AN INVERSE PROBLEM FOR A DOUBLE PHASE IMPLICIT OBSTACLE PROBLEM WITH MULTIVALUED TERMS

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Abstract. In this paper, we study an inverse problem of estimating three discontinuous parameters in a double phase implicit obstacle problem with multivalued terms and mixed boundary conditions which is formulated by a regularized optimal control problem. Under very general assumptions, we introduce a multivalued function called a parameter-to-solution map which admits weakly compact values. Then, by employing the Aubin-Cellina convergence theorem and the theory of nonsmooth analysis, we prove that the parameter-to-solution map is bounded and continuous in the sense of Kuratowski. Finally, a generalized regularization framework for the inverse problem is developed and a new existence theorem is provided.

1. Introduction

Assume that $\Omega \subset \mathbb{R}^N$ (with $N \geq 2$) is a bounded domain such that its boundary $\Gamma := \partial\Omega$ is Lipschitz which is separated into four mutually disjoint parts Γ_1, Γ_2, Γ_3, and Γ_4. Set

$$\mathrm{BV}(\Omega) := \left\{\zeta \in L^1(\Omega) \,:\, \mathrm{TV}(\zeta) < +\infty\right\},$$

where $\mathrm{TV}\colon D(\mathrm{TV}) \subset L^1(\Omega) \to [0, +\infty)$ is defined by

$$\mathrm{TV}(\zeta) := \sup_{\varphi \in C^1(\Omega;\mathbb{R}^N)} \left\{\int_\Omega \zeta(x) \operatorname{div}\varphi(x)\,\mathrm{d}x \,:\, |\varphi(x)| \leq 1 \text{ for all } x \in \Omega\right\} \text{ for all } \zeta \in \mathrm{BV}(\Omega).$$

Given constants $1 < p < N$, $p < q$, $1 < \delta_1 < p^*$ and $1 < \delta_2 < p_*$ (here p^* and p_* are given in (2.1), see below), three nonempty sets $\Sigma \subset \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, $A \subset L^{\delta_1'}(\Omega)$ and $B \subset L^{\delta_2'}(\Gamma_2)$ with $\frac{1}{\delta_1} + \frac{1}{\delta_1'} = 1$ and $\frac{1}{\delta_2} + \frac{1}{\delta_2'} = 1$, in the present, we are interested in the investigation of the nonlinear inverse problem as follows:

Problem 1.1. *Find $\omega^* \in \Sigma$ and $(\alpha^*, \beta^*) \in A \times B$ such that*

$$\inf_{\omega \in \Sigma \text{ and } (\alpha,\beta) \in A \times B} J(\omega, \alpha, \beta) = J(\omega^*, \alpha^*, \beta^*), \tag{1.1}$$

where the cost functional $J\colon \Sigma \times A \times B \to \mathbb{R}$ is given by

$$J(\omega, \alpha, \beta) := \min_{u \in \Lambda(\omega,\alpha,\beta)} C(u) + \kappa\,\mathrm{TV}(a) + \tau G(\alpha, \beta), \tag{1.2}$$

2020 *Mathematics Subject Classification.* 35J20, 35J25, 35J60, 35R30, 49N45, 65J20.

Key words and phrases. Clarke subdifferential, discontinuous parameter, double phase operator, implicit obstacle problem, inverse problem, optimal control, Steklov eigenvalue problem.

$\Lambda(\omega,\alpha,\beta)$ stands for the solution set in the weak sense of the following double phase implicit obstacle problem with respect to $\omega \in L^\infty(\Omega)\cap \mathrm{BV}(\Omega)$ and $(\alpha,\beta)\in A\times B$:

$$\begin{aligned}
-\operatorname{div}\left(\omega(x)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)+g(x,u)+\mu(x)|u|^{q-2}u &\in U_1(x,u)+\alpha(x) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_\omega}&=\beta(x) && \text{on } \Gamma_2,\\
\frac{\partial u}{\partial \nu_\omega}&\in U_2(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_\omega}&\in \partial_c\phi(x,u) && \text{on } \Gamma_4,\\
L(u)&\le H(u),
\end{aligned}\tag{1.3}$$

where $\mu\colon \overline{\Omega}\to[0,\infty)$ is a bounded function,

$$\frac{\partial u}{\partial \nu_\omega}:=\left(\omega(x)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu,$$

with ν being the outward unit normal vector on Γ, $\partial_c\phi$ is the convex subdifferential operator of convex function $s\mapsto\phi(x,s)$, $\tau>0$ and $\kappa>0$ are two given regularized parameters. Here, nonlinear functions $C\colon W^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$, $G\colon A\times B\to\mathbb{R}$, $L\colon W^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$, $H\colon W^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$, $g\colon\Omega\times\mathbb{R}\to\mathbb{R}$, and set-valued operators $U_1\colon\Omega\times\mathbb{R}\to 2^{\mathbb{R}}$ and $U_2\colon\Gamma_3\times\mathbb{R}\to 2^{\mathbb{R}}$ will be specialized in Section 2.

Under very general assumptions, we introduce a multivalued function called a parameter-to-solution map which admits weakly compact values. Then, by using the Aubin-Cellina convergence theorem and the theory of nonsmooth analysis, we prove that the parameter-to-solution map is bounded and continuous in the sense of Kuratowski. Finally, a generalized regularization framework for the inverse problem is developed and a new existence theorem is provided, see Section 3.

Inverse problems of parameter identification in partial differential equations is an important area in mathematics, motivated by several applications in form of equations and inequalities. In this direction we mention the work of Migórski-Khan-Zeng [34] who treated the inverse problem of mixed quasi-variational inequalities in the general form

$$\langle T(a,u),v-u\rangle+\varphi(v)-\varphi(u)\ge\langle m,v-u\rangle\quad\text{for all } v\in K(u),$$

where $K\colon C\to 2^C$ is a set-valued map, $T\colon B\times V\to V^*$ is a nonlinear map, $\varphi\colon V\to\mathbb{R}\cup\{+\infty\}$ is a functional and $m\in V^*$, while V is a real reflexive Banach space, B is another Banach space and C is a nonempty, closed, convex subset of V. Such results are very useful and can be applied to different kind of problems, for example for p-Laplace equations in terms of hemivariational inequalities, see also [33]. Other results can be found in the papers of Clason-Khan-Sama-Tammer [14] for noncoercive variational problems, Gwinner [20] for variational inequalities of second kind, Gwinner-Jadamba-Khan-Sama [21] for an optimization setting, Migórski-Ochal [35] for nonlinear parabolic problems and Papageorgiou-Vetro [44] for existence and relaxation theorems for different types of problems. For more details on this topics the reader is welcome to consult [1, 2, 3, 4, 6, 9, 22, 23, 27] and the references therein.

The nonlinear and nonhomogeneous differential operator involved in (1.3) is the following weighted double phase operator

$$\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\quad\text{for } u\in W^{1,\mathcal{H}}(\Omega).\tag{1.4}$$

Such differential operator has been studied by Liu-Dai [29] when a takes a positive constant. More particularly, if $a\equiv 1$, then it is not hard to verify that the integral form of (1.4) is defined by

$$\int_\Omega\left(|\nabla u|^p+\mu(x)|\nabla u|^q\right)\mathrm{d}x.\tag{1.5}$$

It should be mentioned that the initial motivation for studying the energy functional (1.5) is to apply the unbalance growth formulation for explaining various natural phenomena and describing models from mechanics, physics and engineering sciences which have strongly anisotropic structures (for instance, anisotropic materials), see Zhikov [52, 53, 54]. After that double phase differential/integral operators (1.4) and (1.5) have been widely applied to study of duality theory and of the Lavrentiev phenomenon in the last years. We refer to the papers of Baroni-Colombo-Mingione [7, 8], Byun-Oh [10], Colombo-Mingione [15, 16], Marcellini [31, 32], Mingione-Rădulescu [37]. and Ragusa-Tachikawa [46], see also the references therein.

Another interesting phenomenon of Problem 1.1 and in particular, of (1.3), is the occurrence of multivalued functions in a very general setting which are motivated by several physical applications. We refer to the books of Facchinei-Pang [18], Hu-Papageorgiou [24, 25], Carl-Le [11], Carl-Le-Motreanu [12] and Panagiotopoulos [38, 39] for model problems in mechanics and physics. For existence results for multivalued problems using different tools we refer to the works of Pang [40], Carl-Le-Winkert [13], Pang-Stewart [41, 42], Iannizzotto-Papageorgiou [26], Liu-Zeng-Motreanu [30], Papageorgiou-Rădulescu-Repovš [43] and Zeng-Bai-Gasiński-Winkert [48, 49, 50].

The remainder of the paper is organized as follows. Section 2 is devoted to recall some preliminary materials which will be used in Section 3 and to impose the detailed assumptions on the data of the double phase implicit obstacle problem (1.3). Section 3 states and proves the main results of our paper, see Theorem 3.1, and points out some special cases, see Corollaries 3.3 and 3.4.

2. Mathematical Preliminaries and Hypotheses

The content of this section is twofold. The first part of this section is to recall some basic definitions and preliminaries which contain Musielak-Orlicz Lebesgue and Musielak-Orlicz Sobolev spaces as well as some necessary results in nonsmooth analysis. In the second part, we will impose the general assumptions on the data of the double phase implicit obstacle problem (1.3) and we recall an existence theorem for weak solutions of problem (1.3).

2.1. **Preliminaries.** Under the assumptions of Section 1, in what follows, for any $r \in [1,\infty)$ and $\emptyset \neq \Pi \subset \overline{\Omega}$, by $L^r(\Pi) := L^r(\Pi;\mathbb{R})$ and $\|\cdot\|_{r,\Pi}$, we denote the usual Lebesgue space and its standard r-norm. Set $L^r(\Pi)_+ := \{u \in L^r(\Pi) : u(x) \geq 0 \text{ for a.a. } x \in \Pi\}$, and define $W^{1,r}(\Omega) := \{u \in L^r(\Omega) \mid \nabla u \in L^r(\Omega;\mathbb{R}^N)\}$ and

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

Let $s > 1$, we denote by $s' > 1$ to satisfy $\frac{1}{s} + \frac{1}{s'} = 1$. Also, we adopt the symbols s^* and s_* to stand for the critical exponents of s in the domain and on the boundary, respectively,

$$s^* = \begin{cases} \frac{Ns}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \geq N, \end{cases} \quad \text{and} \quad s_* = \begin{cases} \frac{(N-1)s}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \geq N. \end{cases} \tag{2.1}$$

Consider the r-Laplacian eigenvalue problem with Steklov boundary condition formulated by

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u \qquad &&\text{in } \Omega, \\ |\nabla u|^{r-2}\nabla u \cdot \nu &= \lambda |u|^{r-2}u \qquad &&\text{on } \Gamma, \end{aligned} \tag{2.2}$$

for $1 < r < \infty$. From Lê [28] we know that (2.2) has a smallest eigenvalue $\lambda_{1,r}^S > 0$ which is isolated and simple. Moreover, it can be written as

$$\lambda_{1,r}^S = \inf_{u \in W^{1,r}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{r,\Omega}^r + \|u\|_{r,\Omega}^r}{\|u\|_{r,\Gamma}^r}. \tag{2.3}$$

Next, we formulate the conditions on the exponents p, q and the weight function μ. In the entire paper we assume the following conditions:

$$1 < p < N, \quad p < q < p^* \quad \text{and} \quad 0 \leq \mu(\cdot) \in L^\infty(\Omega). \tag{2.4}$$

Let us consider the following nonlinear and nonhomogeneous function $\mathcal{H}\colon \Omega \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q \quad \text{for all } (x,t) \in \Omega \times [0,\infty).$$

In the sequel, the Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ described by the nonlinear function $\mathcal{H}$ is defined by

$$L^{\mathcal{H}}(\Omega)=\{u\colon \Omega\to\mathbb{R} \text{ is measurable} \,:\, \rho_{\mathcal{H}}(u)<+\infty\},$$

where the modular function $\rho_{\mathcal{H}}(\cdot)$ is given by

$$\rho_{\mathcal{H}}(u):=\int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x=\int_\Omega \left(|u|^p+\mu(x)|u|^q\right)\mathrm{d}x.$$

The space $L^{\mathcal{H}}(\Omega)$ endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}}=\inf\left\{\tau>0\,:\,\rho_{\mathcal{H}}\left(\frac{u}{\tau}\right)\le 1\right\}$$

becomes a reflexive Banach space since it is uniformly convex. We also need the following seminormed space $L^q_\mu(\Omega)$ driven by the weight function $\mu\colon\Omega\to[0,+\infty)$ given by

$$L^q_\mu(\Omega)=\left\{u\colon\Omega\to\mathbb{R} \text{ is measurable} \,:\, \int_\Omega \mu(x)|u|^q\,\mathrm{d}x<+\infty\right\}$$

endowed with the following seminorm

$$\|u\|_{q,\mu}=\left(\int_\Omega \mu(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

In this paper, we use the function space

$$V:=\left\{u\in W^{1,\mathcal{H}}(\Omega)\,:\, u=0 \text{ on } \Gamma_1\right\},$$

which is a closed subspace V of $W^{1,\mathcal{H}}(\Omega)$, where $W^{1,\mathcal{H}}(\Omega)$ is the Musielak-Orlicz Sobolev space defined by

$$W^{1,\mathcal{H}}(\Omega)=\left\{u\in L^{\mathcal{H}}(\Omega)\,:\,|\nabla u|\in L^{\mathcal{H}}(\Omega)\right\}.$$

It is well-known that $W^{1,\mathcal{H}}(\Omega)$ equipped with the norm

$$\|u\|_{1,\mathcal{H}}=\|\nabla u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}},$$

with $\|\nabla u\|_{\mathcal{H}}=\|\,|\nabla u|\,\|_{\mathcal{H}}$ is a reflexive Banach space. This means that V endowed the norm $\|u\|_V=\|u\|_{1,\mathcal{H}}$ for all $u\in V$ becomes a reflexive Banach space as well.

Let us recall some embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, see Liu-Dai [29] and Gasiński-Winkert [19].

Proposition 2.1. *Assume that (2.4) hold and let p^*, p_* be the critical exponents to p as given in (2.1) for $s=p$. Then we have*

(i) *$L^{\mathcal{H}}(\Omega)\hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)\hookrightarrow W^{1,r}(\Omega)$ are continuous for all $r\in[1,p]$;*
(ii) *$W^{1,\mathcal{H}}(\Omega)\hookrightarrow L^r(\Omega)$ is continuous for all $r\in[1,p^*]$ and compact for all $r\in[1,p^*)$;*
(iii) *$W^{1,\mathcal{H}}(\Omega)\hookrightarrow L^r(\Gamma)$ is continuous for all $r\in[1,p_*]$ and compact for all $r\in[1,p_*)$;*
(iv) *$L^{\mathcal{H}}(\Omega)\hookrightarrow L^q_\mu(\Omega)$ is continuous;*
(v) *$L^q(\Omega)\hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous.*

It is obvious to see that the embeddings (ii) and (iii) of Proposition 2.1 still hold, when we replace the space $W^{1,\mathcal{H}}(\Omega)$ by V.

Besides, we recall the following proposition which reveals the essential relationship between the norm $\|\cdot\|_{\mathcal{H}}$ and the modular function $\rho_{\mathcal{H}}\colon L^{\mathcal{H}}(\Omega)\to[0,+\infty)$. Its detailed proof can be found in Liu-Dai[29] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [17].

Proposition 2.2. *Assume that (2.4) hold. For any $y\in L^{\mathcal{H}}(\Omega)$, we have the following assertions:*

(i) *if $y\neq 0$, then $\|y\|_{\mathcal{H}}=\lambda$ if and only if $\rho_{\mathcal{H}}\left(\frac{y}{\lambda}\right)=1$;*
(ii) *$\|y\|_{\mathcal{H}}<1$ (resp. >1 and $=1$) if and only if $\rho_{\mathcal{H}}(y)<1$ (resp. >1 and $=1$);*
(iii) *if $\|y\|_{\mathcal{H}}<1$, then $\|y\|^q_{\mathcal{H}}\le\rho_{\mathcal{H}}(y)\le\|y\|^p_{\mathcal{H}}$;*
(iv) *if $\|y\|_{\mathcal{H}}>1$, then $\|y\|^p_{\mathcal{H}}\le\rho_{\mathcal{H}}(y)\le\|y\|^q_{\mathcal{H}}$;*

(v) $\|y\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(y) \to 0$;
(vi) $\|y\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(y) \to +\infty$.

Given a function $w\colon \Omega \to (0,+\infty)$ such that

$$\omega \in L^\infty(\Omega) \quad \text{and} \quad \inf_{x\in\Omega} \omega(x) > 0, \tag{2.5}$$

we consider the nonlinear operator $\mathcal{A}\colon V \to V^*$ defined by

$$\begin{aligned} \langle \mathcal{A}(u), v\rangle := & \int_\Omega \left(\omega(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x \\ & + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v \,\mathrm{d}x, \end{aligned} \tag{2.6}$$

for $u, v \in V$ where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between V and its dual space V^*. The following proposition delivers the main properties of the operator $\mathcal{A}\colon V \to V^*$. We refer to Liu-Dai [29, Proposition 3.1] for its proof.

Proposition 2.3. *Assume that* (2.4) *and* (2.5) *hold. Then, the operator* $\mathcal{A}$ *defined by* (2.6) *is bounded (maps bounded sets of* V *into bounded sets of* V^**), continuous, monotone (hence maximal monotone) and of type* (S_+)*, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{and} \quad \limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n - u\rangle \le 0,$$

imply $u_n \to u$ *in* V.

In the sequel, we use the symbols "$\xrightarrow{w}$" and "$\to$" to represent the weak and the strong convergences, respectively, in various spaces. We call a function $j\colon E \to \mathbb{R}$ locally Lipschitz at $x \in E$ if there is a neighborhood $O(x)$ of x and a constant $L_x > 0$ such that

$$|j(y) - j(z)| \le L_x\|y - z\|_E \quad \text{for all } y, z \in O(x).$$

We denote by

$$j^\circ(x; y) := \limsup_{z\to x,\, \lambda\downarrow 0} \frac{j(z+\lambda y) - j(z)}{\lambda},$$

the generalized directional derivative of j at the point x in the direction y and $\partial j\colon E \to 2^{E^*}$ given by

$$\partial j(x) := \{\, \xi \in E^* \,:\, j^\circ(x; y) \ge \langle \xi, y\rangle_{E^*\times E} \ \text{ for all } y \in E\} \quad \text{for all } x \in E$$

is the generalized gradient of j at x in the sense of Clarke.

The next proposition summarizes the main properties of generalized gradients and generalized directional derivatives of a locally Lipschitz function. We refer to Migórski-Ochal-Sofonea [36, Proposition 3.23] for its proof.

Proposition 2.4. *Let* $j\colon E \to \mathbb{R}$ *be locally Lipschitz with Lipschitz constant* $L_x > 0$ *at* $x \in E$*. Then we have the following:*

(i) *The function* $y \mapsto j^\circ(x; y)$ *is positively homogeneous, subadditive, and satisfies*

$$|j^\circ(x; y)| \le L_x\|y\|_E \quad \textit{for all } y \in E;$$

(ii) *The function* $(x, y) \mapsto j^\circ(x; y)$ *is upper semicontinuous;*
(iii) *For each* $x \in E$*,* $\partial j(x)$ *is a nonempty, convex, and weak** *compact subset of* E^* *with* $\|\xi\|_{E^*} \le L_x$ *for all* $\xi \in \partial j(x)$*;*
(iv) $j^\circ(x; y) = \max\{\langle \xi, y\rangle_{E^*\times E} \,|\, \xi \in \partial j(x)\}$ *for all* $y \in E$*;*
(v) *The multivalued function* $E \ni x \mapsto \partial j(x) \subset E^*$ *is upper semicontinuous from* E *into the subsets of* E^* *with weak** *topology.*

Finally, let us recall the definition of Kuratowski limits, see, for example, Papageorgiou-Winkert [45, Definition 6.7.4].

Definition 2.5. *Let (X,τ) be a Hausdorff topological space and let $\{A_n\}_{n\in\mathbb{N}}\subset 2^X$ be a sequence of sets. We define the sequential τ-Kuratowski lower limit of the sets A_n by*

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x\in X \, : \, x=\tau\text{-}\lim_{n\to\infty} x_n,\, x_n\in A_n \text{ for all } \; n\geq 1 \right\},$$

and the sequential τ-Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x\in X \, : \, x=\tau\text{-}\lim_{k\to\infty} x_{n_k},\, x_{n_k}\in A_{n_k},\, n_1<n_2<\ldots<n_k<\ldots \right\}.$$

If

$$A=\tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n,$$

then A is called sequential τ-Kuratowski limit of the sets A_n.

2.2. **Functional framework and existence theorem to problem (1.3).** In order to present an existence theorem for problem (1.3), we first impose the following hypotheses on the data of problem (1.3).

H(g): The function $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is such that the following conditions hold:
- (i) the function $x\mapsto g(x,s)$ is measurable for all $s\in\mathbb{R}$;
- (ii) the function $s\mapsto g(x,s)$ is continuous for a. a. $x\in\Omega$;
- (iii) there exist $a_g>0$ and $b_g\in L^1(\Omega)$ such that
$$g(x,s)s\geq a_g|s|^{\varsigma}-b_g(x),$$
for all $s\in\mathbb{R}$ and for a. a. $x\in\Omega$, where $p<\varsigma<p^*$;
- (iv) for any $u,v\in L^{p^*}(\Omega)$, the function $x\mapsto g(x,u(x))v(x)$ belongs to $L^1(\Omega)$;
- (v) the function $s\mapsto g(x,s)$ is nondecreasing for a. a. $x\in\Omega$, i.e.,
$$(g(x,s_1)-g(x,s_2))(s_1-s_2)\geq 0$$
for all $s_1,s_2\in\mathbb{R}$ and for a. a. $x\in\Omega$.

H(U_1): The multivalued mapping $U_1\colon \Omega\times\mathbb{R}\to 2^{\mathbb{R}}$ is such that the following conditions hold:
- (i) for $s\in\mathbb{R}$ and $x\in\Omega$, the set $U_1(x,s)$ is nonempty, bounded, closed and convex in $\mathbb{R}$;
- (ii) the multivalued mapping $x\mapsto U_1(x,s)$ is measurable in Ω for all $s\in\mathbb{R}$;
- (iii) the multivalued mapping $s\mapsto U_1(x,s)$ is upper semicontinuous for a. a. $x\in\Omega$;
- (iv) there exist $a_{U_1}\in L^{p'}(\Omega)_+$ and $b_{U_1}\geq 0$ such that
$$|\eta|\leq a_{U_1}(x)+b_{U_1}|s|^{p-1},$$
for all $\eta\in U_1(x,s)$, for all $s\in\mathbb{R}$ and for a. a. $x\in\Omega$.

H(U_2): $U_2\colon \Gamma_3\times\mathbb{R}\to 2^{\mathbb{R}}$ satisfies the following conditions:
- (i) $U_2(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. a. $x\in\Gamma_3$ and for all $s\in\mathbb{R}$;
- (ii) $x\mapsto U_2(x,s)$ is measurable on Γ_3 for all $s\in\mathbb{R}$;
- (iii) $s\mapsto U_2(x,s)$ is u.s.c. for a. a. $x\in\Gamma_3$;
- (iv) there exist $a_{U_2}\in L^{p'}(\Gamma_3)_+$ and $b_{U_2}\geq 0$ such that
$$|\xi|\leq a_{U_2}(x)+b_{U_2}|s|^{p-1}$$
for all $\xi\in U_2(x,s)$ for a. a. $x\in\Gamma_3$ and for all $s\in\mathbb{R}$.

H(ϕ): $\phi\colon \Gamma_4\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:
- (i) $x\mapsto\phi(x,r)$ is measurable on Γ_4 for all $r\in\mathbb{R}$ such that $x\mapsto\phi(x,0)$ belongs to $L^1(\Gamma_4)$;
- (ii) for a. a. $x\in\Gamma_4$, $r\mapsto\phi(x,r)$ is convex and l.s.c.;
- (iii) for each function $u\in L^p(\Gamma_4)$ the function $x\mapsto\phi(x,u(x))$ belongs to $L^1(\Gamma_4)$.

H(0): $\omega\in L^\infty(\Omega)$ is such that $\inf_{x\in\Omega}\omega(x)\geq c_\Sigma>0$ and $(\alpha,\beta)\in A\times B$.

H(1): The inequality

$$c_\Sigma - b_{U_2}\left(\lambda_{1,p}^S\right)^{-1} > 0$$

holds, where $\lambda_{1,p}^S$ is the first eigenvalue of the p-Laplacian with Steklov boundary condition, see (2.2) and (2.3).

H(L): $L\colon V \to \mathbb{R}$ is positively homogeneous and subadditive such that

$$L(u) \le \limsup_{n\to\infty} L(u_n),$$

whenever $\{u_n\}_{n\in\mathbb{N}} \subset V$ is such that $u_n \xrightarrow{w} u$ in V for some $u \in V$.

H(H): $H\colon V \to (0,+\infty)$ is weakly continuous, that is, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$ such that $u_n \xrightarrow{w} u$ for some $u \in V$, we have

$$H(u_n) \to H(u).$$

Let us introduce a multivalued mapping $K\colon V \to 2^V$ defined by

$$K(u) := \{v \in V \,:\, L(v) \le H(u)\} \tag{2.7}$$

for all $u \in V$. Under the hypotheses H(L) and H(H), we have the following lemma which was proved by Zeng-Rădulescu-Winkert [51].

Lemma 2.6. *Let $H\colon V \to (0,+\infty)$ and $L\colon V \to \mathbb{R}$ be two functions such that* H(L) *and* H(H) *are satisfied. Then, the following statements hold:*

(i) *for each $u \in V$, $K(u)$ is closed and convex in V such that $0 \in K(u)$;*

(ii) *the graph $\operatorname{Gr}(K)$ of K is sequentially closed in $V_w \times V_w$, that is, K is sequentially closed from V with the weak topology into the subsets of V with the weak topology;*

(iii) *if $\{u_n\}_{n\in\mathbb{N}} \subset V$ is a sequence such that*

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{as} \quad n \to \infty$$

for some $u \in V$, then for each $v \in K(u)$ there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$ such that

$$v_n \in K(u_n) \quad \text{and} \quad v_n \to v \quad \text{in } V \quad \text{as} \quad n \to \infty.$$

Moreover, we state the definition of a solution in the weak sense to problem (1.3) as follows.

Definition 2.7. *A function $u \in V$ is said to be a weak solution of problem (1.3), if there exist functions $\eta \in L^1(\Omega)$ and $\xi \in L^1(\Gamma_3)$ with $\eta(x) \in U_1(x,u(x))$ for a. a. $x \in \Omega$ and $\xi(x) \in U_2(x,u(x))$ for a. a. $x \in \Gamma_3$ such that $u \in K(u)$ and the inequality*

$$\begin{aligned}
&\int_\Omega \left(\omega(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x + \int_\Omega g(x,u)(v-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \mu(x)|u|^{q-2}u(v-u)\,\mathrm{d}x + \int_{\Gamma_4}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_4}\phi(x,u)\,\mathrm{d}\Gamma\\
&\ge \int_\Omega (\eta(x)+\alpha(x))(v-u)\,\mathrm{d}x + \int_{\Gamma_2}\beta(x)(v-u)\,\mathrm{d}\Gamma + \int_{\Gamma_3}\xi(x)(v-u)\,\mathrm{d}\Gamma
\end{aligned}$$

is satisfied for all $v \in K(u)$, where the multivalued mapping K is defined by (2.7).

We end this section to deliver a generalized existence theorem of weak solutions to problem (1.3) which can be proved via applying the same arguments as in the proof of Theorem 3.9 of Zeng-Rădulescu-Winkert [51].

Theorem 2.8. *Assume that (2.4),* H(U_1), H(g), H(ϕ), H(U_2), H(L), H(H), H(0) *and* H(1) *are fulfilled. Then, the solution set of problem (1.3) corresponding to $(\omega,\alpha,\beta) \in \Sigma \times A \times B$, denoted by $\Lambda(\omega,\alpha,\beta)$, is nonempty and weakly compact in V.*

3. Main results

This section is devoted to develop a generalized framework to study the inverse problem given in Problem 1.1. More precisely, we are going to establish a generalized theorem to identify the discontinuous parameters $\omega \in \Sigma$ and $\alpha \in A$ in the domain as well as the discontinuous boundary datum $\beta \in B$ for the double phase implicit elliptic obstacle problem given in (1.3).

We suppose the following assumptions to Problem 1.1.

H(2): $\Sigma \subset \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, $A \subset L^{\delta_1'}(\Omega)$ and $B \subset L^{\delta_2'}(\Gamma_2)$ are nonempty, closed and convex sets such that

$$\Sigma := \{\omega \in \mathrm{BV}(\Omega) \,:\, 0 < c_\Sigma \leq \omega(x) \leq d_\Sigma \text{ for a. a. } x \in \Omega\}$$

for some $0 < c_\Sigma \leq d_\Sigma$.

H(C): $C \colon V \to \mathbb{R}$ is a weakly l.s.c. and bounded from below function, i.e.,

$$\liminf_{n\to\infty} C(u_n) \geq C(u) \quad \text{and} \quad C(v) \geq M_C \quad \text{for all } v \in V,$$

whenever $\{u_n\}_{n\in\mathbb{N}} \subset V$ and $u \in V$ are such that $u_n \xrightarrow{w} u$ in V for some $M_C \in \mathbb{R}$.

H(G): $G \colon A \times B \to \mathbb{R}$ is bounded from below such that

(i) G is coercive, that is,

$$G(\alpha,\beta) \to +\infty \quad \text{as } \|\alpha\|_{\delta_1',\Omega} + \|\beta\|_{\delta_2',\Gamma_2} \to +\infty;$$

(ii) G is weakly lower semicontinuous.

The existence theorem to the regularized optimal control problem given in Problem 1.1 is stated as follows

Theorem 3.1. *Assume that all conditions of Theorem 2.8 are satisfied. If, in addition,* H(2)*,* H(C) *and* H(G) *are satisfied, then the solution set of Problem 1.1 is nonempty and weakly compact.*

Proof. The proof of this theorem is divided into five steps.

Step 1: The functional J defined in (1.2) is well-defined.

It is sufficient to show that for each fixed $(\omega,\alpha,\beta) \in \Sigma \times A \times B$, there is at least one function $u^* \in \Lambda(\omega,\alpha,\beta)$ such that the following equality holds

$$\inf_{u\in\Lambda(\omega,\alpha,\beta)} C(u) = C(u^*). \tag{3.1}$$

Recall that C is bounded from below. Thus there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset \Lambda(\omega,\alpha,\beta)$ to the optimal problem $\inf_{u\in\Lambda(\omega,\alpha,\beta)} C(u)$, that is,

$$\inf_{u\in\Lambda(\omega,\alpha,\beta)} C(u) = \lim_{n\to\infty} C(u_n).$$

Keeping in mind that $\Lambda(\omega,\alpha,\beta)$ is weakly compact (see Theorem 2.8), we are able to select a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in \Lambda(\omega,\alpha,\beta)$. From the weakly lower semicontinuity of C (see hypothesis $H(C)$) it follows that

$$\inf_{u\in\Lambda(\omega,\alpha,\beta)} C(u) = \liminf_{n\to\infty} C(u_n) \geq C(u^*) \geq \inf_{u\in\Lambda(\omega,\alpha,\beta)} C(u).$$

This indicates that for every $(\omega,\alpha,\beta) \in \Sigma \times A \times B$ there exists a function $u^* \in \Lambda(\omega,\alpha,\beta)$ such that equality (3.1) is valid. Hence J is well-defined.

Step 2: Λ maps bounded sets of $\Sigma \times A \times B$ to bounded sets of V.

Let $(\omega,\alpha,\beta) \in \Sigma \times A \times B$ and $u \in \Lambda(\omega,\alpha,\beta)$ be arbitrary. From hypotheses H(g), H(ϕ), H(U_1), H(U_2), the definition of Σ and the fact that $0 \in K(u)$, we have

$$\begin{aligned}
0 \geq &\int_\Omega \omega(x)|\nabla u|^p + \mu(x)|\nabla u|^q \,\mathrm{d}x + \int_\Omega g(x,u)u \,\mathrm{d}x + \int_\Omega \mu(x)|u|^q \,\mathrm{d}x - \int_{\Gamma_4} \phi(x,0)\,\mathrm{d}\Gamma \\
&+ \int_{\Gamma_4} \phi(x,u)\,\mathrm{d}\Gamma - \int_\Omega (\eta(x)+\alpha(x))u\,\mathrm{d}x - \int_{\Gamma_2} \beta(x)u\,\mathrm{d}\Gamma - \int_{\Gamma_3} \xi(x)u\,\mathrm{d}\Gamma
\end{aligned}$$

$$
\begin{aligned}
&\geq c_\Lambda \|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{q,\mu}^q + \int_\Omega a_g |u|^\varsigma - b_g(x)\,\mathrm{d}x - \int_\Omega a_{U_1}(x)|u| + b_{U_2}|u|^p\,\mathrm{d}\Gamma\\
&\quad - \int_{\Gamma_3} a_{U_2}(x)|u| + b_{U_2}|u|^p\,\mathrm{d}\Gamma - \int_{\Gamma_4}\phi(x,0)\,\mathrm{d}\Gamma - c_\phi\|u\|_V - d_\phi - m_0\left(\|\alpha\|_{\delta_1',\Omega} + \|\beta\|_{\delta_2',\Gamma_2}\right)\|u\|_V\\
&\geq \left(c_\Lambda - b_{U_2}\left(\lambda_{1,p}^S\right)^{-1}\right)\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + a_g\|u\|_{\varsigma,\Omega}^\varsigma + \|u\|_{q,\mu}^q - \|b_g\|_{1,\Omega} - \int_{\Gamma_4}\phi(x,0)\,\mathrm{d}\Gamma\\
&\quad - c_\phi\|u\|_V - d_0\left(\|c_{U_1}\|_{p',\Omega} + \|c_{U_2}\|_{p',\Gamma_3}\right)\|u\|_V - \left(b_{U_1} + b_{U_2}\left(\lambda_{1,p}^S\right)^{-1}\right)\|u\|_{p,\Omega}^p - d_\phi\\
&\quad - m_0\left(\|\alpha\|_{\delta_1',\Omega} + \|\beta\|_{\delta_2',\Gamma_2}\right)\|u\|_V\\
&\geq \hat{M}_0\left(\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu}^q\right) + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma - \|b_g\|_{1,\Omega} - m_1 - \int_{\Gamma_4}\phi(x,0)\,\mathrm{d}\Gamma\\
&\quad - d_0\left(\|c_{U_1}\|_{p',\Omega} + \|c_{U_2}\|_{p',\Gamma_3}\right)\|u\|_V - c_\phi\|u\|_V - d_\phi - m_0\left(\|\alpha\|_{\delta_1',\Omega} + \|\beta\|_{\delta_2',\Gamma_2}\right)\|u\|_V\\
&= \hat{M}_0\left(\rho_{\mathcal{H}}(\nabla u) + \rho_{\mathcal{H}}(u)\right) + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma - \|b_g\|_{1,\Omega} - d_0\left(\|c_{U_1}\|_{p',\Omega} + \|c_{U_2}\|_{p',\Gamma_3}\right)\|u\|_V - m_1\\
&\quad - \int_{\Gamma_4}\phi(x,0)\,\mathrm{d}\Gamma - c_\phi\|u\|_V - d_\phi - m_0\left(\|\alpha\|_{\delta_1',\Omega} + \|\beta\|_{\delta_2',\Gamma_2}\right)\|u\|_V\\
&\geq \hat{M}_0\min\left\{\|u\|_V^p, \|u\|_V^q\right\} + \frac{a_g}{2}\|u\|_{\varsigma,\Omega}^\varsigma - \|b_g\|_{1,\Omega} - d_0\left(\|c_{U_1}\|_{p',\Omega} + \|c_{U_2}\|_{p',\Gamma_3}\right)\|u\|_V - m_1\\
&\quad - \int_{\Gamma_4}\phi(x,0)\,\mathrm{d}\Gamma - c_\phi\|u\|_V - d_\phi - m_0\left(\|\alpha\|_{\delta_1',\Omega} + \|\beta\|_{\delta_2',\Gamma_2}\right)\|u\|_V
\end{aligned}
$$

for some $m_0, m_1, d_0 > 0$, where we have used Young's inequality, Proposition 2.2, the convexity of

$$
V \ni u \mapsto \int_{\Gamma_4}\phi(x,u)\,\mathrm{d}\Gamma \in \mathbb{R},
$$

and $c_\phi, d_\phi, \hat{M}_0 \geq 0$ are such that

$$
\hat{M}_0 := \min\left\{c_\Lambda - b_{U_2}\left(\lambda_{1,p}^S\right)^{-1}, 1\right\} \quad \text{and} \quad \int_{\Gamma_4}\phi(x,v)\,\mathrm{d}\Gamma \geq -c_\phi\|v\|_V - d_\phi \quad \text{for all } v \in V.
$$

From the estimates above and $1 < p$, it is not difficult to see that Λ maps bounded sets of $\Sigma \times A \times B$ to bounded sets of V.

Step 3: If $\{(\omega_n, \alpha_n, \beta_n)\}_{n\in\mathbb{N}} \subset \Sigma \times A \times B$ is a sequence such that $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $\omega_n \to \omega$ in $L^1(\Omega)$, $\alpha_n \xrightarrow{w} \alpha$ in A and $\beta_n \xrightarrow{w} \beta$ in B for some $(\omega, \alpha, \beta) \in L^1(\Omega) \times A \times B$, then we have $\omega \in \Sigma$ and

$$
\emptyset \neq w\text{-}\limsup_{n\to\infty}\Lambda(\omega_n, \alpha_n, \beta_n) = s\text{-}\limsup_{n\to\infty}\Lambda(\omega_n, \alpha_n, \beta_n) \subset \Lambda(\omega, \alpha, \beta). \tag{3.2}
$$

From the definition of Σ we directly have that $\omega \in \Sigma$. Making use of Step 2, we conclude that $\bigcup_{n\geq 1}\Lambda(\omega_n, \alpha_n, \beta_n)$ is bounded in V. The latter combined with the reflexivity of V implies that the set $w\text{-}\limsup_{n\to\infty}\Lambda(\omega_n, \alpha_n, \beta_n)$ is nonempty. Let $u \in w\text{-}\limsup_{n\to\infty}\Lambda(\omega_n, \alpha_n, \beta_n)$ be arbitrary. Passing to a subsequence if necessary, we are able to find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$ such that

$$
u_n \in \Lambda(\omega_n, \alpha_n, \beta_n) \quad \text{and} \quad u_n \xrightarrow{w} u \quad \text{in } V.
$$

Therefore, for every $n \in \mathbb{N}$, there exist functions $\eta_n \in L^{\delta_1'}(\Omega)$ and $\xi_n \in L^{\delta_2'}(\Gamma_2)$ such that

$$
\begin{aligned}
&\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(v-u_n)\,\mathrm{d}x + \int_\Omega \mu(x)|u_n|^{q-2}u_n(v-u_n)\,\mathrm{d}x\\
&\quad + \int_\Omega g(x,u_n)(v-u_n)\,\mathrm{d}x + \int_{\Gamma_4}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_4}\phi(x,u_n)\,\mathrm{d}\Gamma\\
&\geq \int_\Omega (\eta_n(x) + \alpha_n(x))(v-u_n)\,\mathrm{d}x + \int_{\Gamma_2}\beta_n(x)(v-u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_3}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma
\end{aligned} \tag{3.3}
$$

for all $v \in K(u_n)$. By the weak closedness of the graph of K (see Lemma 2.6), $u_n \in K(u_n)$ and the convergence $u_n \xrightarrow{w} u$ in V, we get $u \in K(u)$. Applying again Lemma 2.6, we are able to find a sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$ such that $v_n \in K(u_n)$ for each $n \in \mathbb{N}$ and $v_n \to u$ in V. Inserting $v = u_n$ in (3.3) yields

$$\begin{aligned}
&\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(v_n-u_n)\,\mathrm{d}x + \int_\Omega \mu(x)|u_n|^{q-2}u_n(v_n-u_n)\,\mathrm{d}x\\
&\quad + \int_\Omega g(x,u_n)(v_n-u_n)\,\mathrm{d}x + \int_{\Gamma_4}\phi(x,v_n)\,\mathrm{d}\Gamma - \int_{\Gamma_4}\phi(x,u_n)\,\mathrm{d}\Gamma\\
&\geq \int_\Omega (\eta_n(x)+\alpha_n(x))(v_n-u_n)\,\mathrm{d}x + \int_{\Gamma_2}\beta_n(x)(v_n-u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_3}\xi_n(x)(v_n-u_n)\,\mathrm{d}\Gamma.
\end{aligned} \tag{3.4}$$

From hypotheses H(U_1)(iv) and H(U_2)(iv), it is not difficult to prove that the sequences $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are bounded in $L^{p'}(\Omega)$ and $L^{p'}(\Gamma_3)$, respectively. Keeping in mind that the embedding of V to $L^\gamma(\Omega)$ is compact for all $1<\gamma<p^*$ and the convexity, lower semicontinuity of $V \ni u \mapsto \int_\Omega \phi(x,u)\,\mathrm{d}\Gamma \in \mathbb{R}$ (see hypotheses H(ϕ), i.e., $V \ni u \mapsto \int_{\Gamma_4}\phi(x,u)\,\mathrm{d}\Gamma \in \mathbb{R}$ is continuous and weakly l.s.c.), we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \mu(x)|u_n|^{q-2}u_n(u_n-v_n)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_\Omega \left(g(x,u_n)-|u_n|^{p-2}u_n\right)(v_n-u_n)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_\Omega (\eta_n(x)+\alpha_n(x))(v_n-u_n)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_{\Gamma_2}\beta_n(x)(v_n-u_n)\,\mathrm{d}\Gamma = 0,\\
&\lim_{n\to\infty}\int_{\Gamma_3}\xi_n(x)(v_n-u_n)\,\mathrm{d}\Gamma = 0,\\
&\liminf_{n\to\infty}\int_{\Gamma_4}\phi(x,v_n)-\phi(x,u_n)\,\mathrm{d}\Gamma \leq \int_{\Gamma_4}\phi(x,u)-\phi(x,u)\,\mathrm{d}\Gamma = 0,
\end{aligned} \tag{3.5}$$

where we have also used the compactness of $V \hookrightarrow L^\gamma(\Gamma)$ for all $1<\gamma<p_*$. Passing to the upper limit as $n\to\infty$ and using (3.5) gives

$$\begin{aligned}
\limsup_{n\to\infty}\bigg(&\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&+\int_\Omega \left(|u_n|^{p-2}u_n+\mu(x)|u_n|^{q-2}u_n\right)(u_n-v_n)\,\mathrm{d}x\bigg) \leq 0.
\end{aligned} \tag{3.6}$$

Next, we are going to prove that $\nabla u_n \to \nabla u$ in $L^p(\Omega;\mathbb{R}^N)$. Because $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are both bounded in V. So, without loss of generality, we may assume that there exists a constant $m_2>0$ such that

$$\left(\|\nabla u_n\|_{\Omega,p}^p + \|\nabla v_n\|_{p,\Omega}^p\right)^{-\frac{2-p}{p}} \geq m_2. \tag{3.7}$$

Indeed, if there exists a subsequence of $\{\|\nabla u_n\|_{\Omega,p}^p+\|\nabla v_n\|_{p,\Omega}^p\}_{n\in\mathbb{N}}$, not relabeled, such that it converges to 0, then from the convergences $v_n \to u$, $u_n \xrightarrow{w} u$ in V and the continuity of the embedding of V to $W^{1,p}(\Omega)$ we obtain $\nabla u_n \to \nabla u = 0$ in $L^p(\Omega;\mathbb{R}^N)$. In this case we would be done. So, we can assume that (3.7) is fulfilled. By Simon [47, formula (2.2)] we have the well-known inequalities

$$M_p|\xi-\eta|^p \leq \left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\cdot(\xi-\eta), \quad \text{if } p\geq 2, \tag{3.8}$$

$$\mathcal{M}_p|\xi-\eta|^2 \leq \left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\cdot(\xi-\eta)\left(|\xi|^p+|\eta|^p\right)^{\frac{2-p}{p}}, \quad \text{if } 1\leq p<2, \tag{3.9}$$

for all $\xi,\,\eta\in\mathbb{R}^N$ with some constants $M_p,\,\mathcal{M}_p>0$ independent of $\xi,\,\eta\in\mathbb{R}^N$.

Let us distinguish the following cases: $1<p<2$ and $p\geq 2$. Assuming $p\geq 2$, it follows from (3.8) that

$$\int_\Omega \omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\geq c_\Lambda M_p\|\nabla(u_n-v_n)\|_{p,\Omega}^p.$$

If $1<p<2$, let us consider the sets

$$\begin{aligned}\Omega_n&=\{x\in\Omega\,:\,\nabla u_n\neq 0\}\cup\{x\in\Omega\,:\,\nabla v_n\neq 0\},\\ \Sigma_n&=\{x\in\Omega\,:\,\nabla v_n=\nabla u_n=0\}.\end{aligned}$$

Then we have $\Omega=\Omega_n\cup\Sigma_n$ and $\Omega_n\cap\Sigma_n=\emptyset$. Invoking the absolute continuity of the Lebesgue integral we get

$$\int_{\Sigma_n}\omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x=0.$$

This means that

$$\begin{aligned}&\int_\Omega \omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\ &=\int_{\Omega_n}\omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\ &\quad+\int_{\Sigma_n}\omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\ &=\int_{\Omega_n}\omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x.\end{aligned}$$

Making use of (3.9) implies

$$\begin{aligned}&\int_\Omega \omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\ &=\int_{\Omega_n}\omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\frac{\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{2-p}{p}}}{\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{2-p}{p}}}\,\mathrm{d}x\\ &\geq\mathcal{M}_p\int_{\Omega_n}\omega_n(x)\left|\nabla u_n-\nabla u\right|^2\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{p-2}{p}}\,\mathrm{d}x\\ &\geq c_\Lambda\mathcal{M}_p\int_{\Omega_n}\left|\nabla u_n-\nabla v_n\right|^2\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{p-2}{p}}\,\mathrm{d}x.\end{aligned}\tag{3.10}$$

Recall that $1<p<2$, so $\frac{2}{p}>1$. Using this and Hölder's inequality we obtain

$$\begin{aligned}&\int_\Omega\left|\nabla u_n-\nabla v_n\right|^p\,\mathrm{d}x=\int_\Omega\left|\nabla u_n-\nabla v_n\right|^{2\cdot\frac{p}{2}}\,\mathrm{d}x\\ &=\int_\Omega\left(\left|\nabla u_n-\nabla v_n\right|^2\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{p-2}{p}}\right)^{\frac{p}{2}}\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{2-p}{2}}\,\mathrm{d}x\\ &\leq\left(\int_\Omega\left|\nabla u_n-\nabla v_n\right|^2\left(|\nabla u_n|^p+|\nabla v_n|^p\right)^{\frac{p-2}{p}}\,\mathrm{d}x\right)^{\frac{p}{2}}\times\left(\int_\Omega\left(|\nabla u_n|^p+|\nabla v_n|^p\right)\,\mathrm{d}x\right)^{\frac{2-p}{2}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_\Omega\left|\nabla u_n-\nabla v_n\right|^2\left(|\nabla u_n|^p+|\nabla u_n|^p\right)^{\frac{p-2}{p}}\,\mathrm{d}x\\ &\geq\left(\int_\Omega\left|\nabla u_n-\nabla v_n\right|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\left(\int_\Omega\left(|\nabla u_n|^p+|\nabla v_n|^p\right)\,\mathrm{d}x\right)^{-\frac{2-p}{p}}.\end{aligned}$$

Inserting the inequality above into (3.10) gives

$$\begin{aligned}
&\int_\Omega \omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&\geq c_\Lambda \mathcal{M}_p\left(\int_\Omega |\nabla u_n-\nabla v_n|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\left(\int_\Omega \left(|\nabla u_n|^p+|\nabla v_n|^p\right)\,\mathrm{d}x\right)^{-\frac{2-p}{p}}\\
&\geq c_\Lambda \mathcal{M}_p m_2\left(\int_\Omega |\nabla u_n-\nabla v_n|^p\,\mathrm{d}x\right)^{\frac{2}{p}}.
\end{aligned}\tag{3.11}$$

From Hölder's inequality it follows that

$$\begin{aligned}
&\int_\Omega \left((\omega_n(x)-\omega(x))|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&\geq -\int_\Omega |\omega_n(x)-\omega(x)||\nabla v_n|^{p-1}|\nabla(u_n-v_n)|\,\mathrm{d}x\\
&= -\int_\Omega |\omega_n(x)-\omega(x)|^{\frac{p-1}{p}}|\nabla v_n|^{p-1}|\omega_n(x)-\omega(x)|^{\frac{1}{p}}|\nabla(u_n-v_n)|\,\mathrm{d}x\\
&\geq -\left(\int_\Omega |\omega_n(x)-\omega(x)||\nabla v_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_\Omega |\omega_n(x)-\omega(x)||\nabla(u_n-v_n)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\geq -(2c_\Lambda)^{\frac{1}{p}}\|\nabla(u_n-v_n)\|_{p,\Omega}\left(\int_\Omega |\omega_n(x)-\omega(x)||\nabla v_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Note that $\omega_n \to \omega$ in $L^1(\Omega)$ and $v_n \to u$ in V. Without loss of generality, we may suppose that $\omega_n(x) \to \omega(x)$ and $\nabla v_n(x) \to \nabla u(x)$ for a. a. $x \in \Omega$. Since $\{u_n - v_n\}_{n\in\mathbb{N}}$ is bounded in V, we pass to the upper limit as $n \to \infty$ for the inequality above and utilize Lebesgue's dominated convergence theorem to find

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left((\omega_n(x)-\omega(x))|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&\geq \lim_{n\to\infty}\left[-(2c_\Lambda)^{\frac{1}{p}}\|\nabla(u_n-v_n)\|_{p,\Omega}\left(\int_\Omega |\omega_n(x)-\omega(x)||\nabla v_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}\right]\\
&=0.
\end{aligned}\tag{3.12}$$

Note that

$$\begin{aligned}
&\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&=\int_\Omega \omega_n(x)\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&\quad+\int_\Omega \left((\omega_n(x)-\omega(x))|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x+\int_\Omega \left(\omega(x)|\nabla v_n|^{p-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&\quad+\int_\Omega \left(\mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x.
\end{aligned}$$

Hence, we can take $\limsup_{n\to\infty}$ in (3.4) and apply (3.5), (3.11), (3.12) as well as

$$\lim_{n\to\infty}\int_\Omega \left(\omega(x)|\nabla v_n|^{p-2}\nabla v_n+\mu(x)|\nabla v_n|^{q-2}\nabla v_n\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x=0,$$

$$\int_\Omega \left(\mu(x)\left(|\nabla u_n|^{q-2}\nabla u_n-|\nabla v_n|^{q-2}\nabla v_n\right)\right)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\geq 0,$$

in order to find that

$$\limsup_{n\to\infty}\|\nabla(u_n-u)\|_{p,\Omega}^p\leq 0 \quad \text{if } p\geq 2,$$

$$\limsup_{n\to\infty} \|\nabla(u-u_n)\|_{p,\Omega}^2 \le 0 \quad \text{ if } 1<p<2.$$

This implies that $\nabla u_n \to \nabla u$ in $L^p(\Omega;\mathbb{R}^N)$.

Moreover, we shall verify that $u_n \to u$ in V. From the convergence $\nabla u_n \to \nabla u$ and $\nabla v_n \to \nabla u$ in $L^p(\Omega;\mathbb{R}^N)$ and $\omega_n \to \omega$ in $L^1(\Omega)$, we apply Lebesgue dominate convergence theorem to get

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left(\omega(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u-v_n)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u-v_n)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_\Omega |\omega(x)-\omega_n(x)||\nabla u_n|^{p-1}|\nabla(u-v_n)|\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_\Omega |\omega_n(x)-\omega(x)||\nabla u_n|^{p-1}|\nabla(u-u_n)|\,\mathrm{d}x = 0.
\end{aligned}$$

The convergences above together with (3.6) implies

$$\begin{aligned}
0 &\ge \limsup_{n\to\infty}\left(\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\right.\\
&\quad\left.+\int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x\right)\\
&\quad+\liminf_{n\to\infty}\left(\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u-v_n)\,\mathrm{d}x\right.\\
&\quad\left.+\int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u-v_n)\,\mathrm{d}x\right)\\
&\ge \limsup_{n\to\infty}\left(\int_\Omega \left(\omega(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\right.\\
&\quad\left.+\int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x\right)\\
&\quad+\lim_{n\to\infty}\left(\int_\Omega \left(\omega(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u-v_n)\,\mathrm{d}x\right.\\
&\quad\left.+\int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u-v_n)\,\mathrm{d}x\right)\\
&\quad-\lim_{n\to\infty}\int_\Omega |\omega(x)-\omega_n(x)||\nabla u_n|^{p-1}|\nabla(u-v_n)|\,\mathrm{d}x\\
&\quad-\lim_{n\to\infty}\int_\Omega |\omega_n(x)-\omega(x)||\nabla u_n|^{p-1}|\nabla(u-u_n)|\,\mathrm{d}x\\
&= \limsup_{n\to\infty}\left(\int_\Omega \left(\omega(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\right.\\
&\quad\left.+\int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x\right).
\end{aligned}$$

Therefore, from the (S_+)-property of $\mathcal{A}$ (see Proposition 2.3), we conclude that $u_n \to u$ in V, that is,

$$s\text{–}\limsup_{n\to\infty}\Lambda(\omega_n,\alpha_n,\beta_n) \neq \emptyset.$$

This means that

$$\emptyset \neq w\text{–}\limsup_{n\to\infty}\Lambda(\omega_n,\alpha_n,\beta_n) = s\text{–}\limsup_{n\to\infty}\Lambda(\omega_n,\alpha_n,\beta_n),$$

due to

$$s\text{–}\limsup_{n\to\infty} \Lambda(\omega_n,\alpha_n,\beta_n) \subset w\text{–}\limsup_{n\to\infty} \Lambda(\omega_n,\alpha_n,\beta_n).$$

Furthermore, we will show that u is a solution of problem (1.3) corresponding to $\omega \in \Sigma$ and $(\alpha,\beta) \in A\times B$, i.e., $u \in \Lambda(\omega,\alpha,\beta)$. Assumptions H$(U_1)$(iv) and H$(U_2)$(iv) guarantee that $\{\eta_n\}_{n\in\mathbb{N}} \subset L^{p'}(\Omega)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset L^{p'}(\Gamma_3)$ are bounded. Employing the reflexivity of $L^{p'}(\Omega)$ and $L^{p'}(\Gamma_3)$, we are able to find functions $\eta \in L^{p'}(\Omega)$ and $\xi \in L^{p'}(\Gamma_3)$ satisfying, by passing to a subsequence if necessary,

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{p'}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{p'}(\Gamma_3).$$

On the other hand, by virtue of hypotheses H(U_1) and H(U_2) and Aubin-Cellina convergence theorem (see e.g. Aubin-Cellina [5, Theorem 1, p.60]), we can prove that

$$\eta(x) \in U_1(x,u(x)) \quad \text{for a. a. } x\in\Omega \quad \text{and} \quad \xi(x) \in U_2(x,u(x)) \quad \text{for a. a. } x \in \Gamma_3.$$

For any fixed $v \in K(u)$, we use Lemma 2.6 to infer that there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset V$ such that $z_n \in K(u_n)$ for each $n\in\mathbb{N}$ and $z_n \to v$ in V. Applying Lebesgue's dominated convergence theorem gives

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(z_n-u_n)\,\mathrm{d}x\\
&=\int_\Omega \lim_{n\to\infty}\left(\omega_n(x)|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(z_n-u_n)\,\mathrm{d}x\\
&=\int_\Omega \left(\omega(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x.
\end{aligned}$$

Putting $v = z_n$ into (3.3), passing to the upper limit as $n\to\infty$ for the resulting inequality (3.3) and applying the convergence properties above, we obtain

$$\begin{aligned}
&\int_\Omega \left(\omega(x)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x + \int_\Omega g(x,u)(v-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \mu(x)|u|^{q-2}u(v-u)\,\mathrm{d}x + \int_{\Gamma_4}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_4}\phi(x,u)\,\mathrm{d}\Gamma\\
&\geq \int_\Omega (\eta(x)+\alpha(x))(v-u)\,\mathrm{d}x + \int_{\Gamma_2}\beta(x)(v-u)\,\mathrm{d}\Gamma + \int_{\Gamma_3}\xi(x)(v-u)\,\mathrm{d}\Gamma
\end{aligned}$$

for all $v \in K(u)$. Therefore, we can observe that $u \in K(u)$ is a solution of problem (1.3) corresponding to $(\omega,\alpha,\beta) \in \Sigma\times A\times B$, that is, $u \in \Lambda(\omega,\alpha,\beta)$. Hence $w\text{–}\limsup_{n\to\infty}\Lambda(\omega_n,\alpha_n,\beta_n) \subset \Lambda(\omega,\alpha,\beta)$ and so we have proved (3.2).

Step 4: If $\{(\omega_n,\alpha_n,\beta_n)\}_{n\in\mathbb{N}} \subset \Sigma\times A\times B$ is such that $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $\omega_n\to\omega$ in $L^1(\Omega)$, $\alpha_n \xrightarrow{w} \alpha$ in $L^{\delta_1'}(\Omega)$ and $\beta_n \xrightarrow{w} \beta$ in $L^{\delta_2'}(\Gamma_2)$ for some $(\omega,\alpha,\beta) \in L^1(\Omega)\times A\times B$, then the inequality

$$J(\omega,\alpha,\beta) \leq \liminf_{n\to\infty} J(\omega_n,\alpha_n,\beta) \tag{3.13}$$

holds.

Let $\{(\omega_n,\alpha_n,\beta_n)\}_{n\in\mathbb{N}} \subset \Sigma\times A\times B$ be such that $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $\omega_n\to\omega$ in $L^1(\Omega)$, $\alpha_n \xrightarrow{w} \alpha$ in $L^{\delta_1'}(\Omega)$ and $\beta_n \xrightarrow{w} \beta$ in $L^{\delta_2'}(\Gamma_2)$ for some $(\omega,\alpha,\beta) \in L^1(\Omega)\times A\times B$. From Step 3 one has $\omega\in\Lambda$. Let $\{u_n\}_{n\in\mathbb{N}} \subset V$ be a sequence satisfying

$$u_n \in \Lambda(\omega_n,\alpha_n,\beta_n) \quad \text{and} \quad \inf_{u\in\Lambda(\omega_n,\alpha_n,\beta_n)} C(u) = C(u_n) \tag{3.14}$$

for each $n\in\mathbb{N}$.

Keeping in mind that $\bigcup_{n\geq 1}\Lambda(\omega_n,\alpha_n,\beta_n)$ is bounded (see Step 2), without loss of any generality, we may suppose that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in V$. Then, from Step 3, we have that $u_n \to u^*$ in V and $u^* \in s\text{–}\limsup_{n\to\infty}\Lambda(\omega_n,\alpha_n,\beta_n) \subset \Lambda(\omega,\alpha,\beta)$. Whereas, from the lower semicontinuity of the function $L^1(\Omega) \ni \omega \mapsto \mathrm{TV}(\omega) \in \mathbb{R}$, the continuity of $V \ni u \mapsto C(u) \in \mathbb{R}$ and the weakly lower semicontinuity of $A \times B \ni (\alpha, \beta) \mapsto G(\alpha, \beta) \in \mathbb{R}$, we get that

$$\begin{aligned}\liminf_{n\to\infty} J(\omega_n, \alpha_n, \beta_n) &= \liminf_{n\to\infty} \left[C(u_n) + \kappa\,\mathrm{TV}(\omega_n) + G(\alpha_n, \beta_n)\right]\\ &\geq \liminf_{n\to\infty} C(u_n) + \liminf_{n\to\infty} \kappa\,\mathrm{TV}(\omega_n) + \liminf_{n\to\infty} G(\alpha_n, \beta_n)\\ &\geq C(u^*) + \kappa\,\mathrm{TV}(\omega) + G(\alpha, \beta)\\ &\geq \inf_{u\in\Lambda(\omega,\alpha,\beta)} C(u) + \kappa\,\mathrm{TV}(\omega) + G(\alpha, \beta)\\ &= J(\omega, \alpha, \beta).\end{aligned}$$

Hence (3.13) follows.

Step 5: The solution set of Problem 1.1 is nonempty and weakly compact.

From the formulation of J and hypotheses H(C) and H(G) we can observe that the cost functional J is bounded from below. Therefore, there exists a minimizing sequence $\{(\omega_n, \alpha_n, \beta_n)\}_{n\in\mathbb{N}} \subset \Sigma \times A \times B$ of (1.1) such that

$$\inf_{\omega\in\Lambda \text{ and } (\alpha,\beta)\in A\times B} J(\omega, \alpha, \beta) = \lim_{n\to\infty} J(\omega_n, \alpha_n, \beta_n). \tag{3.15}$$

By the definition of J we easily see that the sequences $\{\omega_n\}_{n\in\mathbb{N}} \subset \Sigma$ and $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}} \subset A \times B$ are bounded in $\mathrm{BV}(\Omega)$ and $L^{\delta_1'}(\Omega) \times L^{\delta_2'}(\Gamma_2)$, respectively. Passing to a subsequence if necessary we have

$$\omega_n \to \omega^* \quad \text{in } L^1(\Omega), \quad \alpha_n \xrightarrow{w} \alpha^* \quad \text{in } L^{\delta_1'}(\Omega) \quad \text{and} \quad \beta_n \xrightarrow{w} \beta^* \quad \text{in } L^{\delta_2'}(\Gamma_2) \tag{3.16}$$

for some $(\omega^*, \alpha^*, \beta^*) \in \Sigma \times A \times B$, where we have used the closedness of Σ in $L^1(\Omega)$ and the compactness of the embedding $\mathrm{BV}(\Omega)$ to $L^1(\Omega)$.

Let us consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$ satisfying (3.14). Employing the convergence (3.16) and the boundedness of Λ (see Step 2), we get the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}}$ in V. So, we are able to select a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in V$. From Step 3 it is clear that $u^* \in \Lambda(\omega^*, \alpha^*, \beta^*)$. Therefore, we have

$$\begin{aligned}\liminf_{n\to\infty} J(\omega_n, \alpha_n, \beta_n) &= \liminf_{n\to\infty} \left[C(u_n) + \kappa\ \mathrm{TV}(\omega_n) + G(\alpha_n, \beta_n)\right]\\ &\geq \liminf_{n\to\infty} C(u_n) + \kappa \liminf_{n\to\infty} \mathrm{TV}(\omega_n) + \liminf_{n\to\infty} G(\alpha_n, \beta_n)\\ &\geq C(u^*) + \kappa\,\mathrm{TV}(\omega^*) + G(\alpha^*, \beta^*)\\ &\geq \inf_{u\in\Lambda(\omega^*,\alpha^*,\beta^*)} C(u) + \kappa\,\mathrm{TV}(\omega^*) + G(\alpha^*, \beta^*)\\ &= J(\omega^*, \alpha^*, \beta^*)\\ &\geq \inf_{\omega\in\Sigma \text{ and } (\alpha,\beta)\in A\times B} J(\omega, \alpha, \beta).\end{aligned} \tag{3.17}$$

The latter combined with (3.15) implies that $(\omega^*, \alpha^*, \beta^*) \in \Sigma \times A \times B$ is a solution of Problem 1.1.

It remains us to verify that the solution set to Problem 1.1 is weakly compact. For any solution sequence $\{(\omega_n, \alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ of Problem 1.1, we can observe that $\{\omega_n\}_{n\in\mathbb{N}} \subset \Sigma$ is bounded in $\mathrm{BV}(\Omega)$ and $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\delta_1'}(\Omega) \times L^{\delta_2}(\Gamma_2)$, respectively. Arguing as in the proof of existence part, it is possible to suppose that (3.16) holds with some $(\omega^*, \alpha^*, \beta^*) \in \Sigma \times A \times B$. Analogously, we are able to find a sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying (3.14) and $u_n \to u^*$ in V for some $u^* \in \Lambda(\omega^*, \alpha^*, \beta^*)$. Therefore, we have (3.17). This means that $(\omega^*, \alpha^*, \beta^*) \in \Sigma \times A \times B$ is a solution of Problem 1.1, namely, the solution set of Problem 1.1 is weakly compact. □

Let $r_1 : \mathbb{R} \to \mathbb{R}$, $r_2 : \mathbb{R} \to \mathbb{R}$, $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ be functions that satisfy the following conditions:

H(j_1): The functions $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $r_1 : \mathbb{R} \to \mathbb{R}$ are such that the following conditions hold:
- (i) $x \mapsto j_1(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$ with $x \mapsto j_1(x, 0)$ belonging to $L^1(\Omega)$;
- (ii) $s \mapsto j_1(x, s)$ is locally Lipschitz continuous for a. a. $x \in \Omega$ and the function $r_1 : \mathbb{R} \to \mathbb{R}$ is continuous;

(iii) there exist a function $\alpha_{j_1} \in L^{p'}(\Omega)_+$ and a constant $a_{j_1} \geq 0$ such that

$$|r_1(s)\eta| \leq \alpha_{j_1}(x) + a_{j_1}|s|^{p-1}$$

for all $\eta \in \partial j_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

H(j_2): The functions $j_2 \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ and $r_2 \colon \mathbb{R} \to \mathbb{R}$ are such that the following conditions hold:

(i) $x \mapsto j_2(x,s)$ is measurable on Γ_3 for all $s \in \mathbb{R}$ with $x \mapsto j_2(x,0)$ belonging to $L^1(\Gamma_3)$;

(ii) $s \mapsto j_2(x,s)$ is locally Lipschitz continuous for a. a. $x \in \Gamma_3$ and the function $r_2 \colon \mathbb{R} \to \mathbb{R}$ is continuous;

(iii) there exist a function $\alpha_{j_2} \in L^{p'}(\Gamma_3)_+$ and a constant $a_{j_2} \geq 0$ such that

$$|r_2(s)\xi| \leq \alpha_{j_2}(x) + a_{j_2}|s|^{p-1}$$

for all $\xi \in \partial j_2(x,s)$, for a. a. $x \in \Gamma_3$ and for all $s \in \mathbb{R}$.

We have the following lemma.

Lemma 3.2. *Assume that* H(j_1) *and* H(j_2) *are satisfied. Then, the multivalued mappings* $U_1 \colon \Omega\times\mathbb{R} \to 2^{\mathbb{R}}$ *and* $U_2 \colon \Gamma_3 \times \mathbb{R} \to 2^{\mathbb{R}}$ *defined by*

$$U_1(x,s_1) = r_1(s_1)\partial j_1(x,s_1) \quad \textit{and} \quad U_2(y,s_2) = r_2(s_2)\partial j_2(y,s_2)$$

for a. a. $x \in \Omega$*, for all* $y \in \Gamma_3$ *and for all* $s_1, s_2 \in \mathbb{R}$*, satisfy hypotheses* H(U_1) *and* H(U_2)*, respectively, where* $\partial j_1(x,s)$ *(resp.* $\partial j_2(x,s)$*) is the generalized Clarke subdifferential of* $s \mapsto j_1(x,s)$ *(resp.* $s \mapsto j_2(x,s)$*).*

Proof. Via Proposition 2.4, we can see that for a. a. $x \in \Omega$ (resp. for a. a. $x \in \Gamma_3$) and for all $s \in \mathbb{R}$ the set $U_1(x,s)$ (resp. $U_2(x,s)$) is nonempty, bounded, closed and convex in $\mathbb{R}$. This means that H(U_1)(i) (resp. H(U_2)(i)) has been verified. However, hypotheses H(j_1)(i) and H(j_2)(i) reveal that for all $s \in \mathbb{R}$, functions $x \mapsto U_1(x,s) = r_1(s)\ \partial j_1(x,s)$ and $x \mapsto U_2(x,s) = r_2(s)\partial j_2(x,s)$ are measurable in Ω and on Γ_3, respectively. Therefore, H(U_1)(ii) and H(U_2)(ii) are available.

Moreover, we assert that $s \mapsto r_1(s)\partial j_1(x,s)$ is u.s.c. Invoking Proposition 3.8 of Migórski-Ochal-Sofonea [36], we are sufficiently to prove that for each closed set $D \subset \mathbb{R}$ the set $(r_1(\cdot)\partial j_1(x,\cdot))^-(D)$ is closed in $\mathbb{R}$. We take a sequence $\{s_n\}_{n\in\mathbb{N}} \subset (r_1(\cdot)\partial j_1(x,\cdot))^-(D)$ satisfying $s_n \to s$. So, there is a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\eta_n \in r_1(s_n)\partial j_1(x,s_n) \cap D$ for each $n \in \mathbb{N}$. It is obvious there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ having the properties $\eta_n = r_1(s_n)\xi_n$ and $\xi_n \in \partial j_1(x,s_n)$ for all $n \in \mathbb{N}$ and for a. a. $x \in \Omega$. Because of $s_n \to s$, it is now in a position to apply Proposition 2.4(iii) and (v) to find that $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Without loss of generality, it holds $\xi_n \to \xi$ in $\mathbb{R}$ for some $\xi \in D$, because of the closedness of D. Whereas, Proposition 2.4(v) indicates that $\xi \in \partial j_1(x,s)$. The latter together with the continuity of r_1 implies $\eta_n = r_1(s_n)\xi_n \to r_1(s)\xi \in r_1(s)\partial j_1(x,s)$. This means that $s \in (r_1(\cdot)\partial j_1(x,\cdot))^-(D)$, namely, $(r_1(\cdot)\partial j_1(x,\cdot))^-(D)$ is closed. Furthermore, we utilize Proposition 3.8 of Migórski-Ochal-Sofonea [36] to obtain that $s \mapsto r_1(s)\partial j_1(x,s)$ is u.s.c. As before we have done, it is valid as well that $s \mapsto r_2(s)\partial j_2(x,s)$ is u.s.c. Consequently, H(U_1)(iii) and H(U_2)(iii) hold.

Finally, using hypotheses H(j_1)(iii) and H(j_2)(iii), we observe that H(U_1)(iv) and H(U_2)(iv) hold with $r = \delta = p$, $a_{U_1} = \alpha_{j_1}$, $a_{U_2} = \alpha_{j_2}$, $b_{U_1} = a_{j_1}$ and $b_{U_2} = a_{j_2}$. □

It is obvious that when U_1 and U_2 are presented by the functions

$$U_1(x,s_1) = r_1(s_1)\partial j_1(x,s_1) \quad \text{and} \quad U_2(y,s_2) = r_2(s_2)\partial j_2(y,s_2)$$

for a. a. $x \in \Omega$, for a. a. $y \in \Gamma_3$ and for all $s_1, s_2 \in \mathbb{R}$, then problem (1.3) reduces to the following double phase implicit obstacle problem with generalized Clarke subdifferentials:

$$\begin{aligned}
\begin{array}{r} -\operatorname{div}\left(\omega(x)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right) \\ +g(x,u)+\mu(x)|u|^{q-2}u \end{array} &\in r_1(u)\partial j_1(x,u)+\alpha(x) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_\omega} &= \beta(x) && \text{on } \Gamma_2,\\
\frac{\partial u}{\partial \nu_\omega} &\in r_2(u)\partial j_2(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_\omega} &\in \partial_c\phi(x,u) && \text{on } \Gamma_4,\\
L(u) &\leq H(u).
\end{aligned} \tag{3.18}$$

Employing Lemma 3.2 and Theorems 2.8 and 3.1, we have the following corollaries.

Corollary 3.3. *Assume that* (2.4), $\mathrm{H}(j_1)$, $\mathrm{H}(g)$, $\mathrm{H}(\phi)$, $\mathrm{H}(j_2)$, $\mathrm{H}(L)$, $\mathrm{H}(H)$, $\mathrm{H}(0)$ *and the inequality* $c_\Sigma - a_{j_2}\left(\lambda_{1,p}^S\right)^{-1} > 0$ *are fulfilled. Then, the solution set of problem* (3.18) *corresponding to* $(\omega,\alpha,\beta) \in \Sigma \times A \times B$*, denoted by* $\Lambda(\omega,\alpha,\beta)$*, is nonempty and weakly compact in* V*.*

Corollary 3.4. *Assume that all conditions of Corollary 3.3 are satisfied. If, in addition,* $\mathrm{H}(2)$, $\mathrm{H}(C)$ *and* $\mathrm{H}(G)$ *are satisfied, then the solution set of the following problem is nonempty and weakly compact: find* $\omega^* \in \Sigma$ *and* $(\alpha^*,\beta^*) \in A \times B$ *such that*

$$\inf_{\omega\in\Sigma \text{ and } (\alpha,\beta)\in A\times B} J(\omega,\alpha,\beta) = J(\omega^*,\alpha^*,\beta^*),$$

where the cost functional $J\colon \Sigma \times A \times B \to \mathbb{R}$ *is defined in (1.2) and* $\Lambda(\omega,\alpha,\beta)$ *is the solution set in the weak sense of problem (3.18) with respect to* $\omega \in L^\infty(\Omega)\cap \mathrm{BV}(\Omega)$ *and* $(\alpha,\beta) \in A \times B$*.*

Remark 3.5. *In our framework, the functions* C *and* G *have many possibilities. For example,*

$$C(u) = \|\nabla u - z\|_{p,\Omega}^{\zeta_1} \quad \text{and} \quad G(\alpha,\beta) = \|\alpha\|_{\delta_1',\Omega}^{\zeta_2} + \|\beta\|_{\delta_2',\Gamma_2}^{\zeta_3} \tag{3.19}$$

for all $u \in V$ *and* $(\alpha,\beta) \in A \times B$*, where* $z \in L^p(\Omega;\mathbb{R}^N)$ *is the known observed or measured datum and* $\zeta_1,\zeta_2,\zeta_3 \geq 1$*.*

Acknowledgments. This project has received funding from the Natural Science Foundation of Guangxi Grant Nos. 2021GXNSFFA196004, 2022AC21071, and GKAD21220144, NNSF of China Grant Nos. 12001478 and 12101143, the China Postdoctoral Science Foundation funded project No. 2022M-721560, the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Preludium Project No. 2017/25/N/ST1/00611, and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It is also supported by the Ministry of Science and Higher Education of Republic of Poland under Grant No. 440328/PnH2/2019, and the National Science Centre of Poland under Project No. 2021/41/B/ST1/01636. The research of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI–UEFISCDI, project number PCE 137/2021, within PNCDI III.

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