EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR A 1-BIHARMONIC CHOQUARD EQUATION WITH STEEP POTENTIAL WELL IN $\mathbb{R}^N$

HUO TAO, LIN LI, AND PATRICK WINKERT

Abstract. In this paper, we investigate the existence and concentration of solutions for the following 1-biharmonic Choquard equation with steep potential well

$$\begin{cases} \Delta_1^2 - \Delta_1 u + (1+\lambda V(x))\frac{u}{|u|} = (I_\mu * F(u))\, f(u) \quad \text{in } \mathbb{R}^N, \\ u \in \mathrm{BL}(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $\lambda > 0$ is a positive parameter, $V\colon \mathbb{R}^N \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ are continuous functions verifying further conditions, $\Omega = \operatorname{int}(V^{-1}(\{0\}))$ has nonempty interior and $I_\mu\colon \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\mu \in (N-1, N)$. For $\lambda > 0$ large enough we prove the existence of a nontrivial solution u_λ of the problem above via variational methods and the concentration behavior of u_λ which is explored on the set Ω.

1. Introduction

In this work, we consider the existence and concentration of solutions to the following quasilinear elliptic problems with steep potential well

$$\begin{cases} \Delta_1^2 - \Delta_1 u + (1+\lambda V(x))\frac{u}{|u|} = (I_\mu * F(u))\, f(u) \quad \text{in } \mathbb{R}^N, \\ u \in \mathrm{BL}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\lambda > 0$ is a positive parameter, the 1-Laplacian operator is defined as

$$\Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right),$$

and the 1-biharmonic operator is given by

$$\Delta_1^2 u = \Delta\left(\frac{\Delta u}{|\Delta u|}\right).$$

The nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ and the potential $V\colon \mathbb{R}^N \to \mathbb{R}$ satisfy the following assumptions:

(f_1) $f\colon \mathbb{R} \to \mathbb{R}$ is continuous;

(f_2) $\lim\limits_{|s|\to 0} f(s) = 0$;

(f_3) There exist constants $\sigma > 0$ and $1 < q_1 \leq q_2 < \frac{\mu}{N-1}$ such that

$$|f(s)| \leq \sigma(|s|^{q_1-1} + |s|^{q_2-1}) \quad \text{for all } s \in \mathbb{R};$$

2020 *Mathematics Subject Classification.* 35A15, 35J15, 35J60, 35J62.

Key words and phrases. 1-biharmonic operator, Choquard equation, concentration behavior, ground state solution, Nehari manifold.

(f_4) There exists $\kappa \in (1, +\infty)$ such that
$$0 < \kappa F(s) \leq f(s)s, \qquad \text{for } s \neq 0,$$
where $F(s) = \int_0^s f(t)\,\mathrm{d}t$;

(f_5) f is increasing.

(V_1) $V \in C(\mathbb{R}^N)$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;

(V_2) There exists $M_0 > 0$ such that the Lebesgue measure $|\{x \in \mathbb{R}^N : V(x) \leq M_0\}| < +\infty$;

(V_3) $\Omega = \operatorname{int}(V^{-1}(\{0\}))$ is nonempty with smooth boundary and $\overline{\Omega} = V^{-1}(\{0\})$.

Moreover, $I_\mu\colon \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\mu \in (N-1, N)$ on the Euclidean space $\mathbb{R}^N$ of dimension $N \geq 3$, defined for each $x \in \mathbb{R}^N \setminus \{0\}$ by
$$I_\mu(x) = \frac{\Gamma\left(\frac{N-\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)\pi^{\frac{N}{2}} 2^\mu |x|^{N-\mu}},$$
where $\Gamma(\cdot)$ stands for a standard Gamma function. The Choquard equation was introduced by Choquard in 1976 in the modeling of a one-component plasma, see Lieb-Loss [25]. It seems to originate from Fröhlich's and Pekar's model of the polaron, which is a quasiparticle used in condensed matter physics to understand the interactions between electrons and atoms in a solid material, see Fröhlich [19] and Hajaiej [20]. For the study of this equation, we refer, for example, to the papers of Alves-Nóbrega-Yang [3], Alves-Yang [5], Lee-Kim-Bae-Park [23], Liang-Zhang [24], Yang-Tang-Gu [32] and the references therein.

Quasilinear elliptic equations are nonlinear generalizations of linear elliptic partial differential equations. It is well known that linear elliptic equations represent models of various physical problems, such as Laplace and Poisson equation. That is why they have been studied for more than two hundred years and still attract researchers even today. As a branch or evolution of variational calculus, variational methods are almost entirely related to nonlinearity. The earliest origin of variational methods was in the Euler era, and the great development in modern times originated from the pioneering work of Ambrosetti and Rabinowitz in the 1970s. The emergence of modern variational tools such as the mountain path theorem and the symmetric mountain path theorem injected new vitality into ancient variational methods. The variational method has achieved rich results in the existence and multiplicity of solutions for nonlinear elliptic equations or systems. We recommend readers to refer to the works of Anthal-Giacomoni-Sreenadh [6], Bai-Papageorgiou-Zeng [8], Cen-Khan-Motreanu-Zeng [13], Papageorgiou-Rădulescu-Repovš [26], Rădulescu-Repovš [30], Rădulescu-Vetro [31], Zeng-Migorski-Khan [33] and the references therein.

The 1-biharmonic problem is studied in the space of functions $\mathrm{BL}(\Omega)$ with $|\Omega| < +\infty$ or $\mathrm{BL}(\mathbb{R}^N)$. Unlike the usual Sobolev spaces, the space BL is neither reflexive nor uniformly convex and the associated energy functional lacks smoothness. This is the reason why it is so difficult to prove that functionals defined on this space satisfy compactness properties like the Palais-Smale condition and we have to use the critical point theory of nonsmooth functionals. Clearly, the 1-biharmonic problem can also be seen as the limit of the p-biharmonic ones, as the parameter $p \to 1^+$. It is worth noting that the critical exponent for the 1-biharmonic operator is $1^* = \frac{N}{N-1}$ instead of $\frac{N}{N-2}$.

In [27], Parini-Ruf-Tarsi first studied this kind of operator and dealt with the related eigenvalue problem. The authors proved that

$$\Lambda_{1,1}(\Omega) = \inf_{u\in \mathrm{BL}_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\|u\|_1}$$

is attained by a non-negative and superharmonic function v that belongs to the space

$$\mathrm{BL}_0(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) \,:\, \Delta u \in \mathcal{M}(\Omega) \right\},$$

where $\mathcal{M}(\Omega)$ is the space of the Radon measures defined on Ω and $\int_\Omega |\Delta u|$ is defined in (2.1). In fact, their results are more general since they also provide information about the shape of the domain Ω that maximizes $\Lambda_{1,1}(\Omega)$. In [29], the same authors considered the following minimization problem

$$\Lambda_{1,1}^c(\Omega) = \inf_{u\in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\|u\|_1}.$$

and studied the shape of the subset that maximizes the quantity $\Lambda_{1,1}^c(\Omega)$. Furthermore, in Parini-Ruf-Tarsi [28], some optimal constants of Sobolev embeddings in certain function spaces related to the 1-biharmonic operator are proved. In [9], Barile-Pimenta obtained existence results of bounded variation solutions to the following quasilinear fourth-order problem

$$\begin{cases} \Delta_1^2 u = f(x,u) & \text{in } \Omega, \\ u = \frac{\Delta u}{|\Delta u|} = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, Hurtado-Pimenta-Miyagaki [21] proved some compactness results of the $\mathrm{BL}(\mathbb{R}^N)$ of radially symmetric functions and the existence of the ground state solution for the quasilinear elliptic problem

$$\begin{cases} \Delta_1^2 - \Delta_1 u + \frac{u}{|u|} = f(u) & \text{in } \mathbb{R}^N, \\ u \in \mathrm{BL}(\mathbb{R}^N). \end{cases}$$

Moreover, Bartsch, Pankow and Wang studied such a situation for the first time and proved the existence of solutions of a nonlinear Schrödinger equation with steep potential well for λ large enough, see the papers in [10, 11, 12]. In recent years, elliptic equations with steep potential well have attracted much attention. We also refer to the works of Alves-Figueiredo-Pimenta [2], Alves-Nóbrega-Yang [3], Ding-Tanaka [16], Jia-Luo [22] for the subcritical case and Alves-de Morais Filho-Souto [1], Alves-Souto [4], Costa [15], Zhang-Lou [34] for the critical case, see also the references therein.

Motivated by the aforementioned works, in this paper, we consider the 1-biharmonic Choquard problem with the steep potential well. The main results in our paper are the following ones.

Theorem 1.1. *Suppose that assumptions* (f_1)–(f_5) *and* (V_1)–(V_3) *hold. Then there exists* $\lambda^* > 0$ *such that for each* $\lambda \geq \lambda^*$*, problem* (1.1) *has a nontrivial ground state solution* u_λ.

Theorem 1.2. *Suppose that assumptions* (f_1)–(f_5) *and* (V_1)–(V_3) *hold. If* u_λ *is a nontrivial solution obtained by Theorem* 1.1*, then there exists* $u_\Omega \in \mathrm{BL}(\mathbb{R}^N)$ *such* *that, if* $\lambda_n \to +\infty$, *then, up to a subsequence not relabeled,* $u_{\lambda_n} \to u_\Omega$ *in* $L^q_{\text{loc}}(\mathbb{R}^N)$ *for* $1 \leq q < 1^*$ *and*

$$\|u_n\|_{\lambda_n} - \|u_\Omega\|_\Omega \to 0 \quad \text{as } n \to +\infty,$$

where $\|\cdot\|_\lambda$ *and* $\|\cdot\|_\Omega$ *are defined in* (2.4) *and* (4.2). *Furthermore,* $u_\Omega \equiv 0$ *a.e. in* $\mathbb{R}^N \setminus \Omega$ *and* u_Ω *is a solution of*

$$\begin{cases} \Delta_1^2 - \Delta_1 u + \frac{u}{|u|} = (I_\mu * F(u))\, f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This paper is organized as follows. In Section 2 we give a detailed description of the variational framework and the properties of the related function space defined by the energy functional. In Section 3 we give the proof of Theorem 1.1, studying separately the arguments on the existence of solutions for λ large enough. Finally, in Section 4, we prove Theorem 1.2, studying the arguments on the concentration of solutions for $\lambda \to +\infty$.

2. Preliminaries

In this section we recall the basic notions and preliminaries to the underlying function space of problem (1.1). This space is defined by

$$\mathrm{BL}(\mathbb{R}^N) := \left\{ u \in W^{1,1}(\mathbb{R}^N) \,:\, \Delta u \in \mathcal{M}(\mathbb{R}^N) \right\},$$

where $\mathcal{M}(\mathbb{R}^N)$ is the set of all Radon measures on $\mathbb{R}^N$. Parini-Ruf-Tarsi [27] proved that $u \in W^{1,1}(\mathbb{R}^N)$ belongs to $\mathrm{BL}(\mathbb{R}^N)$ if and only if

$$\int_{\mathbb{R}^N} |\Delta u| < +\infty,$$

where

$$\int_{\mathbb{R}^N} |\Delta u| := \sup \left\{ \int_{\mathbb{R}^N} u \Delta \varphi \,\mathrm{d}x \,:\, \varphi \in C_0^\infty(\mathbb{R}^N), \|\varphi\|_\infty \leqslant 1 \right\}. \tag{2.1}$$

The space $\mathrm{BL}(\mathbb{R}^N)$ is a Banach space when endowed with the following norm

$$\|u\| = \int_{\mathbb{R}^N} |\Delta u| + \|\nabla u\|_1 + \|u\|_1,$$

which is continuously embedded into $L^r(\mathbb{R}^N)$ for all $r \in [1, 1^*]$, see Hurtado-Pimenta-Miyagaki [21].

Moreover, the space of smooth functions is not dense in $\mathrm{BL}(\mathbb{R}^N)$ with respect to the topology of the norm. However, it is with respect to the topology induced by the following notion of convergence. This has motivated people to define a weaker sense of convergence in $\mathrm{BL}(\mathbb{R}^N)$. We say that a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BL}(\mathbb{R}^N)$ converges to $u \in \mathrm{BL}(\mathbb{R}^N)$ in the sense of the strict convergence if both of the following conditions are satisfied

$$u_n \to u \quad \text{in } W^{1,1}(\mathbb{R}^N),$$

and

$$\int_{\mathbb{R}^N} |\Delta u_n| \to \int_{\mathbb{R}^N} |\Delta u|,$$

as $n \to +\infty$. In fact, with respect to the strict convergence, $C^\infty(\mathbb{R}^N) \cap \mathrm{BL}(\mathbb{R}^N)$ is dense in $\mathrm{BL}(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathrm{BL}(\mathbb{R}^N)$.

For a vectorial Radon measure $\mu \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual decomposition stated in the Radon-Nikodym Theorem, where μ^a and μ^s are, respectively, the absolute continuous and the singular parts with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$. With $|\mu|$ as the scalar Radon measure, the usual Lebesgue-Radon-Nikodym derivative of μ with respect to $|\mu|$ is given by

$$\frac{\mu}{|\mu|}(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}.$$

It is easy to see that $\mathcal{J}\colon \mathrm{BL}(\mathbb{R}^N) \to \mathbb{R}$, given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| \,\mathrm{d}x + \int_{\mathbb{R}^N} |u| \,\mathrm{d}x$$

is a convex functional which is Lipschitz continuous in its domain and lower semicontinuous with respect to the $W^{1,r}(\mathbb{R}^N)$ topology, for $r \in [1, 1^*]$. Meanwhile, $\mathcal{J}$ is lower semicontinuous with respect to the $L^r(\mathbb{R}^N)$-topology for $r \in [1, 1^*)$, see Hurtado-Pimenta-Miyagaki [21]. Although nonsmooth, the functional $\mathcal{J}$ admits some directional derivatives. More precisely, as is shown by Anzellotti in [7], given $u \in \mathrm{BL}(\mathbb{R}^N)$, for all $v \in \mathrm{BL}(\mathbb{R}^N)$ such that $(\Delta v)^s$ is absolutely continuous with respect to $(\Delta u)^s$, $(\Delta v)^a$ vanishes $\mathcal{L}^N$-a.e. in $\{x \in \mathbb{R}^N : (\Delta u)^a(x) = 0\}$, ∇v vanishes a.e. in the set where ∇u vanishes and $v \equiv 0$, a.e. in the set where u vanishes, it follows that

$$\begin{aligned}\mathcal{J}'(u)v = &\int_{\mathbb{R}^N} \frac{(\Delta u)^a (\Delta v)^a}{|(\Delta u)^a|} \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{\Delta u}{|\Delta u|}(x) \frac{\Delta v}{|\Delta v|}(x)\, |(\Delta v)^s| \\ &+ \int_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{|\nabla u|} \,\mathrm{d}x + \int_{\mathbb{R}^N} \mathrm{sgn}(u) v \,\mathrm{d}x,\end{aligned} \tag{2.2}$$

where $\mathrm{sgn}(u(x)) = 0$ if $u(x) = 0$ and $\mathrm{sgn}(u(x)) = u(x)/|u(x)|$ if $u(x) \neq 0$. In particular, taking (2.2) into account, for all $u \in \mathrm{BL}(\mathbb{R}^N)$, we have

$$\mathcal{J}'(u)u = \mathcal{J}(u). \tag{2.3}$$

Now let X_λ be the subspace of $\mathrm{BL}(\mathbb{R}^N)$ given by

$$X_\lambda = \left\{ u \in \mathrm{BL}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} (1 + \lambda V(x))|u| \,\mathrm{d}x < +\infty \right\}$$

endowed with the norm

$$\|u\|_\lambda = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| \,\mathrm{d}x + \int_{\mathbb{R}^N} (1 + \lambda V(x))|u| \,\mathrm{d}x. \tag{2.4}$$

Note that the embedding $X_\lambda \hookrightarrow \mathrm{BL}(\mathbb{R}^N)$ is continuous in such a way that X_λ is a Banach space that is continuously embedded into $L^r(\mathbb{R}^N)$ for $r \in [1, 1^*]$.

Let us present the energy functional associated with problem (1.1). Let $\Phi_\lambda\colon X_\lambda \to \mathbb{R}$ be given by

$$\Phi_\lambda(u) = \mathcal{J}_\lambda(u) - \mathcal{F}(u), \tag{2.5}$$

where $\mathcal{J}_\lambda = \|u\|_\lambda$ and $\mathcal{F}\colon X_\lambda \to \mathbb{R}$ is defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} (I_\mu * F(u)) F(u) \,\mathrm{d}x.$$

Concerned with the nonlocal type problems with Riesz potential, we need the following well-known Hardy-Littlewood-Sobolev inequality, see Lieb-Loss [25].

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality)**.** *Let $s, r > 1$ and $0 < \alpha < N$ with $1/s + (N-\mu)/N + 1/r = 2$. Let $g \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(s, N, \mu, r)$, independent of g and h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y \leq C(s,N,\mu,r)\|g\|_{L^s(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}.$$

Remark 2.2. *In particular, $F(v) = |v|^{q_1}$ for some $q_1 > 0$. By the Hardy-Littlewood-Sobolev inequality,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{N-\mu}}\,\mathrm{d}y\,\mathrm{d}x$$

is well defined if $F(u) \in L^s(\mathbb{R}^N)$ for $s > 1$ which satisfies

$$s = r \quad \text{and} \quad \frac{2}{s} + \frac{N-\mu}{N} = 2.$$

Since $u \in \mathrm{BL}(\mathbb{R}^N)$, we require that $sq_1 \in [1, 1^*]$. For the subcritical case, we have to assume that

$$\frac{1}{2}\left(2 - \frac{N-\mu}{N}\right) < q_1 \leq q_2 < \frac{1^*}{2}\left(2 - \frac{N-\mu}{N}\right).$$

In our paper, we are assuming a stronger condition on q_1, q_2, and μ, because we intend to study the concentration of the solutions.

Then it is easy to check that $\mathcal{J}_\lambda$ is a convex functional which is Lipschitz continuous in its domain and $\mathcal{F} \in C^1(X_\lambda, \mathbb{R})$. Similar to (2.3), we have

$$\begin{aligned}\mathcal{J}_\lambda'(u)v = &\int_{\mathbb{R}^N} \frac{(\Delta u)^a(\Delta v)^a}{|(\Delta u)^a|}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{\Delta u}{|\Delta u|}(x)\frac{\Delta v}{|\Delta v|}(x)\,|(\Delta v)^s| \\ &+ \int_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{|\nabla u|}\,\mathrm{d}x + \int_{\mathbb{R}^N} (1+\lambda V(x))\,\mathrm{sgn}(u)v\,\mathrm{d}x.\end{aligned} \tag{2.6}$$

In particular, note that, for all $u \in X_\lambda$, $\mathcal{J}_\lambda'(u)u = \mathcal{J}_\lambda(u)$. Moreover, taking $v = u$ in (2.6), it follows that

$$\begin{aligned}\Phi_\lambda'(u)u &= \mathcal{J}_\lambda'(u)u - \int_{\mathbb{R}^N} (I_\mu * F(u))f(u)u\,\mathrm{d}x \\ &= \|u\|_\lambda - \int_{\mathbb{R}^N} (I_\mu * F(u))f(u)u\,\mathrm{d}x.\end{aligned}$$

Let us give a precise definition of the solution we are considering. Since Φ_λ can be written as the difference between the Lipschitz functional $\mathcal{J}_\lambda$ and a smooth functional $\mathcal{F}$, we say that $u_\lambda \in X_\lambda$ is a solution of (1.1) if $0 \in \partial\Phi_\lambda(u_\lambda)$, where $\partial\Phi_\lambda(u_\lambda)$ denotes the subdifferential of Φ_λ in u_λ, as defined, for example, in Chang [14]. This in turn is equivalent to $\mathcal{F}'(u_\lambda) \in \partial\mathcal{J}_\lambda(u_\lambda)$. However, since the convexity of $\mathcal{J}_\lambda$, it implies that $\mathcal{F}'(u_\lambda) \in \partial\mathcal{J}_\lambda(u_\lambda)$ if and only if

$$\mathcal{J}_\lambda(v) - \mathcal{J}_\lambda(u_\lambda) \geq \mathcal{F}'(u_\lambda)(v - u_\lambda) \quad \text{for all } v \in X_\lambda,$$

or equivalently

$$\|v\|_\lambda - \|u_\lambda\|_\lambda \geq \int_{\mathbb{R}} (I_\mu * F(u))f(u_\lambda)(v-u_\lambda)\,\mathrm{d}x \quad \text{for all } v \in X_\lambda. \tag{2.7}$$

Hence, every $u_\lambda \in X_\lambda$ for which (2.7) holds is going to be called a solution of (1.1).

In fact, from Parini-Ruf-Tarsi [27], we know that if $u_\lambda \in X_\lambda$ satisfies (2.7), there exists a function $\gamma \in L_{\infty,N}(\mathbb{R}^N)$ and a vector field $\mathbf{z} \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $|\mathbf{z}|_\infty \leq 1$ and

$$\begin{cases} \operatorname{div}\mathbf{z} \in L_{\infty,N}(\mathbb{R}^N), \Delta\mathbf{z} \in L_{\infty,N}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} u_\lambda \Delta\mathbf{z} - \int_{\mathbb{R}^N} u_\lambda \operatorname{div}\mathbf{z}\,\mathrm{d}x = \int_{\mathbb{R}^N} |\Delta u_\lambda| + \int_{\mathbb{R}^N} |\nabla u_\lambda|\,\mathrm{d}x, \\ \gamma\,|u_\lambda| = (1+\lambda V(x))u_\lambda \quad \text{a.e. in } \mathbb{R}^N, \\ \Delta\mathbf{z} - \operatorname{div}\mathbf{z} + \gamma = (I_\mu * F(u_\lambda))f(u_\lambda), \qquad \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{2.8}$$

where

$$L_{\infty,N}\left(\mathbb{R}^N\right) = \left\{g\colon \mathbb{R}^N \to \mathbb{R} \,|\, g \text{ is measurable and } \|g\|_{\infty,N} < \infty\right\}$$

and

$$\|g\|_{\infty,N} = \sup_{\|\Phi_\lambda\|_1 + \|\Phi_\lambda\|_{1^*} \leq 1} \left|\int_{\mathbb{R}^N} g\Phi_\lambda\,\mathrm{d}x\right|.$$

Hence, (2.8) is the precise version of (1.1).

3. Proof of Theorem 1.1

Let us first recall the Mountain-Pass Theorem in its version from Figueiredo-Pimenta [17].

Theorem 3.1 (Mountain-Pass Theorem)**.** *Let E be a Banach space, $\Psi = I_0 - I$, where $I \in C^1(E,\mathbb{R})$ and I_0 is a locally Lipschitz convex functional defined in E. Suppose that the functional Ψ satisfies the following conditions:*

(g_1) *There exist $\rho > 0$ and $\alpha > \Psi(0)$ such that $\Psi|_{\partial B_\rho(0)} \geq \alpha$.*

(g_2) *$\Psi(e) < \Psi(0)$, for some $e \in E \setminus \overline{B_\rho(0)}$.*

Then for all $\tau > 0$, there exists $x_\tau \in E$ such that

$$c - \tau < \Psi\left(x_\tau\right) < c + \tau,$$

and

$$I_0(y) - I_0\left(x_\tau\right) \geq I'\left(x_\tau\right)\left(y - x_\tau\right) - \tau\left\|y - x_\tau\right\| \quad \textit{for all } y \in E,$$

where $c \geq \alpha$ is characterized by

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \Psi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0$ and $\gamma(1) = e\}$.

Motivated by the paper of Alves-Yang [5] we have the following uniform boundedness results.

Proposition 3.2. *There exists $\mathcal{K} > 0$ such that*

$$|I_\mu * F(u)| \leq \mathcal{K} \quad \textit{for all} \quad u \in X_\lambda. \tag{3.1}$$

Proof. Indeed, by assumptions (f_2) and (f_3), we have that

$$|F(u)| \leq \sigma\left(|u|^{q_1} + |u|^{q_2}\right),$$

and it follows that

$$|I_\mu * F(u)| = \left|\int_{\mathbb{R}^N} \frac{F(u)}{|x-y|^{N-\mu}}\,\mathrm{d}y\right|$$

$$
\begin{aligned}
&= \left| \int_{|x-y|\leq 1} \frac{F(u)}{|x-y|^{N-\mu}} \,\mathrm{d}y \right| + \left| \int_{|x-y|\geq 1} \frac{F(u)}{|x-y|^{N-\mu}} \,\mathrm{d}y \right| \\
&\leq \sigma \int_{|x-y|\leq 1} \frac{|u|^{q_1}+|u|^{q_2}}{|x-y|^{N-\mu}} \,\mathrm{d}y + \sigma \int_{|x-y|\geq 1} \left(|u|^{q_1}+|u|^{q_2}\right) \mathrm{d}y \\
&\leq \sigma \int_{|x-y|\leq 1} \frac{|u|^{q_1}+|u|^{q_2}}{|x-y|^{N-\mu}} \,\mathrm{d}y + C,
\end{aligned}
$$

where we used the fact that $1 < q_1 \leq q_2 < 1^*$. Choosing $t_1 \in (\frac{N}{\mu}, \frac{N}{(N-1)q_1})$ and $t_2 \in (\frac{N}{\mu}, \frac{N}{(N-1)q_2})$, it follows from Hölder's inequality that

$$
\begin{aligned}
&\int_{|x-y|\leq 1} \frac{|u|^{q_1}}{|x-y|^{N-\mu}} \,\mathrm{d}y \\
&\leq \left(\int_{|x-y|\leq 1} |u|^{t_1 q_1} \,\mathrm{d}y \right)^{\frac{1}{t_1}} \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t_1(N-\mu)}{t_1-1}}} \,\mathrm{d}y \right)^{\frac{t_1-1}{t_1}} \\
&\leq C_1 \left(\int_{|r|\leq 1} |r|^{N-1-\frac{t_1(N-\mu)}{t_1-1}} \,dr \right)^{\frac{t_1-1}{t_1}}.
\end{aligned}
$$

Similarly, we get

$$
\int_{|x-y|\leq 1} \frac{|u|^{q_2}}{|x-y|^{N-\mu}} \,\mathrm{d}y \leq C_2 \left(\int_{|r|\leq 1} |r|^{N-1-\frac{t_2(N-\mu)}{t_2-1}} \,dr \right)^{\frac{t_2-1}{t_2}}.
$$

Since $N-1-\frac{t_i(N-\mu)}{t_i-1} > -1$ for $i=1,2$, there exists a constant $C>0$ such that

$$
\int_{|x-y|\leq 1} \frac{|u|^{q_1}+|u|^{q_2}}{|x-y|^{N-\mu}} \,\mathrm{d}y \leq C \quad \text{for all } x \in \mathbb{R}^N.
$$

Hence the inequality implies the uniform boundedness given in (3.1). □

Now let us verify that the functional $\Phi_\lambda\colon X_\lambda \to \mathbb{R}$ defined in (2.5) satisfies the geometrical conditions of the Mountain-Pass Theorem.

Lemma 3.3. *The functional Φ_λ verifies the following properties:*

(g_1) *There exist $\rho > 0$ and $\alpha > \Phi_\lambda(0)$ such that $\Phi_\lambda|_{\partial B_\rho(0)} \geq \alpha$.*

(g_2) *$\Phi_\lambda(e) < \Phi_\lambda(0)$ for some $e \in X_\lambda \setminus \overline{B_\rho(0)}$.*

Proof. We start to verify the first condition. Note that, from (f_2) and (f_3), there exists

$$
|F(u)| \leq \sigma \left(|u|^{q_1} + |u|^{q_2}\right), \tag{3.2}
$$

where q_1, q_2 are as in (f_3). Then, by (3.2) and the Hardy-Littlewood-Sobolev inequality, we get that

$$
\begin{aligned}
\left| \int_{\mathbb{R}^N} (I_\mu * F(u)) F(u) \,\mathrm{d}x \right| &\leq C_1 \|F(u)\|_s \|F(u)\|_s \\
&\leq C_2 \left(\int_{\mathbb{R}^N} \left(|u|^{q_1}+|u|^{q_2}\right)^s \,\mathrm{d}x \right)^{\frac{2}{s}},
\end{aligned}
$$

where $\frac{1}{s} = 1 - \frac{N-\mu}{2N}$. Since $\frac{1}{2}\left(2 - \frac{N-\mu}{N}\right) < q_1 \leq q_2 < \frac{1^*}{2}\left(2 - \frac{N-\mu}{N}\right)$, we can see that $1 < sq_1 \leq sq_2 < 1^*$. By using the continuous embeddings of X_λ, we have that

$$\left(\int_{\mathbb{R}^N} (|u|^{q_1} + |u|^{q_2})^s \, \mathrm{d}x\right)^{\frac{2}{s}} \leq C_3(\|u\|_\lambda^{2q_1} + \|u\|_\lambda^{2q_2}).$$

Therefore,

$$\begin{aligned}
\Phi_\lambda(u) &= \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| \, \mathrm{d}x + \int_{\mathbb{R}^N} (1 + \lambda V(x))|u| \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} (I_\mu * F(u))F(u) \, \mathrm{d}x \\
&= \|u\|_\lambda - \int_{\mathbb{R}^N} (I_\mu * F(u))F(u) \, \mathrm{d}x \\
&\geq \|u\|_\lambda - C_4(\|u\|_\lambda^{2q_1} + \|u\|_\lambda^{2q_2}).
\end{aligned}$$

Since $q_2 \geq q_1 \geq 1$, the claim follows if we choose ρ small enough.

Now let us prove that Φ_λ satisfies (g$_2$). For a fixed positive function $u_0 \in C_0^\infty\left(\mathbb{R}^N\right) \setminus \{0\}$ with $u_0 > 0$, we set

$$\phi(t) := \mathcal{H}\left(\frac{tu_0}{\|u_0\|_\lambda}\right) \qquad \text{for } t > 0,$$

where

$$\mathcal{H}(u) := \int_{\mathbb{R}^N} \left(I_\mu * F(u)\right) F(u) \, \mathrm{d}x.$$

By using the Ambrosetti-Rabinowitz condition (f$_4$), we deduce that

$$\begin{aligned}
\phi'(t) &= \mathcal{H}'\left(\frac{tu_0}{\|u_0\|_\lambda}\right) \frac{u_0}{\|u_0\|_\lambda} \\
&= \int_{\mathbb{R}^N} \left[I_\mu * F\left(\frac{tu_0}{\|u_0\|_\lambda}\right)\right] f\left(\frac{tu_0}{\|u_0\|_\lambda}\right) \frac{u_0}{\|u_0\|_\lambda} \, \mathrm{d}x \\
&\geq \frac{\kappa}{t} \int_{\mathbb{R}^N} \left[I_\mu * F\left(\frac{tu_0}{\|u_0\|_\lambda}\right)\right] F\left(\frac{tu_0}{\|u_0\|_\lambda}\right) \mathrm{d}x \\
&\geq \frac{\kappa}{t}\phi(t).
\end{aligned}$$

Integrating this on $[1, t\|u_0\|_\lambda]$ with $t > \frac{1}{\|u_0\|_\lambda}$, we find

$$\phi\left(t\|u_0\|_\lambda\right) \geq \phi(1)\left(t\|u_0\|_\lambda\right)^\kappa,$$

which implies

$$\mathcal{H}\left(tu_0\right) \geq \mathcal{H}\left(\frac{u_0}{\|u_0\|_\lambda}\right) \|u_0\|_\lambda^\kappa t^\kappa.$$

Thus,

$$\Phi_\lambda(tu_0) \leq t\|u_0\|_\lambda - \mathcal{H}\left(\frac{u_0}{\|u_0\|_\lambda}\right) \|u_0\|_\lambda^\kappa t^\kappa \to -\infty, \tag{3.3}$$

as $t \to +\infty$ since $\kappa > 1$. Then we can choose $e = tu_0 \in X_\lambda$ such that $\Phi_\lambda(e) < 0$. $\square$

From Theorem 3.1 we get that, for all $\lambda > 0$, given a sequence $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \to 0$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \in X_\lambda$ such that

$$\lim_{n\to\infty} \Phi_\lambda(u_n) = c_\lambda$$

and

$$\|v\|_\lambda - \|u_n\|_\lambda \geq \int_{\mathbb{R}^N} (I_\mu * F(u_n))\, f(u_n)(v-u_n)\,\mathrm{d}x - \tau_n\|v-u_n\|_\lambda, \tag{3.4}$$

for all $v \in X_\lambda$ where c_λ is given by

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \sup_{t\in[0,1]} \Phi_\lambda(\gamma(t))$$

and $\Gamma_\lambda = \{\gamma \in C([0,1], X_\lambda) \,:\, \gamma(0) = 0, \Phi_\lambda(\gamma(1)) < 0\}$.

In addition, let us define the Nehari manifold associated to problem (1.1) for $\lambda > 0$ which is given by

$$\mathcal{N}_\lambda = \{u \in X_\lambda \setminus \{0\} \,:\, \Phi'_\lambda(u)u = 0\}\,.$$

From Figueiredo-Pimenta [18] it follows that

$$c_\lambda = \inf_{u\in X_\lambda\setminus\{0\}} \max_{t\geq 0} \Phi_\lambda(tu) = \inf_{u\in\mathcal{N}_\lambda} \Phi_\lambda(u).$$

In the following result, we give lower and upper bounds for c_λ.

Lemma 3.4. *For each $\lambda > 0$, there exist positive constants α_0 and β_0 independent of λ such that*

$$\alpha_0 \leq c_\lambda \leq \beta_0.$$

Proof. From the proof of the property (g_1) in Lemma 3.3, it is obvious that we can take $0 < \alpha_0 < \alpha < c_\lambda$. On the other hand, by $e \in C_0^\infty(\Omega)$, for all $t > 0$, as in (3.3), we have

$$\Phi_\lambda(te) \leq t\left(\int_{\mathbb{R}^N} |\Delta e| + \int_{\mathbb{R}^N} |\nabla e|\,\mathrm{d}x + \int_{\mathbb{R}^N} |e|\,\mathrm{d}x\right) - \mathcal{H}\left(\frac{e}{\|e\|_\lambda}\right) \|e\|_\lambda^\kappa\, t^\kappa \to -\infty,$$

as $t \to \infty$. Thus, there exists a constant $\beta_0 > 0$ such that

$$c_\lambda \leq \max_{t>0} \Phi_\lambda(te) \leq \beta_0.$$

□

Next we are going to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{BL}(\mathbb{R}^N)$.

Lemma 3.5. *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{BL}(\mathbb{R}^N)$.*

Proof. Taking the test function $v = 2u_n$ in (3.4) yields

$$\|u_n\|_\lambda \geq \int_{\mathbb{R}^N} (I_\mu * F(u_n))\, f(u_n)u_n\,\mathrm{d}x - \tau_n\|u_n\|_\lambda,$$

which implies that

$$(1+\tau_n)\|u_n\|_\lambda \geq \int_{\mathbb{R}^N} (I_\mu * F(u_n))\, f(u_n)u_n\,\mathrm{d}x. \tag{3.5}$$

Then, by (f_4) and (3.5), we get

$$\begin{aligned} c_\lambda + o_n(1) &\geq \Phi_\lambda(u_n) \\ &= \|u_n\|_\lambda + \int_{\mathbb{R}^N} (I_\mu * F(u_n))\left(\frac{1}{\kappa} f(u_n)u_n - F(u_n)\right)\mathrm{d}x \end{aligned}$$

$$\begin{aligned} & - \int_{\mathbb{R}^N} \frac{1}{\kappa} \left(I_\mu * F(u_n) \right) f(u_n) u_n \, \mathrm{d}x \\ & \geq \|u_n\|_\lambda \left(1 - \frac{1}{\kappa} - \frac{\tau_n}{\kappa} \right) \\ & \geq C \|u_n\|_\lambda, \end{aligned}$$

for some $C > 0$ which does not depend on $n \in \mathbb{N}$ and $\lambda > 0$. Thus, we conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{BL}(\mathbb{R}^N)$. □

From Lemmas 3.4 and 3.5 we obtain the following result.

Corollary 3.6. *There exists a positive constant $C > 0$ independent of λ such that*

$$\|u_n\|_\lambda \leq C \quad \textit{for all } n \in \mathbb{N}$$

and

$$\liminf_{n\to+\infty} \|u_n\|_\lambda \geq \alpha_0 \quad \textit{for all } \lambda > 0.$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{BL}(\mathbb{R}^N)$ and the compactness of the embedding $\mathrm{BL}(\mathbb{R}^N) \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 \leq r < 1^*$, there exists $u_\lambda \in \mathrm{BL}_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$u_n \to u_\lambda \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for } 1 \leq r < 1^*,$$

and

$$u_n \to u_\lambda \quad \text{a.e. in } \mathbb{R}^N,$$

as $n \to +\infty$. Note that $u_\lambda \in \mathrm{BL}(\mathbb{R}^N)$. Indeed, by Fatou's Lemma, it follows that $u_\lambda \in L^1\mathbb{R}^N$. For a given $R > 0$, from the semicontinuity of the norm in $\mathrm{BL}(B_R(0))$ with respect to the $L^q(B_R(0))$ convergence, we have that

$$\int_{B_R(0)} |\Delta u_{lambda}| \leq \liminf_{n\to+\infty} \int_{B_R(0)} |\Delta u_n| \leq \liminf_{n\to+\infty} \|u_n\|_{\mathrm{BL}(\mathbb{R}^N)} \leq C,$$

where C does not depend on n and on R. Since the last inequality holds for every $R > 0$, then $\Delta u_\lambda \in \mathcal{M}(\mathbb{R}^N)$. Hence, by Hurtado-Pimenta-Miyagaki [21], it follows that $u_\lambda \in \mathrm{BL}(\mathbb{R}^N)$. The following result is crucial for obtaining the compactness properties in our work.

Lemma 3.7. *For all fixed $q \in [1, 1^*)$ and for a given $\varepsilon > 0$, there exist $\lambda^*(q,\varepsilon) > 0$ and $R > 0$ such that*

$$\int_{B^c_R(0)} |u_n|^q \, \mathrm{d}x \leq \varepsilon,$$

for all $\lambda \geq \lambda^*(q,\varepsilon)$ and for all $n \in \mathbb{N}$, where $B^c_R(0) = \{x \in \mathbb{R}^N : |x| > R\}$.

Proof. For a given $R > 0$, let us define the sets

$$\begin{aligned} A(R) &= \{x \in \mathbb{R}^N : |x| > R \text{ and } V(x) \geq M_0\}, \\ B(R) &= \{x \in \mathbb{R}^N : |x| > R \text{ and } V(x) < M_0\}, \end{aligned}$$

where M_0 is given in (V_2).

Note that, by Corollary 3.6, (V_2) and the definition of $\|\cdot\|_\lambda$, we have

$$\int_{A(R)} (1 + \lambda M_0)|u_n| \, \mathrm{d}x \leq \int_{A(R)} (1 + \lambda V(x))|u_n| \, \mathrm{d}x \leq \|u_n\|_\lambda,$$

which implies that

$$\int_{A(R)} |u_n| \, \mathrm{d}x \leq \frac{1}{1+\lambda M_0} \|u_n\|_\lambda \leq \frac{C}{1+\lambda M_0} < \frac{\varepsilon}{2} \tag{3.6}$$

for all $n \in \mathbb{N}$ whenever $\lambda > \lambda^*(\varepsilon)$ and $\lambda^*(\varepsilon) \geq M_0^{-1}(\frac{2C}{\varepsilon} - 1)$.

On the other hand, by Corollary 3.6, (V_2), Hölder's inequality and the embeddings of X_λ, we obtain

$$\int_{B(R)} |u_n| \, \mathrm{d}x \leq C \|u_n\|_{1^*}^{1^*} |B(R)|^{\frac{1}{N}} \leq C |B(R)|^{\frac{1}{N}} < \frac{\varepsilon}{2}, \tag{3.7}$$

where $R > 0$ is large enough and $|B(R)| \to 0$ as $R \to +\infty$.

Then, if $\lambda > \lambda^*(\varepsilon)$ and $R > 0$ is large enough, from (3.6) and (3.7), it follows the result for $q = 1$.

For $q \in (1, 1^*)$, by Corollary 3.6 and interpolation in Lebesgue spaces the estimate follows for λ greater than a certain $\lambda^*(q, \varepsilon)$, since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^{1^*}(\mathbb{R}^N)$. This completes the proof. $\square$

Now we will prove that u_λ is nontrivial.

Lemma 3.8. *There exists $\lambda^* > 0$ such that $u_\lambda \neq 0$ for all $\lambda \geq \lambda^*$.*

Proof. Taking the test function $v = u_n + t u_n$ in (3.4) and letting $t \to 0^\pm$, we get that

$$\Phi_\lambda'(u_n) u_n = o_n(1),$$

which implies that

$$\begin{aligned} \|u_n\|_\lambda &= \int_{\mathbb{R}^N} (I_\mu * F(u_n)) f(u_n) u_n \, \mathrm{d}x + o_n(1) \\ &= \int_{B_R(0)} (I_\mu * F(u_n)) f(u_n) u_n \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N \setminus B_R(0)} (I_\mu * F(u_n)) f(u_n) u_n \, \mathrm{d}x + o_n(1). \end{aligned} \tag{3.8}$$

From (f_3) and Proposition 3.2, we have

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus B_R(0)} (I_\mu * F(u_n)) f(u_n) u_n \, \mathrm{d}x \\ &\leq \mathcal{K} \sigma \int_{\mathbb{R}^N \setminus B_R(0)} |u_n|^{q_1} \, \mathrm{d}x + \mathcal{K} \sigma \int_{\mathbb{R}^N \setminus B_R(0)} |u_n|^{q_2} \, \mathrm{d}x. \end{aligned} \tag{3.9}$$

Then, by Lemma 3.7, taking $\lambda^* \geq \max\{\lambda^*(\frac{\alpha_0}{4\mathcal{K}\sigma}, q_1), \lambda^*(\frac{\alpha_0}{4\mathcal{K}\sigma}, q_2)\}$ where α_0 is as in Corollary 3.6, it follows that (3.9) implies that

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{\mathbb{R}^N \setminus B_R(0)} (I_\mu * F(u_n)) f(u_n) u_n \, \mathrm{d}x \\ &\leq \mathcal{K} \limsup_{n\to+\infty} \int_{\mathbb{R}^N \setminus B_R(0)} f(u_n) u_n \, \mathrm{d}x \leq \frac{\alpha_0}{2}. \end{aligned} \tag{3.10}$$

From the compactness of the embedding $\mathrm{BL}(B_R(0)) \hookrightarrow L^q(B_R(0))$ for $q \in [1, 1^*)$, (f_2) and (f_3), we have that

$$\lim_{n\to+\infty} \int_{B_R(0)} (I_\mu * F(u_n)) f(u_n) u_n \, \mathrm{d}x = \int_{B_R(0)} (I_\mu * F(u_\lambda)) f(u_\lambda) u_\lambda \, \mathrm{d}x. \tag{3.11}$$

Hence, from (3.11), (3.8), (3.10) and Corollary 3.6, we obtain

$$\begin{aligned}
&\int_{B_R(0)} (I_\mu * F(u_\lambda))\, f(u_\lambda)u_\lambda \,\mathrm{d}x \\
&= \lim_{n\to+\infty} \int_{B_R(0)} (I_\mu * F(u_n))\, f(u_n)u_n \,\mathrm{d}x \\
&\geq \liminf_{n\to+\infty} \left(\|u_n\|_\lambda - \int_{\mathbb{R}^N \setminus B_R(0)} (I_\mu * F(u_n))\, f(u_n)u_n \,\mathrm{d}x \right) \\
&\geq \liminf_{n\to+\infty} \|u_n\|_\lambda - \frac{\alpha_0}{2} \\
&\geq \frac{\alpha_0}{2},
\end{aligned}$$

where $\lambda \geq \lambda^*$. Thus $u_\lambda \neq 0$. □

The following result is the pivotal point.

Lemma 3.9. $\Phi'_\lambda(u_\lambda)u_\lambda \leq 0$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \quad \text{in } B_R(0), \quad \varphi \equiv 0 \quad \text{in } B_{2R}^c(0)$$

and let $C > 0$ be a constant such that $|\nabla\varphi| \leq C$ and $|\Delta\varphi| \leq C$, for $\varphi_R := \varphi(\cdot/R)$. Then, for all $u \in \mathrm{BL}(\mathbb{R}^N)$, it follows that

$$(\Delta(\varphi_R u))^s \quad \text{is absolutely continuous w.r.t.}\, (\Delta u)^s. \tag{3.12}$$

Indeed, note that

$$\begin{aligned}
\Delta(\varphi_R u) &= \Delta\varphi_R u + 2\nabla\varphi_R \cdot \nabla u + \varphi_R \Delta u \\
&= \Delta\varphi_R u + 2\nabla\varphi_R \cdot \nabla u + \varphi_R (\Delta u)^a + \varphi_R(\Delta u)^s \quad \text{in } \mathcal{D}'(\mathbb{R}^N).
\end{aligned}$$

Then it follows that

$$(\Delta(\varphi_R u))^s = (\varphi_R(\Delta u)^s)^s = \varphi_R(\Delta u)^s.$$

Taking (3.12) into account and the fact that $\varphi_R u_n$ is equal to 0 a.e. in the set where u_n vanishes, we see that $\varphi_R u_n$ and u_n fulfill two of the three requirements that would allow us to calculate $\Phi'_\lambda(u_n)(\varphi_R u_n)$. However, we have no ensure that

$$(\Delta(\varphi_R u_n))^a = \Delta\varphi_R u + 2\nabla\varphi_R \nabla u + \varphi_R(\Delta u)^a$$

vanishes a.e. in the set

$$\left\{ x \in \mathbb{R}^N \,:\, (\Delta u_n)^a(x) = 0 \right\}.$$

Hence, it might not be possible to calculate the Gateaux derivative $\Phi'_\lambda(u_n)(\varphi_R u_n)$. We have to work in a slightly different way. In fact, it will be enough to work with the left Gateaux derivative

$$\lim_{t\to 0^-} \frac{\Phi_\lambda(u_n + t\varphi_R u_n) - \Phi_\lambda(u_n)}{t},$$

which, by (3.4), satisfy

$$\lim_{t\to 0^-} \frac{\Phi_\lambda(u_n + t\varphi_R u_n) - \Phi_\lambda(u_n)}{t} \leq o_n(1). \tag{3.13}$$

In order to calculate the limit above, let us first calculate separately a part of it. Let us define for all $u \in \mathrm{BL}(\mathbb{R}^N)$,

$$\mathcal{J}_a(u) = \int_{\mathbb{R}^N} |(\Delta u)^a(x)| \,\mathrm{d}x.$$

Then, for all $u, v \in \mathrm{BL}(\mathbb{R}^N)$, we have that

$$\begin{aligned}
&\lim_{t \to 0^-} \frac{\mathcal{J}_a(u+tv) - \mathcal{J}_a(u)}{t}\\
&= \lim_{t \to 0^-} \frac{1}{t} \int_{\mathbb{R}^N} \left(|(\Delta u)^a + t(\Delta v)^a| - |(\Delta u)^a|\right) \mathrm{d}x\\
&= -\int_{T_u} |(\Delta v)^a| \,\mathrm{d}x + \int_{\mathbb{R}^N \setminus T_u} \frac{(\Delta u)^a (\Delta v)^a}{|(\Delta u)^a|} \,\mathrm{d}x,
\end{aligned} \tag{3.14}$$

where $T_u = \left\{x \in \mathbb{R}^N \,:\, (\Delta u)^a(x) = 0\right\}$.

Taking into account (3.13) and (3.14), it follows that

$$\begin{aligned}
o_n(1) \geq &\int_{\mathbb{R}^N \setminus T_{u_n}} \frac{(\Delta u_n)^a [\Delta \varphi_R u_n + 2\nabla \varphi_R \cdot \nabla u_n + \varphi_R (\Delta u_n)^a]}{|(\Delta u_n)^a|} \,\mathrm{d}x\\
&- \int_{T_{u_n}} |(\Delta \varphi_R u_n + 2\nabla \varphi_R \cdot \nabla u_n)| \,\mathrm{d}x\\
&+ \int_{\mathbb{R}^N} \frac{\Delta u_n}{|\Delta u_n|} \frac{\varphi_R (\Delta u_n)^s}{|\varphi_R (\Delta u_n)^s|} |\varphi_R (\Delta u_n)^s|\\
&+ \int_{\mathbb{R}^N} \frac{\nabla u_n \cdot (\nabla \varphi_R u_n + \varphi_R \nabla u_n)}{|\nabla u_n|} \,\mathrm{d}x\\
&+ \int_{\mathbb{R}^N} (1 + \lambda V(x)) \operatorname{sgn}(u_n)(\varphi_R u_n) \,\mathrm{d}x\\
&- \int_{\mathbb{R}^N} (I_\mu * F(u_n)) f(u_n) \varphi_R u_n \,\mathrm{d}x\\
= &\int_{\mathbb{R}^N \setminus T_{u_n}} \varphi_R |(\Delta u_n)^a| \,\mathrm{d}x\\
&+ \int_{\mathbb{R}^N \setminus T_{u_n}} \frac{(\Delta u_n)^a (\Delta \varphi_R u_n + 2\nabla \varphi_R \cdot \nabla u_n)}{|(\Delta u_n)^a|} \,\mathrm{d}x\\
&- \int_{T_{u_n}} |(\Delta \varphi_R u_n + 2\nabla \varphi_R \cdot \nabla u_n)| \,\mathrm{d}x\\
&+ \int_{\mathbb{R}^N} \frac{\Delta u_n}{|\Delta u_n|} \frac{\varphi_R (\Delta u_n)^s}{|\varphi_R (\Delta u_n)^s|} |\varphi_R (\Delta u_n)^s|\\
&+ \int_{\mathbb{R}^N} \frac{\nabla u_n \cdot (\nabla \varphi_R u_n + \varphi_R \nabla u_n)}{|\nabla u_n|} \,\mathrm{d}x\\
&+ \int_{\mathbb{R}^N} (1 + \lambda V(x)) |u_n| \varphi_R \,\mathrm{d}x\\
&- \int_{\mathbb{R}^N} (I_\mu * F(u_n)) f(u_n) \varphi_R u_n \,\mathrm{d}x.
\end{aligned}$$

Noting that $\int_{\mathbb{R}^N\setminus T_{u_n}} \varphi_R \left|(\Delta u_n)^a\right| \mathrm{d}x = \int_{\mathbb{R}^N} \varphi_R \left|(\Delta u_n)^a\right| \mathrm{d}x$ and calculating the $\lim_{n\to+\infty}$ in the inequality above, we have that

$$
\begin{aligned}
0 \geq & \liminf_{n\to+\infty} \left(\int_{\mathbb{R}^N} \varphi_R \left|(\Delta u_n)^a\right| \mathrm{d}x \right.\\
&\qquad \left. + \int_{\mathbb{R}^N} \frac{(\Delta u_n)^s}{|(\Delta u_n)^s|} \frac{\varphi_R(\Delta u_n)^s}{|\varphi_R(\Delta u_n)^s|} \left|\varphi_R(\Delta u_n)^s\right| \right)\\
& + \liminf_{n\to+\infty} \int_{\mathbb{R}^N\setminus T_{u_n}} \frac{(\Delta u_n)^a(\Delta\varphi_R u_n + 2\nabla\varphi_R\cdot\nabla u_n)}{|(\Delta u_n)^a|} \mathrm{d}x\\
& - \limsup_{n\to+\infty} \int_{T_{u_n}} \left|(\Delta\varphi_R u_n + 2\nabla\varphi_R\cdot\nabla u_n)\right| \mathrm{d}x\\
& + \liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{\nabla u_n\cdot(\nabla\varphi_R u_n + \varphi_R\nabla u_n)}{|\nabla u_n|} \mathrm{d}x\\
& + \int_{\mathbb{R}^N} (1+\lambda V(x)) \left|u_\lambda\right| \varphi_R \,\mathrm{d}x - \int_{\mathbb{R}^N} \left(I_\mu * F(u_\lambda)\right) f(u_\lambda)\varphi_R u_\lambda \,\mathrm{d}x.
\end{aligned}
\tag{3.15}
$$

Now, by the lower semicontinuity of the norm in $\mathrm{BL}(B_R(0))$ w.r.t. the $L^1(B_R(0))$-convergence and also by the fact that $\frac{\varphi_R\mu}{|\varphi_R\mu|} = \frac{\mu}{|\mu|}$ a.e. in $B_R(0)$ with (3.15), we have that

$$
\begin{aligned}
& \int_{B_R(0)} |\Delta u_\lambda| \,\mathrm{d}x\\
\leq & -\liminf_{n\to+\infty} \int_{\mathbb{R}^N\setminus T_{u_n}} \frac{(\Delta u_n)^a(\Delta\varphi_R u_n + 2\nabla\varphi_R\cdot\nabla u_n)}{|(\Delta u_n)^a|} \mathrm{d}x\\
& + \limsup_{n\to+\infty} \int_{T_{u_n}} \left|(\Delta\varphi_R u_n + 2\nabla\varphi_R\cdot\nabla u_n)\right| \mathrm{d}x\\
& - \liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{\nabla u_n\cdot(\nabla\varphi_R u_n + \varphi_R\nabla u_n)}{|\nabla u_n|} \mathrm{d}x\\
& - \int_{\mathbb{R}^N} (1+\lambda V(x)) \left|u_\lambda\right| \varphi_R \,\mathrm{d}x + \int_{\mathbb{R}^N} \left(I_\mu * F(u_\lambda)\right) f(u_\lambda)\varphi_R u_\lambda \,\mathrm{d}x.
\end{aligned}
\tag{3.16}
$$

Furthermore, since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^1(\mathbb{R}^N)$, it follows that

$$
\begin{aligned}
& \lim_{R\to+\infty} \left| \lim\ \inf_{n\to\infty} \int_{\mathbb{R}^N\setminus T_{u_n}} \frac{u_n(\Delta u_n)^a\cdot\Delta\varphi_R}{|(\Delta u_n)^a|} \mathrm{d}x \right|\\
\leq & \lim_{R\to+\infty} (\lim\ \inf_{n\to\infty} \int_{\mathbb{R}^N\setminus T_{u_n}} |u_n|\, |\Delta\varphi_R| \ \mathrm{d}x)\\
\leq & \lim_{R\to+\infty} \frac{C}{R} (\lim\ \inf_{n\to\infty} \int_{\mathbb{R}^N\setminus T_{u_n}} |u_n| \ \mathrm{d}x) = 0.
\end{aligned}
\tag{3.17}
$$

Similarly, we can also get that

$$
\begin{aligned}
& \lim_{R\to+\infty} \left| \lim\ \inf_{n\to\infty} \int_{\mathbb{R}^N\setminus T_{u_n}} \frac{(\Delta u_n)^a(2\nabla\varphi_R\cdot\nabla u_n)}{|(\Delta u_n)^a|} \mathrm{d}x \right| = 0,\\
& \lim_{R\to+\infty} \left| \liminf_{n\to+\infty} \int_{T_{u_n}} \left|(u_n\Delta\varphi_R + 2\nabla\varphi_R\cdot\nabla u_n)\right| \mathrm{d}x \right| = 0,
\end{aligned}
\tag{3.18}
$$

and

$$\lim_{R\to+\infty}\left|\liminf_{n\to+\infty}\int_{\mathbb{R}^N}\frac{u_n\nabla u_n\cdot\nabla\varphi_R}{|\nabla u_n|}\,\mathrm{d}x\right|=0. \tag{3.19}$$

Letting $R\to+\infty$ in both sides of (3.16) and taking (3.17), (3.18) and (3.19) into account, we get that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\Delta u_\lambda|+\int_{\mathbb{R}^N}|\nabla u_\lambda|\,\mathrm{d}x+\int_{\mathbb{R}^N}(1+\lambda V(x))\,|u_\lambda|\,\mathrm{d}x\\ &\leq\int_{\mathbb{R}^N}(I_\mu*F(u_\lambda))\,f(u_\lambda)u_\lambda\,\mathrm{d}x.\end{aligned}$$

This shows the assertion of the lemma. □

By the last result, there exists $t_\lambda\in(0,1]$ such that $t_\lambda u_\lambda\in\mathcal{N}_\lambda$. Note also that

$$\begin{aligned}c_\lambda+o_n(1)&=\Phi_\lambda(u_n)+o_n(1)=\Phi_\lambda(u_n)-\Phi_\lambda'(u_n)u_n\\ &=\int_{\mathbb{R}^N}(I_\mu*F(u_n))\,(f(u_n)u_n-F(u_n))\,\mathrm{d}x,\end{aligned}\tag{3.20}$$

and under (f_5), it is easy to see that $t\mapsto f(t)t-F(t)$ is increasing for $t\in(0,+\infty)$ and decreasing for $t\in(-\infty,0)$, then by Fatou's Lemma in the last inequality, we derive that

$$\begin{aligned}c_\lambda&\geq\int_{\mathbb{R}^N}(I_\mu*F(u_\lambda))\,(f(u_\lambda)u_\lambda-F(u_\lambda))\,\mathrm{d}x\\ &\geq\int_{\mathbb{R}^N}(I_\mu*F(u_\lambda))\,(f(t_\lambda u_\lambda)t_\lambda u_\lambda-F(t_\lambda u_\lambda))\,\mathrm{d}x\\ &=\Phi_\lambda(t_\lambda u_\lambda)-\Phi_\lambda'(t_\lambda u_\lambda)t_\lambda u_\lambda\\ &=\Phi_\lambda(t_\lambda u_\lambda)\\ &\geq c_\lambda.\end{aligned}$$

Hence, $t_\lambda=1$, $\Phi_\lambda(u_\lambda)=c_\lambda$, and by (3.20),

$$\begin{aligned}&(I_\mu*F(u_n))\,(f(u_n)u_n-F(u_n))\\ &\to(I_\mu*F(u_\lambda))\,(f(u_\lambda)u_\lambda-F(u_\lambda))\quad\text{in }L^1(\mathbb{R}^N).\end{aligned}\tag{3.21}$$

Moreover, by (f_4), we have

$$0\leq\left(1-\frac{1}{\kappa}\right)f(u_n)u_n\leq f(u_n)u_n-F(u_n),$$

and

$$0\leq(\kappa-1)F(u_n)\leq f(u_n)u_n-F(u_n).$$

Then, by (3.21), we can apply Lebesgue's Dominated Convergence Theorem to get

$$(I_\mu*F(u_n))\,f(u_n)u_n\to(I_\mu*F(u_\lambda))\,f(u_\lambda)u_\lambda\quad\text{in }L^1(\mathbb{R}^N), \tag{3.22}$$

and

$$(I_\mu*F(u_n))\,F(u_n)\to(I_\mu*F(u_\lambda))\,F(u_\lambda)\quad\text{in }L^1(\mathbb{R}^N).$$

Since

$$\|u_\lambda\|_\lambda=\int_{\mathbb{R}^N}(I_\mu*F(u_\lambda))\,f(u_\lambda)u_\lambda\,\mathrm{d}x$$

and

$$\|u_n\|_\lambda = \int_{\mathbb{R}^N} (I_\mu * F(u_n))\, f(u_n) u_n \,\mathrm{d}x + o_n(1),$$

by the limit (3.22), we obtain

$$\|u_n\|_\lambda \to \|u_\lambda\|_\lambda, \tag{3.23}$$

from which we conclude that

$$\|u_n\|_1 \to \|u_\lambda\|_1, \tag{3.24}$$

as $n \to +\infty$.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Based on the previous results, we can finish the proof of Theorem 1.1. Indeed, by (3.4), (3.23), (3.24) and the lower semicontinuity of the norm $\|\cdot\|_\lambda$ w.r.t. the $L^1(\mathbb{R}^N)$-convergence, it follows that

$$\|v\|_\lambda - \|u_\lambda\|_\lambda \geq \int_{\mathbb{R}^N} (I_\mu * F(u_\lambda))\, f(u_\lambda)(v-u_\lambda)\,\mathrm{d}x \quad \text{for all } v \in X_\lambda.$$

Then, u_λ is a nontrivial solution of problem (1.1) and $\Phi_\lambda(u_\lambda) = c_\lambda$. Thus, u_λ is also a ground state solution of problem (1.1). □

4. Proof of Theorem 1.2

In this section, we first consider the problem

$$\begin{cases} \Delta_1^2 - \Delta_1 u + \frac{u}{|u|} = (I_\mu * F(u))\, f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The corresponding energy functional $\Phi_\Omega(u)\colon \mathrm{BL}(\Omega) \to \mathbb{R}$ is given by

$$\Phi_\Omega(u) = \|u\|_\Omega - \int_\Omega (I_\mu * F(u))\, F(u)\,\mathrm{d}x,$$

where

$$\|u\|_\Omega = \int_\Omega |\Delta u| + \int_\Omega |\nabla u|\,\mathrm{d}x + \int_\Omega |u|\,\mathrm{d}x + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}. \tag{4.2}$$

Also, we have that $u \in \mathrm{BL}(\Omega)$ is a solution of (4.1) if

$$\|v\|_\Omega - \|u\|_\Omega \geq \int_\Omega (I_\mu * F(u))\, f(u)(v-u) \quad \text{for all } v \in \mathrm{BL}(\Omega).$$

Definition 4.1. *A sequence $(w_n)_{n\in\mathbb{N}} \subset \mathrm{BL}(\mathbb{R}^N)$ is called a $(\mathrm{PS})_{c,\infty}$-sequence for the family $(\Phi_\lambda)_{\lambda\geq 1}$, if there is a sequence $\lambda_n \to \infty$ such that $u_n \in X_{\lambda_n}$ for $n \in \mathbb{N}$,*

$$\Phi_{\lambda_n}(w_n) \to c,$$

as $n \to +\infty$, and

$$\begin{aligned} &\|v\|_{\lambda_n} - \|w_n\|_{\lambda_n} \\ &\geq \int_{\mathbb{R}^N} (I_\mu * F(w_n))\, f(w_n)(v - w_n) - \tau_n \|v - w_n\|_{\lambda_n} \end{aligned} \tag{4.3}$$

for all $v \in X_{\lambda_n}$, where $\tau_n \to 0$ as $n \to +\infty$.

Similarly to the proof of Lemma 3.3, Φ_Ω also satisfies the geometric conditions of the Mountain-Pass Theorem. Then, the Nehari manifold associated to Φ_Ω is also well defined by

$$\mathcal{N}_\Omega = \left\{ u \in \mathrm{BL}(\Omega) \setminus \{0\} \,:\, \Phi'_\Omega(u)u = 0 \right\},$$

and

$$c_\Omega = \inf_{\mathcal{N}_\Omega} \Phi_\Omega = \inf_{\gamma \in \Gamma_\Omega} \max_{t \in [0,1]} \Phi_\Omega(\gamma(t)),$$

where

$$\Gamma_\Omega = \left\{ \gamma \in C([0,1], \mathrm{BL}(\Omega)) \,:\, \gamma(0) = 0 \text{ and } \Phi_\Omega(\gamma(1)) < 0 \right\}.$$

Lemma 4.2. *Let $(w_n)_{n\in\mathbb{N}} \subset \mathrm{BL}(\mathbb{R}^N)$ be a $(\mathrm{PS})_{d,\infty}$-sequence for $(\Phi_\lambda)_{\lambda \geq 1}$ with $d \in \mathbb{R}$. Then either $d = 0$ or $d \geq c_\Omega$. Moreover, there exists $w_\Omega \in \mathrm{BL}(\mathbb{R}^N)$ such that, up to a subsequence not relabeled, $w_n \to w_\Omega$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, for all $1 \leq q < 1^*, w_\Omega \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and w_Ω is a solution of problem (4.1). Moreover, if $d = c_\Omega$, then*

$$\|w_n\|_{\lambda_n} - \|w_\Omega\|_\Omega \to 0 \quad \text{as } n \to +\infty.$$

Proof. Note that as in the proof of Lemma 3.5, we have that

$$d + o_n(1) \geq C \|w_n\|_{\lambda_n},$$

which implies that $d \geq 0$. We also conclude that $(\|w_n\|_{\lambda_n})_{n\in\mathbb{N}}$ is a bounded sequence and then we know that $(w_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{BL}(\mathbb{R}^N)$.

By the Sobolev embedding, there exists $w_\Omega \in \mathrm{BL}_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$w_n \to w_\Omega \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \leq q < 1^*,$$

and

$$w_n(x) \to w_\Omega(x) \quad \text{a.e. } x \in \mathbb{R},$$

as $n \to +\infty$. Moreover, it is possible to show that in fact w_Ω belongs to $\mathrm{BL}(\mathbb{R}^N)$.

Next let us show that $w_\Omega \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. In fact, for each $m \in \mathbb{N}$, let us define

$$C_m = \left\{ x \in \mathbb{R}^N \,:\, V(x) \geq \frac{1}{m} \right\},$$

and note that $\mathbb{R}^N \setminus \Omega = \bigcup_{i=1}^{+\infty} C_m \cup \partial\Omega$. Then, since $(\|w_n\|_{\lambda_n})_{n\in\mathbb{N}}$ is bounded, we have

$$\begin{aligned} \int_{C_m} |w_n| \,\mathrm{d}x &\leq \frac{m}{\lambda_n} \int_{C_m} \lambda_n V(x) |w_n| \,\mathrm{d}x \\ &\leq \frac{m}{\lambda_n} \|w_n\|_{\lambda_n} \\ &= o_n(1), \end{aligned}$$

which implies by Fatou's Lemma that

$$\int_{C_m} |w_\Omega| \,\mathrm{d}x = 0.$$

Hence, since $\mathbb{R}^N \setminus \Omega = \bigcup_{i=1}^{+\infty} C_m \cup \partial\Omega$ and $|\partial\Omega| = 0$, it follows that

$$\int_{\mathbb{R}^N \setminus \Omega} |w_\Omega| \,\mathrm{d}x = 0,$$

and then that $w_\Omega = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

If $d = 0$, it implies that $\|w_n\|_{\lambda_n} \to 0$ as $n \to +\infty$ and we are done.

If $d > 0$, since

$$d + o_n(1) = \Phi_{\lambda_n}(w_n) \le \|w_n\|_{\lambda_n},$$

it is possible to argue as in Lemma 3.8 in order to show that in fact $w_\Omega \neq 0$.

Similar to the proof of Lemma 3.9, we also get that

$$\Phi'_\Omega(w_\Omega)\, w_\Omega \le 0.$$

From the last conclusion, there exists $t_\Omega \in (0,1]$ such that $t_\Omega w_\Omega \in \mathcal{N}_\Omega$. Note also that

$$\begin{aligned} d + o_n(1) &= \Phi_{\lambda_n}(w_n) + o_n(1) = \Phi_{\lambda_n}(w_n) - \Phi'_{\lambda_n}(w_n)w_n \\ &= \int_{\mathbb{R}^N} (I_\mu * F(w_n))\,(f(w_n)w_n - F(w_n))\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

Then, by Fatou's Lemma in the last inequality, we derive that

$$\begin{aligned} d &\ge \int_{\mathbb{R}^N} (I_\mu * F(w_\Omega))\,(f(w_\Omega)w_\Omega - F(w_\Omega))\,\mathrm{d}x \\ &\ge \int_{\mathbb{R}^N} (I_\mu * F(w_\Omega))\,(f(t_\Omega w_\Omega)t_\Omega w_\Omega - F(t_\Omega w_\Omega))\,\mathrm{d}x \\ &= \Phi_\Omega(t_\Omega w_\Omega) - \Phi'_\Omega(t_\Omega w_\Omega)t_\Omega w_\Omega \\ &= \Phi_\Omega(t_\Omega w_\Omega) \\ &\ge c_\Omega, \end{aligned}$$

which implies that $d \ge c_\Omega$.

Finally, we consider the case $d = c_\Omega$. In this case we have $t_\Omega = 1$, $\Phi_\Omega(w_\Omega) = c_\Omega$ and $w_\Omega \in \mathcal{N}_\Omega$. Then, by (4.4), we obtain

$$\begin{aligned} &(I_\mu * F(w_n)(\, f(w_n)w_n - F(w_n)) \\ &\to (I_\mu * F(w_\Omega))\,(f(w_\Omega)w_\Omega - F(w_\Omega)) \quad \text{in } L^1(\mathbb{R}^N). \end{aligned}$$

Moreover, by (f_4), we also get

$$(I_\mu * F(w_n))\, f(w_n)w_n \to (I_\mu * F(w_\Omega))\, f(w_\Omega)w_\Omega \quad \text{in } L^1(\mathbb{R}^N), \tag{4.5}$$

$$\begin{aligned} (I_\mu * F(w_n))\, F(w_n) &\to (I_\mu * F(w_\Omega))\, F(w_\Omega) \quad \text{in } L^1(\mathbb{R}^N), \\ \|w_n\|_{\lambda_n} &\to \|w_\Omega\|_\Omega, \\ \|w_n\|_1 &\to \|w_\Omega\|_1, \end{aligned} \tag{4.6}$$

as $n \to +\infty$. For each $v \in \mathrm{BL}(\Omega)$, let us consider the extension of $\tilde{v}$ of $v(x)$ given by

$$\tilde{v}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \\ v(x) & \text{if } x \in \Omega, \end{cases}$$

and note that

$$\begin{aligned} \|\tilde{v}\|_{\lambda_n} &= \int_{\mathbb{R}^N} |\Delta \tilde{v}| + \int_{\mathbb{R}^N} |\nabla \tilde{v}|\,\mathrm{d}x + \int_{\mathbb{R}^N} (1 + \lambda_n V(x))\,|\tilde{v}|\,\mathrm{d}x \\ &= \int_\Omega |\Delta \tilde{v}| + \int_\Omega |\nabla \tilde{v}|\,\mathrm{d}x + \int_{\partial\Omega} |\tilde{v}| d\mathcal{H}_{N-1} + \int_\Omega |\tilde{v}|\,\mathrm{d}x \\ &= \|\tilde{v}\|_\Omega. \end{aligned}$$

Then, using the lower limit in (4.3) and taking (4.5) and (4.6) into account, it follows that

$$\|\tilde{v}\|_\Omega - \|w_\Omega\|_\Omega \geq \int_\Omega \left(I_\mu * F(w_\Omega)\right) f\left(w_\Omega\right)\left(\tilde{v} - w_\Omega\right) \mathrm{d}x,$$

which shows that w_Ω is a solution of problem (4.1). The proof is complete. $\square$

Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset [\lambda^*, +\infty)$ be any sequence with $\lambda_n \to +\infty$ and let $u_n := u_{\lambda_n}$ be critical points of Φ_{λ_n} obtained by Theorem 1.1, which implies $\Phi_{\lambda_n}(u_n) = c_{\lambda_n}$.

For a given $u \in \mathrm{BL}(\Omega)$, denoting by $\overline{u}$ its extension by zero on $\mathbb{R}^N \setminus \Omega$, it follows from Green's Formula for BL-functions that

$$\begin{aligned}&\int_{\mathbb{R}^N} |\Delta \overline{u}| + \int_{\mathbb{R}^N} |\nabla \overline{u}| \,\mathrm{d}x + \int_{\mathbb{R}^N} |\overline{u}| \,\mathrm{d}x \\ &= \int_\Omega |\Delta u| + \int_\Omega |\nabla u| \,\mathrm{d}x + \int_\Omega |u| \,\mathrm{d}x + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}.\end{aligned}$$

Then $\overline{u} \in X_\lambda$ and $\Phi_\Omega(u) = \Phi_\lambda(\overline{u})$ for each $\lambda > 0$. Hence, for each $\gamma \in \Gamma_\Omega$, it follows that $\overline{\gamma} \in \Gamma_\lambda$. This fact shows that

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \leq \inf_{\gamma\in\Gamma_\Omega} \max_{t\in[0,1]} \Phi_\Omega(\gamma(t)) = c_\Omega, \tag{4.7}$$

for every $\lambda > 0$, which implies that, up to a subsequence, $\Phi_{\lambda_n}(u_n) = d \in [0, c_\Omega]$ as $n \to +\infty$. Since u_n satisfies (4.3) with $\tau_n = 0$, it follows that $(u_n)_{n\in\mathbb{N}}$ is indeed a $(\mathrm{PS})_{d,\infty}$-sequence.

Finally, by Lemma 3.4, we have $d > 0$, hence $d \geq c_\Omega$ from Lemma 4.2. Then, from the last inequality and (4.7), we obtain $d = c_\Omega$ and $(u_n)_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_{c_\Omega,\infty}$-sequence. Again by Lemma 4.2, there exists $u_\Omega \in \mathrm{BL}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u_\Omega$ in $L^q_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ for $1 \leq q < 1^*$, $u_\Omega \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, u_Ω is a solution of problem (4.1), and

$$\|u_n\|_{\lambda_n} - \|u_\Omega\|_\Omega \to 0 \quad \text{as } n \to +\infty.$$

Hence, Theorem 1.2 is proved. $\square$

Acknowledgements

This work is supported by the Research Fund of the National Natural Science Foundation of China (No. 11861046), Chongqing Municipal Education Commission (No. KJQN20190081), Chongqing Technology and Business University(No. CTBUZDPTTD201909).

References

[1] C.O. Alves, D.C. de Morais Filho, M.A.S. Souto, *Multiplicity of positive solutions for a class of problems with critical growth in* $\mathbb{R}^N$, Proc. Edinb. Math. Soc. (2) **52** (2009), no. 1, 1–21.

[2] C.O. Alves, G. Figueiredo, M.T.O. Pimenta, *Existence and profile of ground-state solutions to a 1-Laplacian problem in* $\mathbb{R}^N$, Bull. Braz. Math. Soc. (N.S.) **51** (2020), no. 3, 863–886.

[3] C.O. Alves, A.B. Nóbrega, M. Yang, *Multi-bump solutions for Choquard equation with deepening potential well*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 48, 28 pp.

[4] C.O. Alves, M.A.S. Souto, *Multiplicity of positive solutions for a class of problems with exponential critical growth in* $\mathbb{R}^2$, J. Differential Equations **244** (2008), no. 6, 1502–1520.

[5] C.O. Alves, M. Yang, *Multiplicity and concentration of solutions for a quasilinear Choquard equation*, J. Math. Phys. **55** (2014), no. 6, 061502, 21 pp.

[6] G.C. Anthal, J. Giacomoni, K. Sreenadh, *Some existence and uniqueness results for logistic Choquard equations*, Rend. Circ. Mat. Palermo (2) **71** (2022), no. 3, 997–1034.

[7] G. Anzellotti, *The Euler equation for functionals with linear growth*, Trans. Amer. Math. Soc. **290** (1985), no. 2, 483–501.

[8] Y. Bai, N.S. Papageorgiou, S. Zeng, *A singular eigenvalue problem for the Dirichlet (p,q)-Laplacian*, Math. Z. **300** (2022), no. 1, 325–345.

[9] S. Barile, M.T.O. Pimenta, *Some existence results of bounded variation solutions to 1-biharmonic problems*, J. Math. Anal. Appl. **463** (2018), no. 2, 726–743.

[10] T. Bartsch, A. Pankov, Z. Wang, *Nonlinear Schrödinger equations with steep potential well*, Commun. Contemp. Math. **3** (2001), no. 4, 549–569.

[11] T. Bartsch, Z. Wang, *Existence and multiplicity results for some superlinear elliptic problems on* $\mathbb{R}^N$, Comm. Partial Differential Equations **20** (1995), no. 9-10, 1725–1741.

[12] T. Bartsch, Z. Wang, *Multiple positive solutions for a nonlinear Schrödinger equation*, Z. Angew. Math. Phys. **51** (2000), no. 3, 366–384.

[13] J. Cen, A.A. Khan, D. Motreanu, S. Zeng, *Inverse problems for generalized quasi-variational inequalities with application to elliptic mixed boundary value systems*, Inverse Problems **38** (2022), no. 6, Paper No. 065006, 28 pp.

[14] K.C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), no. 1, 102–129.

[15] G.S.A. Costa, *Existence and concentration of ground state solutions for an equation with steep potential well and exponential critical growth*, J. Math. Anal. Appl. **518** (2023), no. 2, Paper No. 126708, 17 pp.

[16] Y. Ding, K. Tanaka, *Multiplicity of positive solutions of a nonlinear Schrödinger equation*, Manuscripta Math. **112** (2003), no. 1, 109–135.

[17] G.M. Figueiredo, M.T.O. Pimenta, *Existence of bounded variation solutions for a 1-Laplacian problem with vanishing potentials*, J. Math. Anal. Appl. **459** (2018) no. 2, 861–878.

[18] G.M. Figueiredo, M.T.O. Pimenta, *Nehari method for locally Lipschitz functionals with examples in problems in the space of bounded variation functions*, NoDEA Nonlinear Differential Equations Appl. **25** (2018), no. 5, Paper No. 47, 18 pp.

[19] H. Fröhlich, *Theory of electrical breakdown in ionic crystals*, Proc. Roy. Soc. Edinburgh Sect. A **160** (1937), no. 901, 230–241.

[20] H. Hajaiej, *Schrödinger systems arising in nonlinear optics and quantum mechanics. Part I*, Math. Models Methods Appl. Sci. **22** (2012), no. 7, 1250010, 27 pp.

[21] E.J. Hurtado, M.T.O. Pimenta, O.H. Miyagaki, *On a quasilinear elliptic problem involving the 1-biharmonic operator and a Strauss type compactness result*, ESAIM Control Optim. Calc. Var. **26** (2020), Paper No. 86, 20 pp.

[22] H. Jia, X. Luo, *Existence and concentrating behavior of solutions for Kirchhoff type equations with steep potential well*, J. Math. Anal. Appl. **467** (2018), no. 2, 893–915.

[23] J. Lee, J.M. Kim, J.H. Bae, K. Park, *Existence of nontrivial weak solutions for a quasilinear Choquard equation*, J. Inequal. Appl. **2018**, Paper No. 42, 20 pp.

[24] S. Liang, B. Zhang, *Soliton solutions for quasilinear Schrödinger equations involving convolution and critical nonlinearities*, J. Geom. Anal. **32** (2022), no. 1, Paper No. 9, 48 pp.

[25] E.H. Lieb, M. Loss, "Analysis", American Mathematical Society, Providence, RI, 2001.

[26] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *"Nonlinear analysis—theory and methods"*, Springer, Cham, 2019.

[27] E. Parini, B. Ruf, C. Tarsi, *The eigenvalue problem for the 1-biharmonic operator*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 2, 307–332.

[28] E. Parini, B. Ruf, C. Tarsi, *Limiting Sobolev inequalities and the 1-biharmonic operator*, Adv. Nonlinear Anal. **3** (2014), suppl. 1, s19–s36.

[29] E. Parini, B. Ruf, C. Tarsi, *Higher-order functional inequalities related to the clamped 1-biharmonic operator*, Ann. Mat. Pura Appl. (4) **194** (2015), no. 6, 1835–1858.

[30] V.D. Rădulescu, D.D. Repovš, *"Partial Differential Equations with Variable Exponents"*, CRC, Boca Raton, 2015.

[31] V.D. Rădulescu, C. Vetro, *Anisotropic Navier Kirchhoff problems with convection and Laplacian dependence*, Math. Methods Appl. Sci. **46** (2023), no. 1, 461–478.

[32] X. Yang, X. Tang, G. Gu, *Multiplicity and concentration behavior of positive solutions for a generalized quasilinear Choquard equation*, Complex Var. Elliptic Equ. **65** (2020), no. 9, 1515–1547.

[33] S. Zeng, S.a. Migórski, A.A. Khan, *Nonlinear quasi-hemivariational inequalities: existence and optimal control*, SIAM J. Control Optim. **59** (2021), no. 2, 1246–1274.

[34] J. Zhang, Z. Lou, *Existence and concentration behavior of solutions to Kirchhoff type equation with steep potential well and critical growth*, J. Math. Phys. **62** (2021), no. 1, Paper No. 011506, 14 pp.

(H. Tao) School of Mathematics and Statistics & Chongqing Key Laboratory of Economic and Social Application Statistics, Chongqing Technology and Business University, Chongqing 400067, China

(L. Li) School of Mathematics and Statistics & Chongqing Key Laboratory of Economic and Social Application Statistics, Chongqing Technology and Business University, Chongqing 400067, China
Email address: `linli@ctbu.edu.cn,lilin420@gmail.com`

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
Email address: `winkert@math.tu-berlin.de`