Σ of T is defined as the set of all pairs $(a, b) \in \mathbb{R}^2$ such that the equation

$$Tu = au^+ - bu^-$$

has a nontrivial solution. Here we denoted $u^+ = \max(u, 0)$ (the positive part of u) and $u^- = \max(-u, 0)$ (the negative part of u). Fučík [20] and Dancer [15] were the first authors who recognized that the set Σ plays an important part in the study of semilinear equations of type

$$Tu = g(x, u),$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with jumping nonlinearities satisfying

$$\frac{g(x,s)}{s} \to a \quad \text{as } s \to +\infty, \qquad \frac{g(x,s)}{s} \to b \quad \text{as } s \to -\infty.$$

Initially, a systematic study of this spectrum was developed by Fučík [21] in the case of the negative Laplacian in one-dimension, i.e., for $N = 1$, with periodic boundary condition. He proved that this spectrum is composed of two families of curves in

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[1]Svatopluk Fučík (21th October 1944 - 18th May 1979) was a Czech mathematician.

$\mathbb{R}^2$ emanating from the points (λ_k, λ_k) determined by the eigenvalues λ_k of the negative periodic Laplacian in one-dimension. Afterwards, many authors studied the Fučík spectrum Σ_2 for the negative Laplacian $-\Delta$ with Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbb{R}^N$ (see [2], [5], [14], [24], [25], [28], [29], [34], [35], and the references therein). In this respect, we mention that Dancer [15] proved that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in Σ_2, while de Figueiredo and Gossez [16] constructed a first nontrivial curve in Σ_2 passing through (λ_2, λ_2) and characterized it variationally. Here λ_1 and λ_2 respectively denote the first and second eigenvalue of $-\Delta$ with Dirichlet boundary condition. The next step in this direction was to investigate the Fučík spectrum Σ_p of the negative p-Laplacian (or p-Laplace operator) $-\Delta_p$ aiming to extend the results known for $-\Delta$. We recall that $-\Delta_p$ is given by

$$-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < +\infty,$$

which is a nonlinear operator if $p \neq 2$. If $p = 2$, it reduces to the negative Laplacian $-\Delta$. First, Drábek [18] has shown for $p \neq 2$ and in one-dimension that Σ_p has similar properties as in the linear case, i.e., for $p = 2$.

The aim of this chapter is to give an overview about the Fučík spectrum of the negative p-Laplacian $-\Delta_p$ with $1 < p < +\infty$ and different boundary conditions on the bounded domain Ω in $\mathbb{R}^N$.

Let V be a closed subspace of the Sobolev space $W^{1,p}(\Omega)$ such that $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$ and let V^* denote the dual space with the duality pairing $\langle \cdot, \cdot \rangle$ between V and V^*. It is well known that the operator $-\Delta_p : V \to V^*$ is bounded, continuous, pseudomonotone, and has the (S_+)-property (i.e., from $u_n \rightharpoonup u$ in V and $\limsup_{n\to+\infty}\langle -\Delta_p u_n, u_n - u\rangle \leq 0$ it follows that $u_n \to u$ in V). Note that on $W^{1,p}(\Omega)$ we have

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \quad v \in W^{1,p}(\Omega).$$

We refer to [6] for various nonlinear boundary problems involving $-\Delta_p$.

The Fučík spectrum of $-\Delta_p$ depends strongly on the choice of the boundary condition related to Ω. Specifically, the set Σ_p (resp., Θ_p) is called the Fučík spectrum of $-\Delta_p$ with homogeneous Dirichlet (resp. Neumann) boundary condition if for all pairs $(a, b) \in \Sigma_p$ (resp. Θ_p) the equation

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} \quad \text{in } \Omega$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega \quad \left(\text{resp., } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\right) \tag{1.1}$$

has a nontrivial weak solution. In (1.1), $\partial u/\partial\nu$ stands for the conormal derivative on $\partial\Omega$. If we replace (1.1) by

$$\frac{\partial u}{\partial \nu} = -\beta|u|^{p-2}u \quad \text{on } \partial\Omega$$

with fixed $\beta \geq 0$, we speak of the Fučík spectrum of $-\Delta_p$ with Robin boundary condition denoted by $\widehat{\Sigma}_p$. Finally, we write $\widetilde{\Sigma}_p$ for the Fučík spectrum of $-\Delta_p$ with Steklov boundary condition, which is formed by all $(a, b) \in \mathbb{R}^2$ provided

$$-\Delta_p u = -|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = a(u^+)^{p-1} - b(u^-)^{p-1} \quad \text{on } \partial\Omega$$

is solved nontrivially.

These spectra have intensively been studied in the last years. We will present in Sects. 2 through 5 some of their basic properties. Namely, it will be shown that there exists a close relationship between these spectra and the ordinary spectrum of $-\Delta_p$ subject to different boundary conditions. A fundamental fact is that every Fučík spectrum introduced above contains a first nontrivial curve $\mathcal{C}$ which is Lipschitz continuous and decreasing. However, the asymptotic behavior of these curves is different relative to the imposed boundary condition. Furthermore, we will indicate some applications of these spectra to certain nonlinear elliptic problems with jumping nonlinearities. Subtle phenomena can occur due to the interaction of the involved nonlinearities with these spectra, in particular resonance to spectral elements can appear. We emphasize that these problems and results can be considered beyond the setting of quasilinear elliptic equations. For instance, the field of variational inequalities, as those describing obstacle problems, offers a rich and flexible framework which is highly interesting for its applicability. For different classes of variational inequalities and their applications, we refer to the volume by Pardalos, Rassias, and Khan [33].

2. Dirichlet boundary condition

The Fučík spectrum of the negative p-Laplacian $-\Delta_p$ with homogeneous Dirichlet boundary condition is defined as the set Σ_p of those $(a,b)\in\mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

has a nontrivial (weak) solution u, which means that $u\in W_0^{1,p}(\Omega), u\not\equiv 0$, and it satisfies the equation

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx = \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1})v dx, \quad \forall v\in W_0^{1,p}(\Omega).$$

We note that if $a=b=\lambda$, problem (2.1) reduces to

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{p-2}u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

which is called the Dirichlet eigenvalue problem with respect to the negative p-Laplacian $-\Delta_p$. It is known that the first eigenvalue λ_1 of (2.2) is positive, simple, and its corresponding eigenfunctions have constant sign (see Anane [1] and Lindqvist [23]). In fact, the spectrum $\sigma(-\Delta_p)$ of the negative p-Laplacian $-\Delta_p$ associated to (2.2) includes an unbounded sequence of eigenvalues $(\lambda_k), k\in\mathbb{N}$, called the variational eigenvalues, which fulfills

$$0<\lambda_1<\lambda_2\le\ldots\le\lambda_k\le\ldots\to+\infty.$$

The variational eigenvalues satisfy min-max characterizations.

The Fučík spectrum Σ_p of the negative p-Laplacian $-\Delta_p$ with homogeneous Dirichlet boundary condition contains the two lines $\lambda_1\times\mathbb{R}$ and $\mathbb{R}\times\lambda_1$. Additionally, Σ_p contains the sequence of points $(\lambda_k,\lambda_k), k\in\mathbb{N}$, as can be easily seen from (2.1) and (2.2) by writing $u=u^+-u^-$. The Fučík spectrum Σ_p has been intensively studied by Cuesta, de Figueiredo, and Gossez [13] in the general case of $1<p<+\infty$

through a variational approach using the mountain-pass theorem. In order to give a brief overview of their results, let us set for every $s \geq 0$,

$$J_s(u) = \int_\Omega |\nabla u|^p dx - s \int_\Omega (u^+)^p dx, \quad u \in W_0^{1,p}(\Omega).$$

The function J_s is of class C^1 on $W_0^{1,p}(\Omega)$. Denote $\tilde{J}_s = J_s |_S$, with S given by

$$S = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p = 1 \right\}.$$

Since S is a C^1-submanifold of $W_0^{1,p}(\Omega)$, it follows that $\tilde{J}_s$ is of class C^1 on S in the sense of manifolds. Then the curve $s \in \mathbb{R}^+ \mapsto (s + c(s), c(s)) \in \mathbb{R}^2$ described by the min-max values

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u),$$

where

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}, \tag{2.3}$$

is contained in Σ_p (see [13, Theorem 2.10]). In (2.3), φ_1 denotes the eigenfunction of (2.2) corresponding to λ_1 satisfying $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_p = 1$. Taking into account that Σ_p is symmetric with respect to the diagonal of the plane, it turns out that the curve

$$\mathcal{C} := \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\} \tag{2.4}$$

is contained in Σ_p. It is shown in [13, Theorem 3.1] that $\mathcal{C}$ given in (2.4) is indeed the first nontrivial curve in Σ_p, which means that the first point in Σ_p belonging to the parallel to the diagonal drawn through a point of $(\mathbb{R}_+ \times \{\lambda_1\}) \times (\{\lambda_1\} \times \mathbb{R})$ must be on $\mathcal{C}$ (see Fig. 1). As a consequence, we infer that the curve $\mathcal{C}$ passes through (λ_2, λ_2). In conjunction with the description of $\mathcal{C}$ in (2.4) and the min-max formula for $c(s)$, this yields that λ_2 can be variationally characterized as follows

$$\lambda_2 = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \int_\Omega |\nabla u|^p dx, \tag{2.5}$$

with Γ introduced in (2.3). Moreover, the curve $\mathcal{C}$ in (2.4) is Lipschitz continuous and decreasing as shown in [13, Proposition 4.1]. Finally, we mention that the limit of $c(s)$ as $s \to +\infty$ is equal to the first eigenvalue λ_1 of (2.2), which is proven in [13, Proposition 4.4].

The work of Cuesta, de Figueiredo, and Gossez [13] was the first paper that gave a complete study of the beginning of the Fučík spectrum of $-\Delta_p$ with homogeneous Dirichlet boundary condition and their variational approach was the starting point for investigating the Fučík spectrum under other boundary conditions (Neumann, Steklov, Robin, see the sections below). The knowledge of the properties of Σ_p, especially the existence of the first nontrivial curve $\mathcal{C}$ and its representation, has demonstrated to be very useful in obtaining multiple solutions results for elliptic equations involving the negative p-Laplacian $-\Delta_p$ and jumping nonlinearities.

In order to illustrate the applicability of the Fučík spectrum Σ_p, we consider the following equation with homogeneous Dirichlet boundary condition

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) \quad \text{in } \Omega, \tag{2.6}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$\lim_{t \to 0} \frac{g(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly for a.a. } x \in \Omega.$$

In Carl and Perera [12], it is proven that problem (2.6) has at least three nontrivial solutions provided the point $(a,b) \in \mathbb{R}^2$ lies above the first nontrivial curve $\mathcal{C}$ in Σ_p constructed in (2.4). Moreover, a complete sign information for the three solutions is available: two solutions have opposite constant sign and the third one is sign-changing (nodal solution). This information is obtained by means of the method of sub-supersolution whose application to problem (2.6) strongly relies on the hypothesis that the point $(a,b) \in \mathbb{R}^2$ is situated above the first nontrivial curve $\mathcal{C}$ in Σ_p. The following graphic marks the position of the point $(a,b) \in \mathbb{R}^2$ entering (2.6) and demonstrates the qualitative behavior of the curve $\mathcal{C}$.

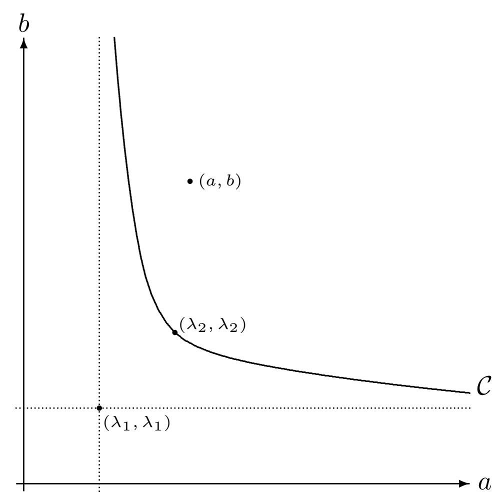

FIGURE 1. The first nontrivial curve $\mathcal{C}$ of the Fučík spectrum of the negative p-Laplacian with Dirichlet boundary condition. Problem (2.6) has multiple solutions if the pair (a,b) is above the curve $\mathcal{C}$.

Multiple solutions results concerning problems of type (2.6) and using the representation of the first nontrivial curve $\mathcal{C}$, in particular the characterization of the second eigenvalue λ_2 of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ as stated in (2.5), can be found in numerous publications; see, for example, [7], [10], [30]. We also refer to versions of such results in the case of nonsmooth potential associated to (2.6) (see, e.g., [8], [9], [11]).

3. Neumann boundary condition

In this section, we give a brief overview of the Fučík spectrum of the negative p-Laplacian $-\Delta_p$ with Neumann boundary condition. In order to avoid misunderstandings, we point out that a Neumann boundary condition stands in this context for a homogeneous Neumann condition. Inhomogeneous Neumann boundary conditions are treated in Section 4 (Steklov boundary condition) and Section 5 (Robin boundary condition). Let us first give the relevant definition of this spectrum. The Fučík spectrum of $-\Delta_p$ with Neumann boundary condition, denoted by Θ_p, consists of all pairs $(a,b) \in \mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

is solved nontrivially, meaning that $u \in W^{1,p}(\Omega), u \not\equiv 0$, and verifies the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1}) v dx, \quad \forall v \in W^{1,p}(\Omega).$$

In (3.1), $\partial u/\partial \nu$ denotes the conormal derivative, that is $\partial u/\partial \nu = |\nabla u|^{p-2}\nabla u \cdot \nu$, where ν is the unit outward normal to $\partial\Omega$. Problem (3.1) is a special case of the Robin Fučík spectrum that will be introduced in Section 5. Clearly, in case where $a = b = \lambda$, problem (3.1) becomes the Neumann eigenvalue problem of the negative p-Laplacian given by

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

As proved in [22], the first eigenvalue $\lambda_1 = 0$ of (3.2) is simple with the corresponding eigenspace $\mathbb{R}$, so all eigenfunctions associated to λ_1 do not change sign in Ω, which does not happen for the higher order eigenvalues. It is easily seen that Θ_p contains in particular $(0,0), (\lambda_2, \lambda_2)$ (λ_2 is the second eigenvalue of (3.2)) and the two lines $0 \times \mathbb{R}$ and $\mathbb{R} \times 0$. The nontrivial part of Θ_p is denoted by $\tilde{\Theta}_p$, that is $\tilde{\Theta}_p = \Theta_p \setminus ((0 \times \mathbb{R}) \cup (\mathbb{R} \times 0))$, which is obviously contained in $\mathbb{R}^+ \times \mathbb{R}^+$.

The basic paper dealing with the Fučík spectrum of the negative Neumann p-Laplacian is due to Arias, Campos, and Gossez [4]. The construction of a first nontrivial curve in $\tilde{\Theta}_p$ can be done similarly to the Dirichlet Fučík spectrum. To this end, for every $s \geq 0$, let $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ be the functional given by

$$J_s(u) = \int_\Omega |\nabla u|^p dx - s \int_\Omega (u^+)^p dx$$

and let $\tilde{J}_s$ be its restriction to

$$S = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega |u|^p = 1 \right\}.$$

Notice that S is a C^1-submanifold of $W^{1,p}(\Omega)$, so $\tilde{J}_s$ is of class C^1 on S in the sense of manifolds. This enables us to consider the notions of critical points and critical values for the functional $\tilde{J}_s$. Then, the first nontrivial curve $\mathcal{C}$ of Θ_p can be determined as in (2.4), whereas

$$\begin{aligned} &c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u),\\ &\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}. \end{aligned}$$

Here we have $\varphi_1 = 1/|\Omega|^{1/p}$, so $\|\varphi_1\|_p = 1$, with $|\Omega|$ denoting the measure of Ω. Arguing as in the case of the Dirichlet Fučík spectrum Σ_p, we see that $\mathcal{C}$ passes through (λ_2, λ_2) (λ_2 denotes the second eigenvalue of (3.2)). Consequently, we get a variational expression of λ_2 as

$$\lambda_2 = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \int_\Omega |\nabla u|^p dx,$$

with Γ introduced above. An important difference between the Dirichlet Fučík spectrum Σ_p and the Neumann Fučík spectrum Θ_p consists in the asymptotic behavior of the first nontrivial curve $\mathcal{C}$. In the Neumann case, to describe the asymptotic properties of the curve $\mathcal{C}$ it is required to consider the situations $p \le N$ and $p > N$ separately. In [4, Theorem 2.3 and Theorem 2.6], it is shown that

$$\lim_{s\to\infty} c(s) = \begin{cases} \lambda_1 = 0 & \text{if } p \le N \\ \overline{\lambda} & \text{if } p > N, \end{cases} \tag{3.3}$$

where

$$\overline{\lambda} = \inf \left\{ \int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), \|u\|_{L^p(\Omega)} = 1,\ u \text{ vanishes somewhere in } \overline{\Omega} \right\}.$$

The definition of $\overline{\lambda}$ is meaningful because for $p > N$ the elements $u \in W^{1,p}(\Omega)$ are continuous functions on $\overline{\Omega}$.

An extension of the previous results to the Fučík spectrum of the negative Neumann p-Laplacian with weights has been achieved by Arias, Campos, Cuesta, and Gossez [3]. Therein, for the weights given by the measurable functions $m(x)$ and $n(x)$ on Ω, the authors consider the set Σ of all pairs $(a, b) \in \mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= am(x)(u^+)^{p-1} - bn(x)(u^-)^{p-1} && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has a nontrivial solution. Under suitable assumptions on the data it is shown that Σ contains a first nontrivial curve.

Recently, Motreanu-Tanaka [31] used the results presented in the first part of this section to study quasilinear elliptic equations of the form

$$\begin{aligned} -\text{div } A(x, \nabla u) &= f(x, u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.4}$$

where, in the principal part of the equation, one has an operator $A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$ of the form $A(x, y) = a(x, |y|)y$, with $a(x, t) > 0$ for all $(x, t) \in \overline{\Omega} \times (0, +\infty)$, which is strictly monotone with respect to the second variable and fulfills some further regularity assumptions, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function having a representation similar to (2.6). They prove existence results for multiple solutions to equation (3.4), the properties of the solution set depending on conditions related to the first nontrivial curve $\mathcal{C}$ in the Neumann Fučík spectrum Θ_p . These results apply in particular to the case of the Neumann p-Laplacian in (3.4), i.e., when div $A(x, \nabla u) = \Delta_p u$.

4. Steklov boundary condition

Now we focus on the Steklov Fučík spectrum of $-\Delta_p$ which addresses $-\Delta_p$ with a special nonhomogeneous boundary condition, known as Steklov boundary condition. This spectrum is defined as the set $\widetilde{\Sigma}_p$ of all pairs $(a,b) \in \mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

has a weak solution $u \not\equiv 0$. Let us recall that $u \in W^{1,p}(\Omega)$ is a weak solution of (4.1) if it satisfies the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = -\int_\Omega |u|^{p-2}uvdx + \int_{\partial\Omega} (a(u^+)^{p-1} - b(u^-)^{p-1})vd\sigma$$

for all $v \in W^{1,p}(\Omega)$. Here the notation $d\sigma$ stands for the $(N-1)$-dimensional surface measure. The name of this spectrum comes from the fact that if $a = b = \lambda$, (4.1) becomes the so-called Steklov eigenvalue problem, namely

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \lambda|u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

The fundamental difference with respect to the Dirichlet and Neumann Fučík spectra is that in the Steklov case a boundary integral is involved, a fact that substantially modifies the analysis regarding the relevant values a and b. The Steklov eigenvalue problem (4.2) was first studied by Martínez and Rossi [26] (see also Lê [22]). They showed that the first eigenvalue is positive, simple and every eigenfunction corresponding to the first eigenvalue does not change sign in Ω. Actually, we may find an eigenfunction associated to the first eigenvalue λ_1 belonging to $\operatorname{int}(C^1(\overline{\Omega})_+)$, where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, which is nonempty and given by

$$\operatorname{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\right\}.$$

Furthermore, in [19] it is established that there exists a sequence of eigenvalues λ_n of (4.2) such that $\lambda_n \to +\infty$ as $n \to +\infty$. The Steklov Fučík spectrum defined in (4.1) has been studied by Martínez and Rossi [27]. Their approach is mainly based on the ideas of Cuesta, de Figueiredo, and Gossez [13]. Precisely, for each $s \geq 0$, one defines a C^1 functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$J_s(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx - s\int_{\partial\Omega} (u^+)^p d\sigma.$$

Restricting J_s to

$$S = \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p d\sigma = 1\right\},$$

one obtains a C^1-functional $\tilde{J}_s$ on the C^1-submanifold S of $W^{1,p}(\Omega)$. Then, the first nontrivial curve in $\tilde{\Sigma}_p$ is expressed as

$$\mathcal{C} = \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\},$$

where

$$\begin{aligned} c(s) &= \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u), \\ \Gamma &= \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\} \end{aligned}$$

(cf. [27, Theorem 2.1]), where $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ with $\|\varphi_1\|_p = 1$. In particular, we derive the following variational characterization of the second eigenvalue λ_2 of the Steklov eigenvalue problem (4.2) which results in

$$\lambda_2 = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \int_\Omega \left(|\nabla u|^p + |u|^p \right) dx. \tag{4.3}$$

As before, the first nontrivial curve $\mathcal{C}$ is Lipschitz continuous and decreasing (cf. [27, Proposition 4.1]). Similar to the Neumann Fučík spectrum, in order to state the asymptotic properties of $\mathcal{C}$, which means in fact to determine the limit of $c(s)$ as $s \to +\infty$, it is needed to take into account two cases, $p \leq N$ and $p > N$. The following holds (see [27, Theorem 4.1])

$$\lim_{s \to \infty} c(s) = \begin{cases} \lambda_1 & \text{if } p \leq N, \\ \overline{\lambda} > \lambda_1 & \text{if } p > N, \end{cases}$$

where

$$\overline{\lambda} = \inf_{u \in L} \max_{r \in \mathbb{R}} \frac{\|r\varphi_1 + u\|^p_{W^{1,p}(\Omega)}}{\|r\varphi_1 + u\|^p_{L^p(\partial\Omega)}}$$

with

$$L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere on } \partial\Omega\}.$$

As an application of the results in [27], consider the following nonlinear elliptic equation subject to Steklov-type boundary condition with perturbation

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

for Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ which are bounded on bounded sets and satisfy

(A) $\displaystyle\lim_{s \to 0} \frac{f(x,s)}{|s|^{p-1}} = 0$ uniformly for a.a. $x \in \Omega$,

(B) $\displaystyle\lim_{s \to 0} \frac{g(x,s)}{|s|^{p-1}} = 0$ uniformly for a.a. $x \in \partial\Omega$,

(C) $\displaystyle\lim_{|s| \to \infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty$ uniformly for a.a. $x \in \Omega$,

(D) $\displaystyle\lim_{|s| \to \infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$ uniformly for a.a. $x \in \partial\Omega$.

(E) There exists $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.

(F) g satisfies the condition

$$|g(x_1,s_1) - g(x_2,s_2)| \leq L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs $(x_1,s_1), (x_2,s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0,1]$.

If the point (a,b) is above the first nontrivial curve $\mathcal{C}$ in $\tilde{\Sigma}_p$, problem (4.4) possesses three nontrivial solutions: one solution with positive sign, one solution with negative sign, and the third one being sign-changing (cf. Winkert [38], see also [36] if $a =$

$b = \lambda > \lambda_2$ using the representation in (4.3)). An extension of this result for a nonsmooth problem corresponding to (4.4) can be found in Winkert [37].

5. Robin boundary condition

Finally, we discuss the Fučík spectrum of $-\Delta_p$ with a Robin boundary condition. To this end, we consider weak solutions $u \in W^{1,p}(\Omega)$ of the problem

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

meaning that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \beta \int_{\partial\Omega} |u|^{p-2}uv d\sigma = \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1})v dx$$

for all $v \in W^{1,p}(\Omega)$. In the formulation of (5.1), the parameter β is supposed to be a fixed, nonnegative constant. The Fučík spectrum of the negative p-Laplacian with Robin boundary condition is defined as the set $\widehat{\Sigma}_p$ of all pairs $(a, b) \in \mathbb{R}^2$ for which a nontrivial solution $u \in W^{1,p}(\Omega)$ of (5.1) exists. Clearly, if $\beta = 0$, it reduces to the Fučík spectrum Θ_p of the negative Neumann p-Laplacian (see Section 3). As before, the special case $a = b = \lambda$ leads to

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

which is the Robin eigenvalue problem of the negative p-Laplacian.

Problem (5.2) was studied in the important publication of Lê [22] devoted to the eigenvalue problems for the negative p-Laplacian. In the Robin case he proved similar results as they hold for the other eigenvalue problems. The first eigenvalue in (5.2), denoted as usually by λ_1, is simple, isolated, and can be variationally characterized as follows:

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma : \int_\Omega |u|^p dx = 1 \right\}.$$

It is also known that the eigenfunctions corresponding to λ_1 are of constant sign and belong to $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.

Recently in [32], the authors of the present text investigated the Fučík spectrum introduced in (5.1) with the aim to complete the picture of the Fučík spectrum involving the negative p-Laplacian by extending to the case of Robin boundary condition the information previously known for Dirichlet problem (see Section 2), Steklov problem (see Section 4), and homogeneous Neumann problem (see Section 3).

The approach in [32] is variational relying on the C^1-functional associated to problem (5.1), which is expressed on $W^{1,p}(\Omega)$ by

$$J(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma - \int_\Omega (a(u^+)^p + b(u^-)^p) dx.$$

It is clear that the critical points of J are exactly the (weak) solutions of problem (5.1). In comparison with the corresponding functionals related to the Fučík spectrum for the Dirichlet and Steklov problems, the functional J exhibits an essential difference because its expression does not incorporate the norm of the space

$W^{1,p}(\Omega)$, and it is also different from the functional used to treat the Neumann problem because it has the additional boundary term involving β.

The results in [32] can be summarized as follows. Applying various ideas and techniques on the pattern of [13], [4], [27], it is shown that $\widehat{\Sigma}_p$ contains a first nontrivial curve, denoted again by $\mathcal{C}$, and expressed as

$$\mathcal{C} = \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\},$$

where $c(s)$ is given by

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u),$$
$$\Gamma = \{\gamma \in C([-1, 1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\},$$

(see [32, Theorem 3.3]), with φ_1 standing for the eigenfunction of (5.2) associated to λ_1 which is normalized as $\|\varphi_1\|_{L^p(\Omega)} = 1$ and satisfies $\varphi_1 > 0$ on $\overline{\Omega}$. In the above formula of $c(s)$, $\tilde{J}_s$ is equal to the restriction of the C^1-functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_s(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma - s \int_\Omega (u^+)^p dx$$

to the C^1-submanifold

$$S = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega |u|^p dx = 1 \right\}$$

of $W^{1,p}(\Omega)$. It is shown in [32, Proposition 4.2] that the curve $\mathcal{C}$ is Lipschitz continuous and decreasing. The asymptotic behavior of $\mathcal{C}$ requires, as in the Neumann and Steklov cases, some more considerations. In case $p \leq N$, the following holds:

$$\lim_{s \to +\infty} c(s) = \lambda_1$$

(see [32, Theorem 4.3]). If $p > N$, one can suppose that $\beta > 0$ (the case $\beta = 0$ is included in Section 3, see (3.3)). In this respect, the key idea is to work with an adequate equivalent norm on the space $W^{1,p}(\Omega)$. So, for $\beta > 0$ one introduces the norm

$$\|u\|_\beta = \|\nabla u\|_{L^p(\Omega)} + \beta \|u\|_{L^p(\partial\Omega)},$$

which is an equivalent norm on $W^{1,p}(\Omega)$ (see also Deng [17, Theorem 2.1]). Then in [32, Theorem 4.4] one obtains that the limit of $c(s)$ as $s \to +\infty$ is

$$\overline{\lambda} = \inf_{u \in L} \max_{r \in \mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p d\sigma}{\int_\Omega |r\varphi_1 + u|^p dx},$$

where

$$L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere in } \overline{\Omega}, u \not\equiv 0\}.$$

Moreover, there holds $\overline{\lambda} > \lambda_1$.

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Département de Mathématiques, Université de Perpignan, Avenue Paul Alduy 52, 66860 Perpignan Cedex, France
E-mail address: motreanu@univ-perp.fr

Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
E-mail address: winkert@math.tu-berlin.de