PARAMETRIC ROBIN DOUBLE PHASE PROBLEMS WITH CRITICAL GROWTH ON THE BOUNDARY

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Abstract. The aim of this paper is to study a double phase problem with nonlinear boundary condition of critical growth and with a superlinear right-hand side that does not satisfy the Ambrosetti-Rabinowitz condition. Based on an equivalent norm in the Musielak-Orlicz Sobolev space along with variational tools and critical point theory, we prove the existence of at least two nontrivial, bounded weak solutions.

1. Introduction

Given a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ with Lipschitz boundary $\partial\Omega$, this paper is devoted to the study of the following parametric double phase problem with nonlinear boundary condition given by

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + \alpha(x)|u|^{p-2}u &= \lambda f(x,u) \qquad && \text{in } \Omega,\\ \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nu + \beta(x)|u|^{p_*-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, $\nu(x)$ denotes the unit normal of Ω at the point $x \in \partial\Omega$, $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth and a certain behavior at $\pm\infty$ (see (H2)) and we suppose that

(H1) (i) $1 < p < N$, $p < q < p^*$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$, where p^* is the critical Sobolev exponent to p given by

$$p^* = \frac{Np}{N-p};$$

(ii) $\alpha \in L^\infty(\Omega)$ with $\alpha(x) \geq 0$ for a. a. $x \in \Omega$ and $\alpha \not\equiv 0$;

(iii) $\beta \in L^\infty(\partial\Omega)$ with $\beta(x) \geq 0$ for a. a. $x \in \partial\Omega$.

The exponent p_* denotes the critical exponent of p on the boundary given by

$$p_* = \frac{(N-1)p}{N-p}. \tag{1.2}$$

If $\beta(x) \equiv 0$, then (1.1) reduces to the homogeneous Neumann problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + \alpha(x)|u|^{p-2}u &= \lambda f(x,u) \qquad && \text{in } \Omega,\\ \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nu &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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The divergence operator in (1.1) is known as the so-called double phase operator which is given by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right) \tag{1.3}$$

for functions u defined in an appropriate Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, see Section 2 for its precise definition. If $\inf_\Omega \mu \geq \mu_0 > 0$ or $\mu(x)\equiv 0$, then (1.3) becomes the (q,p)-Laplacian or the p-Laplacian, respectively.

The operator (1.3) is related to the two-phase integral functional

$$\omega \mapsto \int_\Omega \left(|\nabla \omega|^p+\mu(x)|\nabla \omega|^q\right)\,\mathrm{d}x, \tag{1.4}$$

which was first introduced by Zhikov [21] in order to provide models for strongly anisotropic materials. It is clear that the integrand of (1.4) has unbalanced growth and the functional changes its ellipticity on the set where the weight function vanishes, so on the set $\{x\in\Omega\,:\,\mu(x)=0\}$.

The aim of this paper is to present an existence result of problem (1.1) under very general assumptions on the perturbation term. The idea is to use an abstract critical point theorem applied to the energy functional of (1.1) in order to get the existence of two bounded, nontrivial weak solutions with different energy sign and located within a precise interval depending on the data.

Our work extends a recent paper of the first three authors published in [7] in several ways. Indeed, we are dealing with a double phase operator instead of the p-Laplacian and we can drop the Ambrosetti-Rabinowitz condition of the perturbation term by weaker assumptions near $\pm\infty$. Moreover, we also allow critical growth on the boundary in contrast to [7]. In order to overcome with this critical term, we use an equivalent norm on the space $W^{1,\mathcal{H}}(\Omega)$ recently obtained in Crespo-Blanco-Papageorgiou-Winkert [5].

So far, there are only few works for double phase problems with nonlinear boundary condition. Gasiński-Winkert [10] studied the superlinear problem

$$\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right) &= f(x,u)-|u|^{p-2}u-\mu(x)|u|^{q-2}u \quad && \text{in } \Omega,\\
\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu &= g(x,u) && \text{on } \partial\Omega,
\end{aligned}$$

where they showed the existence of constant-sign and sign changing solutions by using the Nehari manifold technique. Papageorgiou-Rădulescu-Repovš, [15] obtained several existence results for superlinear and also resonant reactions of the problem

$$\begin{aligned}
-\operatorname{div}\left(a_0(x)|\nabla u|^{p-2}\nabla u\right)-\Delta_q u+\xi(x)|u|^{p-2}u &= f(x,u) \quad && \text{in } \Omega,\\
\frac{\partial u}{\partial n_\vartheta}+|u|^{p-2}u &= 0 && \text{on } \partial\Omega.
\end{aligned}$$

Their proofs are mainly based on Morse theory using the notion of homological local linking. Other results for Neumann double phase problems can be found in the works of Crespo-Blanco-Papageorgiou-Winkert [5] (singular problems), El Manouni-Marino-Winkert [8] (parametric problems depending on p-Laplacian eigenvalues), Farkas-Fiscella-Winkert [9] (singular Finsler problems) and Papageorgiou-Vetro-Vetro [16] (parametric Robin problems). All these papers use different techniques than ours.

Finally we mention related works for Neumann or Robin problems involving the p-Laplacian or the (p,q)-Laplacian applying different techniques as truncation methods, critical point theory or comparison principles. We refer to the papers of Bonanno-Candito [1], Bonanno-D'Aguì [2], D'Aguì-Sciammetta [6], Guarnotta-Marano-Motreanu [11], Motreanu-Sciammetta-Tornatore [13], Papageorgiou-Rădulescu [14], Winkert [20] and the references therein.

The paper is organized as follows. In Section 2 we present some preliminaries about Musielak-Orlicz Sobolev spaces and equip the space $W^{1,\mathcal{H}}(\Omega)$ with an equivalent norm. Furthermore, we present an abstract critical point theorem that we are going to use in our main proof. In Section 3 we present the main existence result including several special cases when f is nonnegative and/or independent of x. Moreover a concrete example is given.

2. Preliminaries

In this section we state some facts about Musielak-Orlicz Sobolev spaces including an equivalent norm for our function space $W^{1,\mathcal{H}}(\Omega)$ and we recall the main properties of the double phase operator. Moreover, we are going to state an abstract critical point result due to Bonanno-D'Aguì [3] which is used in our proofs.

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary. We denote by $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ for any $1 \leq r \leq \infty$. The corresponding Sobolev space $W^{1,r}(\Omega)$ for $1 < r < \infty$ is equipped with the equivalent norm

$$\|u\|_{1,r} = \left(\|\nabla u\|_r^r + \int_\Omega \alpha(x)|u|^r \,\mathrm{d}x\right)^{\frac{1}{r}}, \tag{2.1}$$

where α fulfills hypothesis (H1)(ii). The proof of such result is similar to those proofs as in Papageorgiou-Winkert [17, Proposition 4.5.34] and [18, Proposition 2.8].

Let $1 < p < \infty$ and $\kappa \in [1, p^*]$. Then the Sobolev embedding theorem ensures that the embedding $W^{1,p}(\Omega) \hookrightarrow L^\kappa(\Omega)$ is continuous, that is,

$$\|u\|_\kappa \leq c_\kappa \|u\|_{1,p} \quad \text{for all } u \in W^{1,p}(\Omega), \tag{2.2}$$

where $c_\kappa > 0$ is the best Sobolev constant. If $\kappa < p^*$, this embedding is compact and from Hölder's inequality we directly obtain that

$$c_\kappa \leq c_{p^*}|\Omega|^{\frac{p^*-\kappa}{p^*\kappa}} \quad \text{for } \kappa < p^*, \tag{2.3}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω in $\mathbb{R}^N$.

For our further treatment we need the following number

$$\vartheta := \frac{c_{p^*}^p}{|\Omega|^{\frac{p}{p^*}}} \max\left\{\|\alpha\|_\infty|\Omega|, \|\beta\|_\infty|\partial\Omega|\right\}, \tag{2.4}$$

where α, β satisfy (H1)(ii) and (H1)(iii), respectively, and $|\partial\Omega|$ denotes the $(N-1)$-dimensional Hausdorff surface measure of $\partial\Omega$.

Suppose (H1)(i) hold true, we introduce the nonlinear function $\mathcal{H}\colon \Omega\times[0,\infty) \to [0,\infty)$ given by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q.$$

Then, denoting by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$, the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is given by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) \,:\, \varrho_{\mathcal{H}}(u) < +\infty\}$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 \,:\, \varrho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

Here, the modular function $\varrho_{\mathcal{H}}$ is defined by

$$\varrho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x.$$

The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. It is known that $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$ are both reflexive Banach spaces and we can equip the space $W^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\begin{aligned}\|u\| = \inf\Bigg\{\tau > 0 \,:\, &\int_\Omega \left(\left(\frac{|\nabla u|}{\tau}\right)^p + \mu(x)\left(\frac{|\nabla u|}{\tau}\right)^q\right)\mathrm{d}x + \int_\Omega \alpha(x)\left(\frac{|u|}{\tau}\right)^p \mathrm{d}x \\ &+ \int_{\partial\Omega} \beta(x)\left(\frac{|u|}{\tau}\right)^{p_*} \mathrm{d}\sigma \leq 1\Bigg\},\end{aligned}$$

see Crespo-Blanco-Papageorgiou-Winkert [5, Proposition 2.2]. Using (2.1) for $r = p$, the corresponding modular ρ to $\|\cdot\|$ is given by

$$\begin{aligned}\rho(u) &= \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\alpha}^p + \|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\ &= \|u\|_{1,p}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p_*,\beta,\partial\Omega}^{p_*} \quad \text{for } u \in W^{1,\mathcal{H}}(\Omega),\end{aligned} \tag{2.5}$$

where

$$L^q_\mu(\Omega) = \left\{u \in M(\Omega) \,:\, \int_\Omega \mu(x)|u|^q\,\mathrm{d}x < +\infty\right\}$$

is endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}$$

and $\|\cdot\|_{p,\alpha}$ as well as $\|\cdot\|_{p_*,\beta,\partial\Omega}$ are the seminorms

$$\|u\|_{p,\alpha} = \left(\int_\Omega \alpha(x)|u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{p_*,\beta,\partial\Omega} = \left(\int_{\partial\Omega} \beta(x)|u|^{p_*}\,\mathrm{d}\sigma\right)^{\frac{1}{p_*}}.$$

The following proposition shows the relation between the norm $\|\cdot\|$ and the modular function ρ. We refer to Liu-Dai [12, Proposition 2.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [4, Proposition 2.16] for similar proofs of these results. Let

$$\ell := \max\{q, p_*\} \tag{2.6}$$

with p_* given in (1.2).

Proposition 2.1. *Let hypotheses (H1) be satisfied, let $u \in W^{1,\mathcal{H}}(\Omega)$, let ρ be defined by (2.5) and let ℓ be given in (2.6). Then the following hold:*

(i) *If $u \neq 0$, then $\|u\| = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
(ii) *$\|u\| < 1$ (resp. > 1, $= 1$) if and only if $\rho(u) < 1$ (resp. > 1, $= 1$);*
(iii) *If $\|u\| < 1$, then $\|u\|^\ell \leq \rho(u) \leq \|u\|^p$;*

(iv) *If* $\|u\| > 1$, *then* $\|u\|^p \le \rho(u) \le \|u\|^\ell$;
(v) $\|u\| \to 0$ *if and only if* $\rho(u) \to 0$;
(vi) $\|u\| \to +\infty$ *if and only if* $\rho(u) \to +\infty$.

Next, we state some useful embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [4, Proposition 2.16] or Gasiński-Winkert [10].

Proposition 2.2. *Let hypotheses (H1) be satisfied. Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ *are continuous for all* $r \in [1,p]$;
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1,p^*]$;
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is compact for all* $r \in [1,p^*)$;
(iv) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is continuous for all* $r \in [1,p_*]$;
(v) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is compact for all* $r \in [1,p_*)$;
(vi) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_\mu(\Omega)$ *is continuous;*
(vii) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

Next, we consider the operator $A\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ defined by

$$\begin{aligned}\langle A(u),v\rangle &:= \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x\\ &\qquad + \int_\Omega \alpha(x)|u|^{p-2}uv\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u|^{p_*-2}uv\,\mathrm{d}\sigma\end{aligned}\tag{2.7}$$

for all $u,v \in W^{1,\mathcal{H}}(\Omega)$ with $\langle\cdot\,,\cdot\rangle$ being the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$. The properties of the operator $A\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ are summarized in the next proposition. The proof is similar to those in Liu-Dai [12, Proposition 3.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [4, Proposition 3.5].

Proposition 2.3. *Let hypotheses (H1) be satisfied. Then, the operator A defined in (2.7) is bounded, continuous, strictly monotone and of type* (S_+), *that is,*

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\,\langle Au_n, u_n - u\rangle \le 0,$$

imply $u_n \to u$ *in* $W^{1,\mathcal{H}}(\Omega)$.

Given a Banach space X and its dual space X^*, we say that a functional $\varphi \in C^1(X)$ satisfies the Cerami-condition (C-condition for short), if every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\ge 1} \subseteq \mathbb{R}$ is bounded and

$$(1+\|x_n\|_X)\,\varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence.

Finally, we mention the following abstract critical point result due to Bonanno-D'Aguì [3, see Theorem 2.1 and Remark 2.2].

Theorem 2.4. *Let X be a real Banach space and let Φ, $\Psi\colon X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf\limits_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\hat{u} \in X$, with $0 < \Phi(\hat{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})},\tag{2.8}$$

and, for each

$$\lambda \in \Lambda := \left] \frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the C*-condition and it is unbounded from below. Moreover, Φ is supposed to be coercive.*

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Main Result

In this section we are going to prove that problem (1.1) has at least two nontrivial, bounded weak solutions. First, we assume the following hypotheses:

(H2) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

- (i) there exist $\kappa \in (\ell, p^*)$ and constants $\eta_1, \eta_2 > 0$ such that
$$|f(x,s)| \le \eta_1 + \eta_2 |s|^{\kappa-1}$$
for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where ℓ is given in (2.6);
- (ii) if $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$, then
$$\lim_{s\to\pm\infty} \frac{F(x,s)}{|s|^\ell} = +\infty$$
uniformly for a. a. $x \in \Omega$, where ℓ is given in (2.6);
- (iii) there exists
$$\zeta \in \left((\kappa - p)\frac{N}{p}, p^* \right) \tag{3.1}$$
such that
$$0 < \zeta_0 \le \liminf_{s\to\pm\infty} \frac{f(x,s)s - \ell F(x,s)}{|s|^\zeta}$$
uniformly for a. a. $x \in \Omega$, where ℓ is given in (2.6).

Remark 3.1. *From (H2)(ii) and (H2)(iii) it is easy to see that $f(x,\cdot)$ is $(\ell-1)$-superlinear at $\pm\infty$. Note that the conditions in (H2)(ii) and (H2)(iii) are weaker than the Ambrosetti-Rabinowitz condition which is usually supposed in the literature. The function*

$$f(s) = \begin{cases} |s|^{\beta_1-2}s & \text{if } |s| \le 1, \\ |s|^{q-2}s\ln(|s|) + |s|^{\beta_2-2}s & \text{if } 1 < |s|, \end{cases}$$

where $1 < \beta_1 < p$ and $1 < \beta_2 < \ell$, see (2.6), fulfills hypotheses (H2) but fails to satisfy the Ambrosetti-Rabinowitz condition.

The corresponding energy functional $I_\lambda\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ related to (1.1) is defined by

$$I_\lambda(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|\nabla u\|_{q,\mu}^q + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\int_\Omega F(x,u)\,\mathrm{d}x$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$. It is easy to verify that $I_\lambda \in C^1(W^{1,\mathcal{H}}(\Omega))$ and the critical points of I_λ are the weak solutions of (1.1). Let

$$I_\lambda = \Phi(u) - \lambda \Psi(u) \quad \text{for } u \in W^{1,\mathcal{H}}(\Omega),$$

where $\Phi,\ \Psi\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ are given by

$$\begin{aligned} \Phi(u) &= \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|\nabla u\|_{q,\mu}^q + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}, \\ \Psi(u) &= \int_\Omega F(x,u)\,\mathrm{d}x \end{aligned} \tag{3.2}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$. Clearly we have

$$\begin{aligned} \langle I_\lambda'(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla v\,\mathrm{d}x + \int_\Omega \alpha(x)|u|^{p-2}uv\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \beta(x)|u|^{p_*-2}uv\,\mathrm{d}\sigma - \lambda\int_\Omega f(x,u)v\,\mathrm{d}x, \\ \langle \Phi'(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla v\,\mathrm{d}x + \int_\Omega \alpha(x)|u|^{p-2}uv\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \beta(x)|u|^{p_*-2}uv\,\mathrm{d}\sigma, \\ \langle \Psi'(u), v\rangle &= \int_\Omega f(x,u)v\,\mathrm{d}x \end{aligned}$$

for all $u, v \in W^{1,\mathcal{H}}(\Omega)$.

The next proposition shows that the energy functional of problem (1.1) fulfills the C-condition.

Proposition 3.2. *Let hypotheses (H1) and (H2) be satisfied. Then the functional $I_\lambda\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}}(\Omega)$ be a sequence such that

$$|I_\lambda(u_n)| \le c_1 \quad \text{for some } c_1 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.3}$$

$$(1 + \|u_n\|)\, I_\lambda'(u_n) \to 0 \quad \text{in } W^{1,\mathcal{H}}(\Omega)^* \text{ as } n \to \infty. \tag{3.4}$$

From (3.4) we obtain

$$\left|\langle A(u_n), h\rangle - \lambda\int_\Omega f(x,u_n)\,h\,\mathrm{d}x\right| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{3.5}$$

for all $h \in W^{1,\mathcal{H}}(\Omega)$ with $\varepsilon_n \to 0^+$. If we take $h = u_n \in W^{1,\mathcal{H}}(\Omega)$ in (3.5), we get

$$-\|u_n\|_{1,p}^p - \|\nabla u_n\|_{q,\mu}^q - \|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \le \varepsilon_n \tag{3.6}$$

for all $n \in \mathbb{N}$. On the other hand, taking (3.3) into account, it follows that

$$\frac{\ell}{p}\|u_n\|_{1,p}^p + \frac{\ell}{q}\|\nabla u_n\|_{q,\mu}^q + \frac{\ell}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\int_\Omega \ell F(x,u_n)\,\mathrm{d}x \le \ell c_1, \tag{3.7}$$

where ℓ is given in (2.6). Note that $p < q \le \ell$ and $p_* \le \ell$. Using this and adding (3.6) and (3.7) gives

$$\lambda\int_\Omega \left(f(x,u_n)\,u_n - \ell F(x,u_n)\right)\mathrm{d}x \le c_2 \tag{3.8}$$

for some $c_2 > 0$ and for all $n \in \mathbb{N}$.

Hypotheses (H2)(i) and (H2)(iii) imply that we can find $c_3 \in (0, \zeta_0)$ and $c_4 > 0$ such that

$$c_3|s|^\zeta - c_4 \leq f(x,s)s - \ell F(x,s) \tag{3.9}$$

for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$. From (3.9) and (3.8) it follows that

$$\|u_n\|_\zeta^\zeta \leq c_5 \quad \text{for some } c_5 > 0 \text{ and for all } n \in \mathbb{N}.$$

This shows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq L^\zeta(\Omega) \text{ is bounded.} \tag{3.10}$$

It is clear that we may assume $\zeta < \kappa < p^*$, see (H2)(iii). Hence, there exists $t \in (0,1)$ such that

$$\frac{1}{\kappa} = \frac{1-t}{\zeta} + \frac{t}{p^*}. \tag{3.11}$$

From the interpolation inequality (see Papageorgiou-Winkert [17, p. 116]), we obtain that

$$\|u_n\|_\kappa \leq \|u_n\|_\zeta^{1-t} \|u_n\|_{p^*}^t \quad \text{for all } n \in \mathbb{N}.$$

Using this along with (3.10) and the embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, see Proposition 2.2(ii), yields

$$\|u_n\|_\kappa^\kappa \leq c_6 \|u_n\|^{t\kappa} \quad \text{for all } n \in \mathbb{N} \tag{3.12}$$

with some $c_6 > 0$. Now we choose $h = u_n \in W^{1,\mathcal{H}}(\Omega)$ in (3.5) to get

$$\|u_n\|_{1,p}^p + \|\nabla u_n\|_{q,\mu}^q + \|u_n\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda \int_\Omega f(x,u_n)u_n \,\mathrm{d}x \leq \varepsilon_n$$

for all $n \in \mathbb{N}$. From this together with the growth condition in (H2)(i), Proposition 2.1(iii), (iv) and (3.12) it follows that

$$\begin{aligned} \min\left\{\|u_n\|^p, \|u_n\|^\ell\right\} &\leq \lambda \int_\Omega f(x,u_n)u_n \,\mathrm{d}x + \varepsilon_n \\ &\leq \lambda c_7 \left[1 + \|u_n\|^{t\kappa}\right] + \varepsilon_n \end{aligned} \tag{3.13}$$

for some $c_7 > 0$ and for all $n \in \mathbb{N}$.

Using (3.11) and (3.1) we obtain

$$\begin{aligned} t\kappa &= \frac{p^*(\kappa - \zeta)}{p^* - \zeta} = \frac{Np(\kappa-\zeta)}{Np - N\zeta + \zeta p} \\ &< \frac{Np(\kappa - \zeta)}{Np - N\zeta + (\kappa - p)\frac{N}{p}p} = p < \ell. \end{aligned} \tag{3.14}$$

With a view to (3.13) and (3.14) we see that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}}(\Omega) \text{ is bounded.}$$

Hence, we can find a subsequence, not relabeled, such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^\kappa(\Omega), \tag{3.15}$$

since $\kappa < p^*$, see Proposition 2.2(iii). Choosing $h = u_n - u \in W^{1,\mathcal{H}}(\Omega)$ in (3.5), passing to the limit as $n \to \infty$ and using the convergence properties in (3.15), we conclude that

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0.$$

Finally, by the (S_+)-property of the operator A (see Proposition 2.3) we know that $u_n \to u$ in $W^{1,\mathcal{H}}(\Omega)$ which shows the assertion of the proposition. □

The main result in this paper reads as follows.

Theorem 3.3. *Let hypotheses (H1), (H2) be satisfied and let $\tau, \omega > 0$ be two constants with $\tau > \omega$ such that*

$$\eta_1 \tau^{1-p} + \frac{\eta_2}{\kappa}\tau^{\kappa-p} < \frac{1}{\vartheta|\Omega|}\cdot \frac{\displaystyle\int_\Omega F(x,\omega)\,\mathrm{d}x}{\omega^p + \omega^{p_*}}, \tag{3.16}$$

where η_1, η_2, κ and ϑ are given in (H2)(i) and (2.4), respectively. Then, for each

$$\lambda \in \Lambda_1 := \left] \frac{\vartheta|\Omega|^{\frac{p}{p^*}}}{p c_{p^*}^p} \cdot \frac{\omega^p + \omega^{p_*}}{\displaystyle\int_\Omega F(x,\omega)\,\mathrm{d}x}, \frac{1}{p c_{p^*}^p |\Omega|^{\frac{p}{N}}} \cdot \frac{1}{\eta_1\tau^{1-p} + \dfrac{\eta_2}{\kappa}\tau^{\kappa-p}} \right[, \tag{3.17}$$

problem (1.1) has at least two nontrivial bounded weak solutions $u_\lambda, v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ such that $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$.

Proof. We are going to apply Theorem 2.4 in order to show the assertions of the theorem. First, let Φ and Ψ be the C^1-functionals given in (3.2). Due to Proposition 2.1(iv) it is clear that Φ is coercive and taking hypothesis (H2)(ii) into account, we see that I_λ is unbounded from below. Obviously, we have

$$\inf_{u\in W^{1,\mathcal{H}}(\Omega)} \Psi(u) = \Psi(0) = \Phi(0).$$

Now, we fix $\lambda \in \Lambda_1$ which is possible because of (3.16). Let

$$r := \frac{1}{p}\frac{|\Omega|^{\frac{p}{p^*}}}{c_{p^*}^p}\tau^p, \tag{3.18}$$

where c_{p^*} is the best constant of the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and let

$$\hat{u}(x) = \omega. \tag{3.19}$$

Clearly, $\hat{u} \in W^{1,\mathcal{H}}(\Omega)$.

First we are going to show that $0 < \Phi(\hat{u}) < r$. From (3.19) and (2.4) we have that

$$\begin{aligned}
0 < \Phi(\hat{u}) &= \frac{1}{p}\|\hat{u}\|_{p,\alpha}^p + \frac{1}{p_*}\|\hat{u}\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&\le \frac{\omega^p}{p}\|\alpha\|_\infty|\Omega| + \frac{\omega^{p_*}}{p_*}\|\beta\|_\infty|\partial\Omega| \\
&\le \frac{1}{p}\left(\omega^p + \omega^{p_*}\right)\max\left\{\|\alpha\|_\infty|\Omega|, \|\beta\|_\infty|\partial\Omega|\right\} \\
&= \frac{\vartheta|\Omega|^{\frac{p}{p_*}}}{p c_{p^*}^p}\left(\omega^p + \omega^{p_*}\right).
\end{aligned} \tag{3.20}$$

In order to show $\Phi(\hat{u}) < r$, with view to (3.18) and (3.20), we have to verify that

$$\vartheta\left(\omega^p + \omega^{p_*}\right) < \tau^p. \tag{3.21}$$

Let us suppose the inequality in (3.21) is not satisfied, so we have

$$\vartheta\left(\omega^p + \omega^{p_*}\right) \geq \tau^p. \tag{3.22}$$

Using the growth condition of f given in (H2)(i) we easily obtain that

$$\int_\Omega F(x,\omega)\,\mathrm{d}x \leq \int_\Omega \left(\eta_1\omega + \frac{\eta_2}{\kappa}\omega^\kappa\right)\,\mathrm{d}x = \left(\eta_1\omega + \frac{\eta_2}{\kappa}\omega^\kappa\right)|\Omega|. \tag{3.23}$$

From $\tau > \omega$ along with (3.22) and (3.23) it follows that

$$\begin{aligned}
\eta_1\tau^{1-p} + \frac{\eta_2}{\kappa}\tau^{\kappa-p} &= \frac{\eta_1\tau + \frac{\eta_2}{\kappa}\tau^\kappa}{\tau^p} \geq \frac{\eta_1\tau + \frac{\eta_2}{\kappa}\tau^\kappa}{\vartheta\left(\omega^p + \omega^{p_*}\right)} \geq \frac{|\Omega|\left(\eta_1\omega + \frac{\eta_2}{\kappa}\omega^\kappa\right)}{\vartheta|\Omega|\left(\omega^p + \omega^{p_*}\right)}\\
&\geq \frac{\displaystyle\int_\Omega F(x,\omega)\,\mathrm{d}x}{\vartheta|\Omega|\left(\omega^p + \omega^{p_*}\right)},
\end{aligned}$$

which is a contradiction to (3.16). Hence, (3.21) is fulfilled and so we have $0 < \Phi(\hat{u}) < r$.

Next, we have to show that (2.8) is satisfied for r and $\hat{u}$ defined in (3.18) and (3.19), respectively.

First, from the representation of r in (3.18), we have

$$\tau = \left(\frac{c_{p^*}^p pr}{|\Omega|^{\frac{p}{p^*}}}\right)^{\frac{1}{p}}. \tag{3.24}$$

Note that

$$\Phi^{-1}\left(]-\infty, r[\right) \subseteq \left\{u \in W^{1,\mathcal{H}}(\Omega)\,:\, \|u\|_{1,p} \leq (pr)^{\frac{1}{p}}\right\}. \tag{3.25}$$

Applying the growth condition in (H2)(i) along with (2.3), (2.2), (3.25) as well as (3.24) yields

$$\begin{aligned}
&\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}\\
&\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\left(\eta_1\|u\|_1 + \frac{\eta_2}{\kappa}\|u\|_\kappa^\kappa\right)}{r}\\
&\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\left(\eta_1 c_{p^*}|\Omega|^{\frac{p^*-1}{p^*}}\|u\|_{1,p} + \frac{\eta_2}{\kappa}c_{p^*}^\kappa|\Omega|^{\frac{p^*-\kappa}{p^*}}\|u\|_{1,p}^\kappa\right)}{r}\\
&\leq \frac{\eta_1 c_{p^*}|\Omega|^{\frac{p^*-1}{p^*}}(pr)^{\frac{1}{p}} + \frac{\eta_2}{\kappa}c_{p^*}^\kappa|\Omega|^{\frac{p^*-\kappa}{p^*}}(pr)^{\frac{\kappa}{p}}}{r}\\
&= pc_{p^*}^p|\Omega|^{\frac{p^*-p}{p^*}}\left[\eta_1\left(\frac{c_{p^*}^p pr}{|\Omega|^{\frac{p}{p^*}}}\right)^{\frac{1-p}{p}} + \frac{\eta_2}{\kappa}\left(\frac{c_{p^*}^p pr}{|\Omega|^{\frac{p}{p^*}}}\right)^{\frac{\kappa-p}{p}}\right]\\
&= pc_{p^*}^p|\Omega|^{\frac{p}{N}}\left[\eta_1\tau^{1-p} + \frac{\eta_2}{\kappa}\tau^{\kappa-p}\right].
\end{aligned} \tag{3.26}$$

Combining (3.26), (3.16) and (3.20) it follows that

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &\leq pc_{p^*}^p|\Omega|^{\frac{p}{N}}\left[\eta_1\tau^{1-p}+\frac{\eta_2}{\kappa}\tau^{\kappa-p}\right]\\
&< pc_{p^*}^p|\Omega|^{\frac{p}{N}}\left[\frac{1}{\vartheta|\Omega|}\cdot\frac{\displaystyle\int_\Omega F(x,\omega)\,\mathrm{d}x}{\omega^p+\omega^{p_*}}\right]\\
&= \frac{\displaystyle\int_\Omega F(x,\omega)\,\mathrm{d}x}{\frac{\vartheta|\Omega|^{\frac{p}{p^*}}}{pc_{p^*}^p}\left(\omega^p+\omega^{p_*}\right)}\\
&\leq \frac{\Psi(\hat{u})}{\Phi(\hat{u})}.
\end{aligned}$$

Hence, (2.8) is satisfied.

Since I_λ satisfies the C-condition, see Proposition 3.2, we are in the position to apply Theorem 2.4 and get the existence of at least two nontrivial weak solutions $u_\lambda, v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ of problem (1.1) such that $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$. The boundedness of u_λ, v_λ follows from Gasiński-Winkert [10, Theorem 3.1]. This finishes the proof. ☐

Let us now mention some important special cases of Theorem 3.3. When f is nonnegative we have the following corollary.

Corollary 3.4. *Let hypotheses (H1), (H2) be satisfied, where in addition we assume that $f(x,s) \geq 0$ for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$. Moreover, let $\tau,\omega > 0$ be two constants with $\tau > \omega$ such that*

$$\eta_1\tau^{1-p}+\frac{\eta_2}{\kappa}\tau^{\kappa-p} < \frac{1}{\vartheta|\Omega|}\cdot\frac{\displaystyle\int_\Omega F(x,\omega)\,\mathrm{d}x}{\omega^p+\omega^{p_*}},$$

where η_1,η_2, κ and ϑ are given in (H2)(i) and (2.4), respectively. Then, for each $\lambda \in \Lambda_1$, where Λ_1 is given in (3.17), problem (1.1) has at least two nontrivial nonnegative bounded weak solutions $u_\lambda, v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ such that $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$.

The proof is a direct consequence of Theorem 3.3 and the fact that we can test the weak formulation of (1.1) with the negative part of the function to obtain its nonnegativity.

When f is independent of x, we get the following corollary.

Corollary 3.5. *Assume (H1) and let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function such that (H2) is satisfied. Moreover, let $\tau,\omega > 0$ be two constants with $\tau > \omega$ such that*

$$\eta_1\tau^{1-p}+\frac{\eta_2}{\kappa}\tau^{\kappa-p} < \frac{1}{\vartheta}\cdot\frac{F(\omega)}{\omega^p+\omega^{p_*}}, \tag{3.27}$$

where η_1,η_2, κ and ϑ are given in (H2)(i) and (2.4), respectively. Then, for each

$$\lambda \in \Lambda_2 := \left]\frac{\vartheta}{pc_{p^*}^p|\Omega|^{\frac{p}{N}}}\cdot\frac{\omega^p+\omega^{p_*}}{F(\omega)}, \frac{1}{pc_{p^*}^p|\Omega|^{\frac{p}{N}}}\cdot\frac{1}{\eta_1\tau^{1-p}+\frac{\eta_2}{\kappa}\tau^{\kappa-p}}\right[,$$

problem (1.1) *has at least two nontrivial bounded weak solutions* $u_\lambda, v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ *such that* $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$. *If* f *is in addition nonnegative, then* $u_\lambda, v_\lambda \geq 0$.

Next, we replace (3.27) by a weaker condition on F near zero.

Corollary 3.6. *Assume* (H1) *and let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* (H2) *holds. Moreover, suppose that*

$$\limsup_{t\to 0^+} \frac{F(t)}{t^p} = +\infty. \tag{3.28}$$

Then, for each

$$\lambda \in \Lambda_3 = \left]0, \frac{1}{pc_{p^*}^p |\Omega|^{\frac{p}{N}}} \cdot \sup_{\tau>0} \frac{1}{\eta_1 \tau^{1-p} + \dfrac{\eta_2}{\kappa}\tau^{\kappa-p}}\right[,$$

problem (1.1) *has at least two nontrivial bounded weak solutions* $u_\lambda, v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ *such that* $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$. *If* f *is in addition nonnegative, then* $u_\lambda, v_\lambda \geq 0$.

Proof. Fix $\lambda \in \Lambda_3$, there exists $\tau > 0$ such that

$$\lambda < \frac{1}{pc_{p^*}^p |\Omega|^{\frac{p}{N}}} \cdot \frac{1}{\eta_1 \tau^{1-p} + \dfrac{\eta_2}{\kappa}\tau^{\kappa-p}}.$$

Taking (3.28) into account, we see that

$$\limsup_{t\to 0^+} \frac{F(t)}{t^p + t^{p_*}} = +\infty,$$

which implies the existence of $\omega \in (0, \tau)$ such that

$$\frac{1}{\lambda} < \frac{pc_{p^*}^p |\Omega|^{\frac{p}{N}}}{\vartheta} \cdot \frac{F(\omega)}{\omega^p + \omega^{p_*}}.$$

Thus, we can apply Corollary 3.5 in order to get the assertion. $\square$

Example 3.7. *Let* $p = 2$, $N = 3$ *and* $q = 4$, *then* $1 < p < N$, $p < q < p^* = 6$, $p_* = 4$ *and* $\ell = \max\{q, p_*\} = 4$. *Let* $\Omega = B\left(0, \left(\frac{2}{3}\right)^{\frac{5}{6}}\right) \subset \mathbb{R}^3$. *Then* $\left|B\left(0, \left(\frac{2}{3}\right)^{\frac{5}{6}}\right)\right| = \dfrac{2^{\frac{11}{3}}}{3^{\frac{5}{3}}}\pi$. *We consider the function*

$$f(t) = \begin{cases} (1+\ln 2)\sqrt{t} & \text{for } 0 \leq t < 1, \\ t^3 \ln(1+t) + t^4 & \text{for } t \geq 1. \end{cases}$$

Then we have

$$\begin{aligned} F(t) &= \int_0^t f(s)\,\mathrm{d}s \\ &= \int_0^1 (1+\ln 2)\sqrt{s}\,\mathrm{d}s + \int_0^t \left[s^3\ln(1+s) + s^4\right]\,\mathrm{d}s \\ &= \frac{t^4-1}{4}\ln(1+t) - \frac{t}{48}\left(3t^3 - 4t^2 + 6t - 12\right) + \frac{t^5}{5} + \frac{81}{80} + \frac{2}{3}\ln 2. \end{aligned}$$

For each

$$\lambda \in \left]0, \frac{3^{\frac{49}{12}} \cdot \pi^{\frac{2}{3}}}{2^{\frac{22}{3}}}\right[,$$

problem (1.1) *admits at least two nonnegative, nontrivial bounded weak solutions.*

If fact, taking $\eta_1 = \eta_2 = 2$, $\kappa = 5$ *and* $\zeta = 5$ *we observe that* (H2) *and* (3.28) *hold. Moreover, owing to Talenti* [19], *one has that*

$$c_{p^*} \leq \pi^{-\frac{1}{2}} 3^{-\frac{1}{2}} \left(\frac{\Gamma\left(\dfrac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)} \right)^{\frac{1}{3}} = \frac{2^{\frac{2}{3}}}{3^{\frac{1}{3}} \cdot \pi^{\frac{2}{3}}}.$$

Therefore

$$\frac{1}{p c_{p^*}^p |\Omega|^{\frac{p}{N}}} \sup_{\xi > 0} \frac{1}{\eta_1 \xi^{1-p} + \dfrac{\eta_2}{k} \xi^{-p}} = \frac{3^{\frac{49}{12}} \cdot \pi^{\frac{2}{3}}}{2^{\frac{22}{3}}}.$$

Then, we can apply Corollary 3.6 *and get the required result.*

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