MULTIPLE BOUND STATES FOR A CLASS OF FRACTIONAL CRITICAL SCHRÖDINGER-POISSON SYSTEMS WITH CRITICAL FREQUENCY

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Abstract. In this paper we study the fractional Schrödinger-Poisson system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases}$$

where $s \in (0,1)$, $\varepsilon > 0$ is a small parameter, $2_s^* = \frac{6}{3-2s}$ is the critical Sobolev exponent and $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$ is a nonnegative function which may be zero in some regions of $\mathbb{R}^3$, e.g., it is of the critical frequency case. By virtue of a new global compactness lemma, and the Lusternik-Schnirelmann category theory, we relate the number of bound state solutions with the topology of the zero set where V attains its minimum for small values of ε.

1. Introduction and main results

This paper deals with the following fractional Schödinger-Poisson system with doubly critical growth and critical frequency of the form

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$ and $2_s^* = \frac{6}{3-2s}$ denotes the critical Sobolev exponent. Here, the fractional Laplacian operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s w(x) = C_s \text{ P.V.} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}y, \quad w \in \mathcal{S}(\mathbb{R}^3),$$

where P.V. stands for the Cauchy principal value, C_s is a normalizing constant and $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying functions. In (1.1), the first equation is a nonlinear fractional Schrödinger equation introduced by Laskin [29] which appeared in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes, see Laskin [28], in which the potential satisfies a nonlinear fractional Poisson equation. For this reason, system (1.1) is called a fractional Schrödinger-Poisson system, also known as the fractional Schrödinger-Maxwell system, as a model describing solitary waves for the nonlinear stationary fractional Schrödinger equation interacting with the electrostatic field, which arises in many mathematical physics context and which is not only a physically relevant generalization of the classical NLS but also an important model in the study of fractional quantum mechanics. It is worth pointing out that the fractional Laplace operator $(-\Delta)^s$ becomes the classical Laplace operator $-\Delta$ as $s \to 1^-$, see Proposition 4.4 of Di Nezza-Palatucci-Valdinoci [18]. From a probabilistic point of view, the fractional Laplace operator could be viewed as the infinitesimal generator of a Lévy process, see, for example, Applebaum [5] and Bertoin [8]. This operator arises in the description of various phenomena in applied sciences, such as plasma physics (see Kurihura [27]), flame propagation (see Caffarelli-Roquejoffre-Sire [10]), financial (see Cont-Tankov [17]), anomalous diffusion (see Metzler-Klafter [37]), obstacle problems (see Silvestre [47]), conformal geometry, and minimal

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surfaces (see Chang-González [15]). For more details on fractional Laplacian operators we refer the readers to Di Nezza-Palatucci-Valdinoci [18], Molica Bisci-Rădulescu-Servadei [38] and the references therein.

Note that, even if $s = 1$, (1.1) reduces to the following classical Schödinger-Poisson system with nonlocal critical term

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \phi|u|^3u = f(u) + |u|^4u, & x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

which was of interest to a number of authors. Indeed it can be used as a model to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and in plasma physics. We refer to Li-Li-Shi [30, 31] for more details in the physics aspects. In 2020, Feng [19] considered (1.2) with $f \in C(\mathbb{R},\mathbb{R})$ being subcritical growth and $V \in C(\mathbb{R}^3,\mathbb{R})$ satisfying the global condition

(V) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0.$

This condition was first introduced by Rabinowitz [44]. By employing the concentration-compactness principle and the mountain pass theorem, Feng [19] proved the existence of positive ground state solutions under condition (V) and under some appropriate subcritical growth conditions on the nonlinearity f. Almost at the same time, Feng [20] revisited problem (1.2) (with $\varepsilon = 1$), when the potential V is neither coercive nor periodic and asymptotic periodic. Based on the modified concentration-compactness principle and the mountain pass theorem the existence of ground state solution has been shown.

There are also some papers concerning the Schrödinger-Poisson system with nonlocal critical term, but without the Sobolev critical term $|u|^4u$. Li-Li-Shi [30, 31] studied the existence of positive solutions to the following Schrödinger-Poisson-type system with nonlocal critical term

$$\begin{cases} -\Delta u + bu + q\phi|u|^3u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

whereby the proofs are using variational methods which do not require any compactness conditions. Moreover, Guo [23] obtained two positive solutions of (1.3) if $b = 0$, $q = -1$ and f is a sublinear term of the form $\lambda Q(x)|u|^{p-2}u$ by applying the Nehari manifold and variational methods. In [35], Liu investigated the following Schrödinger-Poisson-type system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

and obtained the existence of positive solutions by using the mountain pass theorem and the concentration-compactness principle. Furthermore, Li-He [34] studied the existence and multiplicity of semiclassical state solutions for (1.4) by using variational methods along with the Lusternik-Schnirelmann category theory.

From another perspective, systems (1.2), (1.3) and (1.4) can be rewritten as the following Choquard-type equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}(I_\alpha * F(u))f(u) + h(u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

which was introduced as an approximation to the Hartree-Fock theory of one component plasma, where I_α stands for the Riesz potential. Such a class of equations was already proposed by Pekar in 1954 in [41] to study the quantum theory of a polaron at rest. Later, by using symmetry decreasing rearrangement inequalities, Lieb [32] investigated the existence and uniqueness for a class of nonlinear Choquard equation. Recently, Cassani-Zhang [12] showed the existence and concentration behavior of ground state solutions for equation (1.5). Under appropriate conditions on α and f, Alves-Yang [1] presented a new concentration behavior of solutions for

(1.5) through variational methods. By means of a new nonlocal penalization technique, Moroz-Van Schaftingen [39] proved that (1.5) admits a family of solutions concentrating to the local minimum of V.

Note that in the aforementioned Schrödinger-Poisson systems in (1.2)-(1.4), the second equation has a nonlocal critical growth in the sense of the Hardy-Littlewood-Sobolev inequality. On the other hand, the following classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

with the Poisson equation being square growth, has been studied extensively by many authors over the past decades. As a model describing the interaction of a charge particle with an electromagnetic field, it arises in many mathematical physics context, see, for example, Benci-Fortunato [7]. The existence and multiplicity of solutions for the Schrödinger-Poisson system (1.6) under variant assumptions on V, K and f has been widely investigated by numerous authors, which have been developed many effective methods to study equations or systems with nonlocal terms. We refer the readers to works of Ambrosetti [2], Ambrosetti-Ruiz [3], Azzollini-d'Avenia-Vaira [6], Cerami-Vaira [13], Ruiz [45], Wang-Chen-Liao [49], Zhang-Ma-Xie [54] and the references cited therein.

For the case $s \in (0,1)$, only few recent papers considered the fractional Schrödinger-Poisson system like (1.1). It is a fundamental equation in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes, see Caffarelli-Silvestre [11] and Laskin [28, 29]. Zhang-do Ó-Squassina [53] considered, by using a perturbation approach, the existence and the asymptotical behaviors of positive solutions to the fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.7}$$

with $V(x) = 0$, a parameter $K(x) = \lambda > 0$, and a general subcritical or critical nonlinearity f. Teng [48] studied the existence of ground state solutions of (1.7) with $K(x) = 1$ and $f(x,u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u, q \in (2, 2_s^*)$, by using the Pohozaev-Nehari manifold, the monotonic trick and the global compactness lemma. For further results on the existence and multiplicity of semiclassical bound states of (1.7), we refer to Liu-Zhang [36], Murcia-Siciliano [40] and Yang-Yu-Zhao [52], see also the papers of Ambrosio [4], Chen-Li-Peng [16], Ghimenti-Van Schaftingen [22], Guo-Li [24], Zhang-Zhang [55] and the references therein.

Recently, Qu-He [43] studied the following fractional Schrödinger-Poisson system with doubly critical growth

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u + f(u), & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \tag{1.8}$$

and showed the existence and multiplicity of concentrating solutions by using the Nehari manifold and the Lusternik-Schnirelmann category theory. In 2021, He [25] obtained a mountain pass solution to (1.8) (if $\varepsilon = 1$) with the help of the concentration-compactness principle. Feng-Yeng [21] considered a class of fractional Schrödinger-Poisson type systems with doubly critical terms

$$\begin{cases} (-\Delta)^s u + V(x)u - \phi|u|^{2_s^*-3}u = K(x)|u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $s \in (\frac{3}{4}, 1)$, $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$, $K \in L^\infty(\mathbb{R}^3)$ and $\inf_{x\in\mathbb{R}^3} K(x) = K_0 > 0$. By applying the concentration-compactness principle and variational methods, the authors derived the existence of one ground state solution to the system (1.9). We remark that the novelty of (1.8) and (1.9) is the fact that the second equation has nonlocal critical growth and is driven by the nonlocal operator $(-\Delta)^s$, which makes the study of problem (1.8) and (1.9) more interesting and challenging. As we already observed, the previous results on the existence and multiplicity of solutions for systems (1.8) and (1.9) were mainly focused on the existence of ground state solutions under the condition (V). A natural question arisen spontaneously is that, whether or not it admits multiple high energy solutions for (1.9) with critical frequency, that is, $V(\cdot) \geq 0$, $V \not\equiv 0$ on $\mathbb{R}^3$, and doubly critical growth. This is still an open problem and the purpose of this paper is to fill this gap. To be more precise, we would like to establish the existence and multiplicity of bound state solutions to (1.1) when V is of critical frequency case.

Throughout this paper, we suppose that V satisfies the following assumptions:

(V_1) $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$ and $V(\cdot) \geq 0$ on $\mathbb{R}^3$;
(V_2) the set $M = \{x \in \mathbb{R}^3 : V(x) = 0\}$ is nonempty and bounded.

Based on the assumptions on $V(\cdot)$ above, it is easy to verify that the following equation can be seen as the limit equation of (1.1):

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u - \phi|u|^{2_s^*-3}u = |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3. \end{cases} \tag{1.10}$$

In the sequel, for each $\varepsilon > 0$, we define the energy functionals associated to (1.1) and (1.10) by J_ε and $J_{\varepsilon,\infty}$, respectively,

$$J_\varepsilon(u) = \frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x - \frac{\varepsilon^{-2s}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x,$$

and

$$J_{\varepsilon,\infty}(u) = \frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \,\mathrm{d}x - \frac{\varepsilon^{-2s}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}\,\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\,\mathrm{d}x,$$

where ϕ_u is given in (2.1). Furthermore, we introduce the Nehari manifold associated to J_ε, $J_{\varepsilon,\infty}$, respectively, as

$$\mathcal{N}_\varepsilon = \left\{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\} \,:\, J'_\varepsilon(u)u = 0\right\}, \quad \mathcal{N}_\varepsilon^\infty = \left\{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\} \,:\, J'_{\varepsilon,\infty}(u)u = 0\right\}.$$

For every $\varepsilon > 0$, we define the infimum

$$m_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) \quad \text{and} \quad m_\varepsilon^\infty := \inf_{u\in\mathcal{N}_{\varepsilon,\infty}} J_{\varepsilon,\infty}(u).$$

If Y is a closed set of a topological space X, we denote by $\mathrm{cat}_X(Y)$ the Lusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. For any fixed $\mu > 0$, denote $M_\mu = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \leq \mu\}$. Now, our main result can be formulated as follows.

Theorem 1.1. *Let $s \in (\frac{1}{2}, 1)$ and suppose that the conditions* (V_1), (V_2) *are satisfied. Then, the following properties hold:*

(i) *$m_\varepsilon = m_\varepsilon^\infty$ and m_ε is not achieved, for each $\varepsilon > 0$.*
(ii) *For any fixed $\mu > 0$, there exist $\varepsilon_\mu > 0$ such that for any $\varepsilon \in (0, \varepsilon_\mu)$, system* (1.1) *possesses at least $\mathrm{cat}_{M_\mu}(M)$ bound states in $D^{s,2}(\mathbb{R}^3)$.*

The proof of Theorem 1.1 is of variational character. From a technical perspective, the characteristics of the double non-localities from nonlocal critical convolution term $\phi_u|u|^{2_s^*-3}u$ and the Sobolev critical term $|u|^{2_s^*-2}u$ in system (1.1) make it more complicated to deal with the convergence of (PS)-sequences, due to the lack of compactness for the embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$. Moreover, it is difficult to check the $(\mathrm{PS})_c$-condition since the nodal solutions of (1.1) no longer has the double energy properties, as we know the energy doubling characteristic plays a key role in proving the main results, see, for example, Chen-Li-Peng [16], Qu-He [42] and Zhang-Zhang [55]. In order to overcome these difficulties, we shall employ some ideas from Qu-He [43] and establish a new global compactness lemma in the fractional case and some estimates become more subtle and delicate to be established. We shall construct two barycenter functions and use the Lusternik-Schnirelmann category theory to obtain the desired results. As far as we know, the multiplicity of high energy solutions for system (1.1) has not been studied in the literature.

The paper is organized as follows: In Section 2 we give some preliminary results while in Section 3, we introduce the corresponding limit problem and prove some useful lemmas. In Section 4, we present a new global compactness lemma which describes the behavior of (PS)-sequences and we regain the compactness if the functional energy lies in a suitable interval. In Section 5, we first define two barycenter functions and give some necessary estimates. Then, we give the proof of Theorem 1.1 by means of the Lusternik-Schnirelmann category theory.

2. Notations and preliminary results

To begin with, we present some notations which will be used throughout this paper. First, we denote by $C, C_i > 0$, $i = 1, 2, \cdots$, different positive constants whose values may vary from line to line and are not essential to the proofs of our results. We denote by $L^p = L^p(\mathbb{R}^3)$ for $p \in (1, \infty]$ the Lebesgue spaces with the standard norm $\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$.

We recall that the fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined as

$$D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) \,:\, \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x < \infty \right\},$$

equipped with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{3-2s}} \,\mathrm{d}x \,\mathrm{d}y,$$

see Di Nezza-Palatucci-Valdinoci [18] and Molica Bisci-Rădulescu-Servadei [38]. The Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ is continuous and the best constant S is given by

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} \,\mathrm{d}x\right)^{\frac{2}{2_s^*}}},$$

see Servadei-Valdinoci [46]. By the Lax-Milgram theorem, we know that for each $u \in D^{s,2}(\mathbb{R}^3)$, there exists a unique solution $\phi = \phi_u \in D^{s,2}(\mathbb{R}^3)$ such that $(-\Delta)^s \phi_u = |u|^{2_s^*-1}$ in a weak sense. The function ϕ_u can been expressed explicitly as

$$\phi_u(x) = K_s \int_{\mathbb{R}^3} \frac{|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}} \,\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^3, \tag{2.1}$$

where $K_s = \pi^{-\frac{3}{2}} 2^{-2s} \Gamma(3-2s)(\Gamma(s))^{-1}$, see He [25] for example. The function ϕ_u has the following characteristics, see Qu-He [43] for the proof.

Lemma 2.1. *Let $u \in D^{s,2}(\mathbb{R}^3)$, then the following hold:*

(i) $\phi_{tu} = t^{2_s^*-1} \phi_u$ *for all* $t \geq 0$.

(ii) *For any $u \in D^{s,2}(\mathbb{R}^3)$, we have* $\|\phi_u\| \leq S^{-\frac{1}{2}} \|u\|_{2_s^*}^{2_s^*-1}$ *and*

$$\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} \,\mathrm{d}x \leq S^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} \,\mathrm{d}x \right)^{\frac{2_s^*-1}{2_s^*}} \|\phi_u\| \leq S^{-1} \|u\|_{2_s^*}^{2(2_s^*-1)}.$$

(iii) *If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$ and $\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0$ in $D^{s,2}(\mathbb{R}^3)$.*

(iv) *If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then*

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} \,\mathrm{d}x - \int_{\mathbb{R}^3} \phi_{u_n-u} |u_n - u|^{2_s^*-1} \,\mathrm{d}x - \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} \,\mathrm{d}x \to 0,$$

and

$$\phi_{u_n} |u_n|^{2_s^*-3} u_n - \phi_{u_n-u} |u_n - u|^{2_s^*-3}(u_n - u) - \phi_u |u|^{2_s^*-3} u \rightharpoonup 0$$

in $(D^{s,2}(\mathbb{R}^3))^*$, where $(D^{s,2}(\mathbb{R}^3))^*$ denotes the dual space of $D^{s,2}(\mathbb{R}^3)$.

Lemma 2.2. *If $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$, then the functional*

$$G(u) = \int_{\mathbb{R}^3} V(x)u^2 \, \mathrm{d}x$$

is weakly continuous in $D^{s,2}(\mathbb{R}^3)$.

Proof. The proof is analogous to that of Lemma 2.13 in Willem [51], we sketch it here for convenience. First, the functional G is well defined by Hölder's inequality. Let $u_n \rightharpoonup u$ be weakly in $D^{s,2}(\mathbb{R}^3)$. By virtue of the continuity of the Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, we see that $u_n \rightharpoonup u$ weakly in $L^{2_s^*}(\mathbb{R}^3)$. Thus, $u_n^2 \rightharpoonup u^2$ weakly in $L^{\frac{3}{3-2s}}(\mathbb{R}^3)$. Notice that $V \in L^{\frac{3}{2s}}(\mathbb{R}^3) = (L^{\frac{3}{3-2s}}(\mathbb{R}^3))^*$. Consequently,

$$\int_{\mathbb{R}^3} V(x)u^2 \, \mathrm{d}x \to \int_{\mathbb{R}^3} V(x)u^2 \, \mathrm{d}x \quad \text{as } n \to \infty,$$

which completes the proof. □

At the end of this paragraph, we introduce the following version of the Hardy-Littlewood-Sobolev inequality, see Lieb-Loss [33].

Proposition 2.3. *Assume that $t, r > 1$ and $0 < \alpha < N$ with $1/t + \alpha/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(t, N, \alpha, r)$ independent of f, h satisfying*

$$\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\alpha} \, \mathrm{d}x \, \mathrm{d}y \le C(t, N, \alpha, r)\|f\|_t \|h\|_r. \tag{2.2}$$

If $t = r = \frac{2N}{2N-\alpha}$, then

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{\pi}{2} - \frac{\alpha}{2})}{\Gamma(N - \frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{\pi}{2})}{\Gamma(N)} \right\}^{-1+\frac{\alpha}{N}}.$$

In this case there is equality in (2.2) if and only if $f \equiv Ch$ and

$$h(x) = A(\delta^2 + |x - x_0|^2)^{-\frac{2N-\alpha}{2}}$$

for some $A \in \mathbb{C}$, $\delta \in \mathbb{R} \setminus \{0\}$ and $x_0 \in \mathbb{R}^N$.

3. Limit problem

From the assumptions (V_1) and (V_2), the limit equation of (1.1) takes the form as

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u - \phi|u|^{2_s^*-3}u = |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

Recall that for each $\varepsilon > 0$, we have the Nehari manifolds

$$\mathcal{N}_\varepsilon = \left\{ u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} \, : \, J_\varepsilon'(u)u = 0 \right\}, \quad \mathcal{N}_\varepsilon^\infty = \left\{ u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} \, : \, J_{\varepsilon,\infty}'(u)u = 0 \right\},$$

and for each $\varepsilon > 0$, we define the infimums

$$m_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) \quad \text{and} \quad m_\varepsilon^\infty := \inf_{u \in \mathcal{N}_{\varepsilon,\infty}} J_{\varepsilon,\infty}(u).$$

Now, we study the correspondence between the ground state solutions of (3.1) with $\varepsilon = 1$, i.e.,

$$\begin{cases} (-\Delta)^s u - \phi|u|^{2_s^*-3}u = |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \tag{3.2}$$

and the ground state solutions to the following equation

$$(-\Delta)^s u = |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3. \tag{3.3}$$

The following observation is useful in the energy estimation of the corresponding functionals.

Lemma 3.1.

(i) *The ground state solutions of problem (3.2) and problem (3.3) are one-to-one correspondences.*

(ii) *Assume that u, ϕ are positive solutions of (3.2), then we have*

$$u(x) = \phi(x) = \widetilde{U}_{\delta,x_0}(x) := \kappa U_{\delta,x_0}(x) \quad \text{for some } x_0 \in \mathbb{R}^3 \text{ and } \delta > 0,$$

where the function

$$U_{\delta,x_0}(x) := \frac{d_s \delta^{\frac{3-2s}{2}}}{(\delta^2 + |x - x_0|^2)^{\frac{3-2s}{2}}}$$

solves (3.3) with the numbers

$$d_s = \left(\frac{S^{3/(2s)}\Gamma(3)}{\pi^{3/2}\Gamma(3/2)} \right)^{\frac{3-2s}{6}}, \quad \kappa = \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{4s}}.$$

Moreover, U_{δ,x_0} satisfies

$$\|U_{\delta,x_0}\|^2_{D^{s,2}} = \|U_{\delta,x_0}\|^{2^*_s}_{2^*_s} = S^{\frac{3}{2s}}.$$

Proof. (i) Assume that $w_1(x)$ is a solution of problem (3.3), then

$$w_2(x) = D_{w_1} w_1(x),$$

where D_{w_1} is given by

$$D_{w_1}^{2-2^*_s} = \frac{1}{2} \left(1 + \sqrt{1 + 4 \frac{\int_{\mathbb{R}^3} \phi_{w_1} |w_1|^{2^*_s - 1} \, dx}{\int_{\mathbb{R}^3} |w_1|^{2^*_s} \, dx}} \right) \tag{3.4}$$

solving problem (3.2). For any solutions $w_{11}(x)$ and $w_{12}(x)$ of problem (3.3), if $D_{w_{11}} w_{11}(x) = D_{w_{12}} w_{12}(x)$ holds, then we infer to

$$w_{11}(x) = \frac{D_{w_{12}}}{D_{w_{11}}} w_{12}(x).$$

Since both $w_{11}(x)$ and $w_{12}(x)$ are solutions of problem (3.3), we have $D_{w_{12}}/D_{w_{11}} = 1$ and hence $w_{11}(x) = w_{12}(x)$. On the other hand, assume that $w_2(x)$ is a solution of problem (3.2), then

$$w_1(x) = T_{w_2} w_2(x)$$

solves problem (3.3), where $T_{w_2} > 0$ fulfills

$$T_{w_2}^{2^*_s - 2} = \frac{\int_{\mathbb{R}^3} \phi_{w_2} |w_2|^{2^*_s - 1} \, dx + \int_{\mathbb{R}^3} |w_2|^{2^*_s} \, dx}{\int_{\mathbb{R}^3} |w_2|^{2^*_s} \, dx}.$$

From the observations above, we infer that $w_1(x) \xrightarrow{D_{w_1}} w_2(x) \xrightarrow{T_{w_2}} w_1(x)$.

(ii) Now, we introduce the following equation

$$\begin{cases} (-\Delta)^s u = \phi |u|^{2^*_s - 3} u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2^*_s - 1}, & x \in \mathbb{R}^3. \end{cases} \tag{3.5}$$

Assume that u, ϕ are positive solutions for (3.5), then

$$(-\Delta)^s (u - \phi) = (\phi - u) |u|^{2^*_s - 2}, \quad x \in \mathbb{R}^3.$$

Multiplying both sides of this equation by $(u - \phi)$ and integrating by part gives

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} (u - \phi)|^2 \, dx + \int_{\mathbb{R}^3} (u - \phi)^2 |u|^{2^*_s - 2} \, dx = 0.$$

Hence we can conclude that $u = \phi = U_{\delta,x_0}$ and so, (3.3) is equivalent to (3.5). Thus, the function U_{δ,x_0} satisfies

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} U_{\delta,x_0}|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \phi_{U_{\delta,x_0}} |U_{\delta,x_0}|^{2_s^*-1} \,\mathrm{d}x = \int_{\mathbb{R}^3} |U_{\delta,x_0}|^{2_s^*} \,\mathrm{d}x = S^{\frac{3}{2s}}, \tag{3.6}$$

taking the form

$$U_{\delta,x_0}(x) := \frac{d_s \delta^{\frac{3-2s}{2}}}{(\delta^2 + |x - x_0|^2)^{\frac{3-2s}{2}}}.$$

Now, testing (3.4) by $w_1(x) = U_{\delta,x_0}(x)$, by a direct calculation, we obtain

$$D_{w_1} = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{4s}} := \kappa. \tag{3.7}$$

Therefore, the ground state solution of (3.2) has the form

$$w_2(x) := \widetilde{U}_{\delta,x_0}(x) := \kappa U_{\delta,x_0}(x) \quad \text{for some } x_0 \in \mathbb{R}^3 \text{ and } \delta > 0.$$

The assertions follows. ◻

Remark 3.2. *From (3.6) and (3.7), by a simple computation, we have that*

$$\|\widetilde{U}_{\delta,x_0}\|^2 = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}$$

and $U^*_{\delta,x_0}(x) := \varepsilon^{\frac{3-2s}{2}} \widetilde{U}_{\delta,x_0}(x)$ *is the unique ground state solution of equation (3.1).*

Lemma 3.3. *The infimum* m_ε^∞ *is achieved by* $U^*_{\delta,x_0}(x) = \varepsilon^{\frac{3-2s}{2}} \widetilde{U}_{\delta,x_0}(x)$ *with*

$$m_\varepsilon^\infty = J_{\varepsilon,\infty}(U^*_{\delta,x_0}) = \varepsilon^3 \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left[12 + (1-\sqrt{5})(3-2s)\right]}{6(3+2s)} S^{\frac{3}{2s}}.$$

Proof. By Lemma 3.1 and Remark 3.2, we see that

$$U^*_{\delta,x_0}(x) = \varepsilon^{\frac{3-2s}{2}} \widetilde{U}_{\delta,x_0}(x) \tag{3.8}$$

is the positive ground solution of equation (3.1). Moreover, by (3.6), (3.7) and (3.8), a direct computation shows that

$$\begin{aligned}
&J_{\varepsilon,\infty}(U^*_{\delta,x_0}) \\
&= \frac{\varepsilon^3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \widetilde{U}_{\delta,x_0}|^2 \,\mathrm{d}x - \frac{\varepsilon^3}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{\widetilde{U}_{\varepsilon,x_0}} |\widetilde{U}_{\varepsilon,x_0}|^{2_s^*-1} \,\mathrm{d}x - \frac{\varepsilon^3}{2_s^*} \int_{\mathbb{R}^3} |\widetilde{U}_{\varepsilon,x_0}|^{2_s^*} \,\mathrm{d}x \\
&= \varepsilon^3 \left(\frac{\kappa^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} U_{\delta,x_0}|^2 \,\mathrm{d}x - \frac{\kappa^{2(2_s^*-1)}}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{U_{\varepsilon,x_0}} |U_{\varepsilon,x_0}|^{2_s^*-1} \,\mathrm{d}x - \frac{\kappa^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |U_{\varepsilon,x_0}|^{2_s^*} \,\mathrm{d}x \right) \\
&= \varepsilon^3 \left(\frac{\kappa^2}{2} S^{\frac{3}{2s}} - \frac{\kappa^{2(2_s^*-1)}}{2(2_s^*-1)} S^{\frac{3}{2s}} - \frac{\kappa^{2_s^*}}{2_s^*} S^{\frac{3}{2s}} \right) \\
&= \varepsilon^3 \left(\frac{\kappa^2}{2} - \frac{\kappa^{2(2_s^*-1)}}{2(2_s^*-1)} - \frac{\kappa^{2_s^*}}{2_s^*} \right) S^{\frac{3}{2s}} \\
&= \varepsilon^3 \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left[12 + (1-\sqrt{5})(3-2s)\right]}{6(3+2s)} S^{\frac{3}{2s}}.
\end{aligned}$$

◻

Lemma 3.4. *Suppose that the conditions* (V_1) *and* (V_2) *are satisfied. Then* $m_\varepsilon = m_\varepsilon^\infty$ *and* m_ε *is not attained.*

Proof. Let $u \in D^{s,2}(\mathbb{R}^3)$, we define the function

$$\begin{aligned}\Psi(t) &:= \langle J_\varepsilon'(tu), tu\rangle \\ &= t^2\varepsilon^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x + t^2\int_{\mathbb{R}^3}V(x)u^2\,\mathrm{d}x - \varepsilon^{-2s}t^{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}\,\mathrm{d}x \\ &\quad - t^{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\,\mathrm{d}x \\ &= at^2 - bt^{22^*_{\alpha,s}} - ct^{2_s^*},\end{aligned}$$

where

$$a = \int_{\mathbb{R}^3}(\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2)\,\mathrm{d}x,\quad b = \varepsilon^{-2s}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}\,\mathrm{d}x \quad\text{and}\quad c = \int_{\mathbb{R}^3}|u|^{2_s^*}\,\mathrm{d}x.$$

It is easy to check that there exist unique $t(u) > 0, s(u) > 0$ such that $t(u)u \in \mathcal{N}_\varepsilon$, $s(u)u \in \mathcal{N}_\varepsilon^\infty$ and

$$J_\varepsilon(t(u)u) = \max_{t>0} J_\varepsilon(tu),\quad J_{\varepsilon,\infty}(s(u)u) = \max_{t>0} J_{\varepsilon,\infty}(tu).$$

Moreover, for each $u \in \mathcal{N}_\varepsilon$, we derive that

$$\begin{aligned}m_\varepsilon^\infty &\le J_{\varepsilon,\infty}(s(u)u) \\ &\le \frac{\varepsilon^{2s}}{2}\|s(u)u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|s(u)u|^2\,\mathrm{d}x \\ &\quad - \varepsilon^{-2s}\frac{|s(u)|^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}\,\mathrm{d}x - \frac{|s(u)|^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\,\mathrm{d}x \\ &= J_\varepsilon(s(u)u) \\ &\le J_\varepsilon(u),\end{aligned}$$

consequently, one has

$$m_\varepsilon^\infty \le m_\varepsilon. \tag{3.9}$$

Let w be a positive solution of (1.10) centered at the origin and take a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $|x_n| \to \infty$ as $n \to \infty$. Set $w_n(x) = \varepsilon^{\frac{3-2s}{2}}w(x - x_n)$ and $t_n = t(w_n)$. Clearly, $w_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$ and by Lemma 2.2, we have

$$\int_{\mathbb{R}^3}V(x)w_n^2\,\mathrm{d}x \to 0.$$

At this point, we have

$$\begin{aligned}J_\varepsilon(t_nw_n) = \frac{t_n^2\varepsilon^{2s}}{2}\|w_n\|^2 + \frac{t_n^2}{2}o_n(1) - \frac{\varepsilon^{-2s}|t_n|^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2_s^*-1}\,\mathrm{d}x \\ - \frac{|t_n|^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|w_n|^{2_s^*}\,\mathrm{d}x.\end{aligned} \tag{3.10}$$

Since $w_n \in \mathcal{N}_\varepsilon^\infty$ and $t_nw_n \in \mathcal{N}_\varepsilon$, we have

$$\varepsilon^{2s}\|w_n\|^2 = \varepsilon^{-2s}\int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2_s^*-1}\,\mathrm{d}x + \int_{\mathbb{R}^3}|w_n|^{2_s^*}\,\mathrm{d}x, \tag{3.11}$$

and

$$\varepsilon^{2s}t_n^2\|w_n\|^2 + t_n^2\int_{\mathbb{R}^3}V(x)w_n^2\,\mathrm{d}x = \varepsilon^{-2s}t^{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2_s^*-1}\,\mathrm{d}x + t^{2_s^*}\int_{\mathbb{R}^3}|w_n|^{2_s^*}\,\mathrm{d}x. \tag{3.12}$$

From (3.11) and (3.12) it follows

$$\varepsilon^{2s}\left[t_n^{2(2_s^*-2)} - 1\right]\|w_n\|^2 + \left[t_n^{2_s^*-2} - t_n^{2(2_s^*-2)}\right]\int_{\mathbb{R}^3}|w_n|^{2_s^*}\,\mathrm{d}x = \int_{\mathbb{R}^3}V(x)w_n^2\,\mathrm{d}x = o_n(1),$$

which implies that $t_n \to 1$ as $n \to \infty$. By (3.10), we arrive that $\lim_{n\to\infty} J_\varepsilon(w_n) = m_\varepsilon^\infty$. Hence, we deduce to $m_\varepsilon \le m_\varepsilon^\infty$, and then $m_\varepsilon = m_\varepsilon^\infty$ by (3.9).

Next, we show that m_ε cannot be achieved. Assuming the conclusion is not true, then there exists $u_0 \in \mathcal{N}_\varepsilon$ such that $J_\varepsilon(u_0) = m_\varepsilon = m_\varepsilon^\infty$. Therefore, by $s(u_0)u_0 \in \mathcal{N}_\varepsilon^\infty$, we infer to

$$\begin{aligned} m_\varepsilon^\infty &\le J_{\varepsilon,\infty}(s(u_0)u_0) \\ &\le \frac{\varepsilon^{2s}}{2}\|s(u_0)u_0\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|s(u_0)u_0|^2\,\mathrm{d}x \\ &\quad - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{s(u_0)u_0}|s(u_0)u_0|^{2_s^*-1}\,\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|s(u_0)u_0|^{2_s^*}\,\mathrm{d}x \\ &= J_\varepsilon(s(u_0)u_0) \le J_\varepsilon(u_0) = m_\varepsilon^\infty. \end{aligned}$$

Inevitably, we conclude that

$$\int_{\mathbb{R}^3} V(x)|s(u_0)u_0|^2\,\mathrm{d}x = 0, \quad s(u_0) = 1 \quad \text{imply} \quad u_0 \equiv 0 \quad \text{on } \mathbb{R}^3 \setminus M.$$

Therefore, $u_0 \in \mathcal{N}_\varepsilon^\infty$ with $J_\varepsilon(u_0) = m_\varepsilon^\infty$. Then, by Lemma 3.3, $u_0(x) = \varepsilon^{\frac{3-2s}{2}}\widetilde{U}_{\delta,x_0}(x) > 0$ for all $x \in \mathbb{R}^3$ and for some $\delta > 0$, $x_0 \in \mathbb{R}^3$, which contradicts $u_0 \equiv 0$ on $\mathbb{R}^3 \setminus M$. □

Corollary 3.5. *Let $c > 0$ and $\{u_n\}_{n\in\mathbb{N}}$ be a $(\mathrm{PS})_c$-sequence of J_ε restricted on $\mathcal{N}_\varepsilon$. Then $\{u_n\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_c$-sequence of J_ε in $D^{s,2}(\mathbb{R}^3)$. If u is a critical point of J_ε restricted on $\mathcal{N}_\varepsilon$, then u must be a critical point of J_ε in $D^{s,2}(\mathbb{R}^3)$.*

Proof. If $\{u_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence of J_ε restricted on $\mathcal{N}_\varepsilon$, then we have

$$\begin{aligned} c + o_n(1) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\mathrm{d}x - \frac{\varepsilon^{-2s}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x \\ &\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\,\mathrm{d}x, \end{aligned}$$

and

$$0 = \int_{\mathbb{R}^3}\left(\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\mathrm{d}x - \varepsilon^{-2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x - \int_{\mathbb{R}^3}|u_n|^{2_s^*}\,\mathrm{d}x.$$

Consequently, we obtain

$$\begin{aligned} c + o_n(1) &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}\left[\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right]\mathrm{d}x \\ &\quad + \left(\frac{1}{2_s^*} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x. \end{aligned}$$

Thus, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $D^{s,2}(\mathbb{R}^3)$. By virtue of $u_n \in \mathcal{N}_\varepsilon$ and by Lemma 2.1, we derive that

$$\begin{aligned} &\varepsilon^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x \\ &\le \varepsilon^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}V(x)|u_n|^2\,\mathrm{d}x \\ &= \varepsilon^{-2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x + \int_{\mathbb{R}^3}|u_n|^{2_s^*}\,\mathrm{d}x \le \varepsilon^{-2s}S^{-1}|u_n|_{2_s^*}^{2(2_s^*-1)} + |u_n|_{2_s^*}^{2_s^*} \\ &\le \varepsilon^{-2s}S^{-2_s^*}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x\right)^{2_s^*-1} + S^{-\frac{2_s^*}{2}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x\right)^{\frac{2_s^*}{2}}, \end{aligned}$$

which implies that

$$\varepsilon^{-2s}S^{-2_s^*}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x\right)^{2_s^*-2} + S^{-\frac{2_s^*}{2}}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x\right)^{\frac{2_s^*-2}{2}} \ge \varepsilon^{2s}.$$

Hence

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \,\mathrm{d}x \geq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s}. \tag{3.13}$$

On the other hand, by the Lagrangian multiplier rule, we have for $\lambda_n \in \mathbb{R}$ such that

$$o_n(1) = J'_\varepsilon(u_n) - \lambda_n F'_\varepsilon(u_n),$$

with $F_\varepsilon(u) = J'_\varepsilon(u)u$. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $D^{s,2}(\mathbb{R}^3)$, we get

$$o_n(1) = J'_\varepsilon(u_n)u_n - \lambda_n F'_\varepsilon(u_n)u_n = -\lambda_n F'_\varepsilon(u_n)u_n.$$

In view of $2_s^* > 2$, and by (3.13), we infer that

$$\begin{aligned} F'_\varepsilon(u_n)u_n &= 2\varepsilon^{2s}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \,\mathrm{d}x + 2\int_{\mathbb{R}^3} V(x)u_n^2 \,\mathrm{d}x \\ &\quad - 2(2_s^*-1)\varepsilon^{-2s}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x - 2_s^*\int_{\mathbb{R}^3} |u_n|^{2_s^*}\,\mathrm{d}x \\ &= (2-2_s^*)\varepsilon^{2s}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \,\mathrm{d}x + (2-2_s^*)\int_{\mathbb{R}^3} V(x)|u_n|^2 \,\mathrm{d}x \\ &\quad + (2-2_s^*)\varepsilon^{-2s}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x \\ &\leq (2-2_s^*)\varepsilon^{2s}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \,\mathrm{d}x \\ &\leq [2-2_s^*]\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s} < 0. \end{aligned} \tag{3.14}$$

As a result, $\lambda_n \to 0$ as $n \to \infty$. Moreover, using the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, we deduce that $\{F'_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is bounded. So, $J'_\varepsilon(u_n) \to 0$ as $n \to \infty$ and then $\{u_n\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_c$-sequence of J_ε in $D^{s,2}(\mathbb{R}^3)$.

Now, if u is a critical point of J_ε restricted on $\mathcal{N}_\varepsilon$, then there exists $\lambda \in \mathbb{R}$ so as to $J'_\varepsilon(u) = \lambda F'_\varepsilon(u)$, and

$$0 = F_\varepsilon(u) = J'_\varepsilon(u)u = \lambda F'_\varepsilon(u)u.$$

By the same calculation as in (3.14), we can obtain that

$$F'_\varepsilon(u)u \leq [2-2_s^*]\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} \varepsilon^{3-2s} < 0.$$

Hence, $\lambda = 0$ and so, $J'_\varepsilon(u) = 0$. □

4. A global compactness lemma

In this section, we intend to establish a global compactness lemma, which is useful in the analyzing the decomposition of the (PS)-sequences, and proving its compactness.

Lemma 4.1. *Let the conditions* (V_1), (V_2) *be satisfied and let, for each* $\varepsilon > 0$, $\{u_n\}_{n\in\mathbb{N}} \subset D^{s,2}(\mathbb{R}^3)$ *be a* (PS)*-sequence of* J_ε *at the level* $c > 0$. *Then, replacing* u_n *with a subsequence if necessary, there exist a number* $k \in \mathbb{N}$, *sequences of points* $x_n^1, \cdots, x_n^k \in \mathbb{R}^3$ *and radii* $r_n^1, \cdots, r_n^k$ *such that the following hold:*

(i) $u_n^0 \equiv u_n \rightharpoonup u^0$ *in* $D^{s,2}(\mathbb{R}^3)$;
(ii) $u_n^j \equiv (u_n^{j-1} - u^{j-1})_{r_n^j, x_n^j} \rightharpoonup u^j$ *in* $D^{s,2}(\mathbb{R}^3)$, $j = 1, 2, \cdots, k$;
(iii) $\|u_n\|^2 \to \sum_{j=0}^k \|u^j\|^2$;
(iv) $J_\varepsilon(u_n) \to J_\varepsilon(u^0) + \sum_{j=1}^k J_{\varepsilon,\infty}(u^j)$,

as $n \to \infty$, *where* u^0 *is a solution of equation* (1.1) *and* $u^j, 1 \le j \le k$, *are the nontrivial solutions of equation* (1.10). *Moreover, in case* $k = 0$ *the above holds without* u^j.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(\mathrm{PS})_c$-sequence for J_ε. Then, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $D^{s,2}(\mathbb{R}^3)$. Thus, we may extract a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denote by itself, such that $u_n \rightharpoonup u^0$ in $D^{s,2}(\mathbb{R}^3)$ as $n \to \infty$ and $u_n \to u^0$ a.e. in $\mathbb{R}^3$. Moreover, $J'_\varepsilon(u^0) = 0$. Indeed, for each $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} J'_\varepsilon(u_n)\varphi &= \varepsilon^{2s}\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi\,\mathrm{d}x + \int_{\mathbb{R}^3}V(x)u_n\varphi\,\mathrm{d}x \\ &\quad - \varepsilon^{-2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi\,\mathrm{d}x - \int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi\,\mathrm{d}x. \end{aligned}$$

By Lemma 2.1, it follows that $\phi_{u_n} \rightharpoonup \phi_{u^0}$ in $D^{s,2}(\mathbb{R}^3)$ and then $\phi_{u_n} \rightharpoonup \phi_{u^0}$ in $L^{2_s^*}(\mathbb{R}^3)$. Thus,

$$\int_{\mathbb{R}^3}(\phi_{u_n} - \phi_{u^0})|u^0|^{2_s^*-3}u^0\varphi \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

In view of $u_n \to u$ a.e. in $\mathbb{R}^3$ and using Hölder's inequality, we infer to

$$\begin{aligned} &\int_{\mathbb{R}^3}\left|\phi_{u_n}(|u_n|^{2_s^*-3}u_n - |u^0|^{2_s^*-3}u^0)\right|^{\frac{2_s^*}{2_s^*-1}}\,\mathrm{d}x \\ &\le C\left(|\phi_{u_n}|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}}|u_n|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}} + |\phi_{u_n}|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}}|u^0|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}}\right) \le C. \end{aligned}$$

By Proposition 5.4.7 in Willem [50], one has $\phi_{u_n}(|u_n|^{2_s^*-3}u_n - |u^0|^{2_s^*-3}u) \rightharpoonup 0$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$ and then

$$\int_{\mathbb{R}^3}\phi_{u_n}(|u_n|^{2_s^*-3}u_n - |u^0|^{2_s^*-3}u^0)\varphi\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

This together with (4.1) implies

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^3}\phi_{u^0}|u^0|^{2_s^*-3}u\varphi\,\mathrm{d}x \quad \text{as } n \to \infty. \tag{4.2}$$

Since $|u_n|^{2_s^*-2}u_n \rightharpoonup |u^0|^{2_s^*-2}u^0$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^3}|u^0|^{2_s^*-2}u^0\varphi\,\mathrm{d}x.$$

Therefore, combining (4.2) with the weak convergence of $u_n \rightharpoonup u^0$ in $D^{s,2}(\mathbb{R}^3)$, we derive to

$$J'_\varepsilon(u^0)\varphi = \lim_{n\to\infty}J'_\varepsilon(u_n)\varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3).$$

This implies that $J'_\varepsilon(u^0) = 0$ and u is a critical point of J_ε by the density of $C_0^\infty(\mathbb{R}^3)$ in $D^{s,2}(\mathbb{R}^3)$. Set

$$v_n^1(x) := u_n(x) - u^0(x).$$

Applying the Brezis-Lieb Lemma [9] and Lemma 2.1 (iv), we can easily deduce that

$$\|v_n^1\|^2 = \|u_n\|^2 - \|u^0\|^2 + o_n(1), \quad \|v_n^1\|_{2_s^*}^{2_s^*} = \|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*} + o_n(1), \tag{4.3}$$

$$J_\varepsilon(v_n^1) = J_\varepsilon(u_n) - J_\varepsilon(u^0) + o_n(1),$$

and

$$J'_\varepsilon(v_n^1) = J'_\varepsilon(u_n) - J'_\varepsilon(u^0) + o_n(1).$$

By $v_n^1 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$ and Lemma 2.2, one has

$$\int_{\mathbb{R}^3}V(x)|v_n^1|^2\,\mathrm{d}x = o_n(1) \quad \text{and} \quad \int_{\mathbb{R}^3}V(x)v_n^1\varphi\,\mathrm{d}x = o_n(1)\|\varphi\|$$

for each $\varphi \in D^{s,2}(\mathbb{R}^3)$. Consequently, we arrive that

$$\begin{aligned} J_{\varepsilon,\infty}(v_n^1) &= J_\varepsilon(v_n^1) + o_n(1) = J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1) \\ J'_{\varepsilon,\infty}(v_n^1) &= J'_\varepsilon(v_n^1) + o_n(1) = o_n(1). \end{aligned} \tag{4.4}$$

If $v_n^1 \to 0$ in $D^{s,2}(\mathbb{R}^3)$, then we are done. If $v_n^1 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$, then there exists $\eta > 0$ satisfying

$$J_\varepsilon(v_n^1) > \eta > 0. \tag{4.5}$$

We claim that there exist two sequences $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ verifying

$$h_n := (v_n^1)_{r_n,y_n} \rightharpoonup \omega \neq 0 \quad \text{in } D^{s,2}(\mathbb{R}^3),$$

where $(v_n^1)_{r_n,y_n} = r_n^{\frac{3-2s}{2}} v_n^1(r_n x + y_n)$. Indeed, by (4.4) we get

$$\varepsilon^{2s}\|v_n^1\|^2 = \varepsilon^{-2s}\int_{\mathbb{R}^3} \phi_{v_n^1}|v_n^1|^{2_s^*-1}\,\mathrm{d}x + \int_{\mathbb{R}^3}|v_n^1|^{2_s^*}\,\mathrm{d}x + o_n(1)$$

and

$$J_{\varepsilon,\infty}(v_n^1) = \varepsilon^{-2s}\left(\frac{1}{2} - \frac{1}{2(2_s^*-1)}\right)\int_{\mathbb{R}^3}\phi_{v_n^1}|v_n^1|^{2_s^*-1}\,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|v_n^1|^{2_s^*}\,\mathrm{d}x + o_n(1).$$

Combining (4.5), Lemma 2.1(ii) and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, we obtain that

$$0 < d_1 < |v_n^1|_{2_s^*}^{2_s^*-1} < D_1,$$

for some $d_1, D_1 > 0$. Now, we introduce the Lévy concentration function

$$\mathcal{Q}_n(r) := \sup_{x\in\mathbb{R}^3}\int_{B_r(z)}|v_n^1|^{2_s^*}\,\mathrm{d}x.$$

In view of $\mathcal{Q}_n(0) = 0$ and $\mathcal{Q}_n(\infty) > d_1^{\frac{6}{3+2s}}$, then there exist sequences $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ so as to

$$\sup_{z\in\mathbb{R}^3}\int_{B_r(z)}|v_n^1|^{2_s^*}\,\mathrm{d}x = \int_{B_{r_n}(y_n)}|v_n^1|^{2_s^*}\,\mathrm{d}x = b,$$

where

$$0 < b < \min\left\{d_1^{\frac{6}{3+2s}}, \left(\frac{S}{2C_{3,s}D_1}\right)^{\frac{6}{6s-3}}\right\}.$$

Recalling that $h_n = (v_n^1)_{r_n,y_n}$, without any loss of generality, we may assume that $h_n \rightharpoonup h$ in $D^{s,2}(\mathbb{R}^3)$ and $h_n \to h$ a.e. in $\mathbb{R}^3$. A simple calculation yields

$$\sup_{z\in\mathbb{R}^3}\int_{B_1(z)}|h_n(x)|^{2_s^*}\,\mathrm{d}x = \int_{B_1(0)}|h_n(x)|^{2_s^*}\,\mathrm{d}x = \int_{B_{r_n}(y_n)}|v_n^1|^{2_s^*}\,\mathrm{d}x = b. \tag{4.6}$$

By virtue of the invariance of the $D^{s,2}(\mathbb{R}^3)$-norms under translation and dilation, we have

$$\|v_n^1\|^2 = \|h_n\|^2, \quad \|v_n^1\|_{2_s^*}^{2_s^*} = \|h_n\|_{2_s^*}^{2_s^*}, \quad \int_{\mathbb{R}^3}\phi_{h_n}|h_n|^{2_s^*-1}\,\mathrm{d}x = \int_{\mathbb{R}^3}\phi_{v_n^1}|v_n^1|^{2_s^*-1}\,\mathrm{d}x,$$

which lead to

$$J_{\varepsilon,\infty}(h_n) = J_{\varepsilon,\infty}(v_n^1) = J_{\varepsilon,\infty}(u_n) - J_{\varepsilon,\infty}(u^0) + o_n(1), \tag{4.7}$$

and by (4.4), we get

$$J'_{\varepsilon,\infty}(h_n) = J'_{\varepsilon,\infty}(v_n^1) = o_n(1).$$

If $h=0$, then $h_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$. Suppose that $\psi \in C_0^\infty(\mathbb{R}^3)$ such that $\operatorname{supp}\psi \subset B_1(y^0)$ for some $y^0 \in \mathbb{R}^3$ and $|\nabla\psi(x)| \leq C$ for all $x \in \mathbb{R}^3$, and we obtain

$$\begin{aligned}\|\psi h_n\|^2 &= \iint_{\mathbb{R}^6} \frac{|\psi(x)h_n(x)-\psi(y)h_n(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{\mathbb{R}^6} \frac{(h_n(x)-h_n(y))(\psi^2(x)h_n(x)-\psi^2(y)h_n(y))}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+ \iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n(x)h_n(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.8}$$

By Hölder's inequality, we have

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n(x)h_n(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq \left(\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac12} \times \left(\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac12}.\end{aligned} \tag{4.9}$$

We claim that

$$\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n^2(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y = o_n(1). \tag{4.10}$$

To this aim, we derive out

$$\begin{aligned}&\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{B_1(y^0)}\int_{B_1(y^0)} \frac{|\psi(x)-\psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+2\int_{B_1(y^0)}\int_{B_1^c(y^0)} \frac{|\psi(x)-\psi(y)|^2 h_n^2(x)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq \int_{B_1(y^0)} h_n^2(x)\,\mathrm{d}x \int_{B_1(y^0)} \frac{|\nabla\psi(y+\tau(x-y))|^2|x-y|^2}{|x-y|^{3+2s}}\,\mathrm{d}y\\ &\quad+C\int_{B_1(y^0)} h_n^2(x)\,\mathrm{d}x\int_{B_1^c(y^0)} \frac{1}{|x-y|^{3+2s}}\,\mathrm{d}y\\ &\leq C_1\int_{B_1(y^0)} h_n^2(x)\,\mathrm{d}x\int_0^2 \frac{r^2}{r^{1+2s}}dr + C\int_{B_1(y^0)} h_n^2(x)\,\mathrm{d}x\int_1^\infty \frac{r^2}{r^{3+2s}}dr\\ &\leq C_2\int_{B_1(y^0)} h_n^2(x)\,\mathrm{d}x \to 0\end{aligned} \tag{4.11}$$

as $n\to\infty$, by virtue of $h_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$, where $\tau=\tau(y)\in(0,1)$. By a similar argument, we can prove

$$\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 h_n^2(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y \to 0 \quad \text{as } n\to\infty. \tag{4.12}$$

By (4.8)–(4.12), we have

$$\|\psi h_n\|^2 = \iint_{\mathbb{R}^6} \frac{(h_n(x)-h_n(y))(\psi^2(x)h_n(x)-\psi^2(y)h_n(y))}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y + o_n(1).$$

Therefore, combining the results above and Proposition 2.3, we have

$$
\begin{aligned}
\varepsilon^{2s} S\|\psi h_n\|_{2_s^*}^2 &\le \varepsilon^{2s}\|\psi h_n\|^2 \\
&= \varepsilon^{2s}\iint_{\mathbb{R}^6} \frac{(h_n(x)-h_n(y))(\psi^2(x)h_n(x)-\psi^2(y)h_n(y))}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y + o_n(1) \\
&= \varepsilon^{2s}\int_{\mathbb{R}^3} \phi_{h_n}|h_n|^{2_s^*-1}\psi^2\,\mathrm{d}x + o_n(1) \\
&\le \varepsilon^{2s} C_{3,s}\|h_n\|_{2_s^*}^{2_s^*-1}\left(\int_{\mathbb{R}^3}(|h_n|^{2_s^*-1}\psi^2)^{\frac{6}{3+2s}}\,\mathrm{d}x\right)^{\frac{3+2s}{6}} + o_n(1) \\
&= \varepsilon^{2s} C_{3,s}\|h_n\|_{2_s^*}^{2_s^*-1}\left(\int_{\mathbb{R}^3}|h_n|^{\frac{6s-3}{3-2s}\frac{6}{3+2s}}|h_n\psi|^{\frac{12}{3+2s}}\,\mathrm{d}x\right)^{\frac{3+2s}{6}} + o_n(1) \\
&\le \varepsilon^{2s} C_{3,s}|h_n|_{2_s^*}^{2_s^*-1}\left(\int_{\mathbb{R}^3}|\psi h_n|^{2_s^*}\,\mathrm{d}x\right)^{\frac{3-2s}{3}}\left(\int_{B_1(y^0)}|h_n|^{2_s^*}\,\mathrm{d}x\right)^{\frac{6s-3}{6}} + o_n(1) \\
&\le \varepsilon^{2s} C_{3,s} D_1 b^{\frac{6s-3}{6}}\left(\int_{\mathbb{R}^3}|\psi h_n|^{2_s^*}\,\mathrm{d}x\right)^{\frac{3-2s}{3}} + o_n(1) < \frac{1}{2}\varepsilon^{2s} S\|\psi h_n\|_{2_s^*}^2 + o_n(1),
\end{aligned}
$$

which shows that $h_n \to 0$ in $L^{2_s^*}_{loc}(\mathbb{R}^3)$, a contradiction to (4.6). Thus, $h \neq 0$. By (4.4) and the weakly sequentially continuity of $J'_{\varepsilon,\infty}$, we deduce $J'_{\varepsilon,\infty}(h) = 0$ and so $\{h_n\}_{n\in\mathbb{N}}$, $\{r_n^1\}_{n\in\mathbb{N}}$ and $\{y_n^1\}_{n\in\mathbb{N}}$ are the required sequences.

By iterating this procedure, we have the sequences $v_n^j = u_n^{j-1} - u^{j-1}, j \ge 2$, and the rescaled functions $u_n^j = (v_n^j)_{r_n^j, y_n^j} \rightharpoonup u^j$ in $D^{s,2}(\mathbb{R}^3)$, where u^j is a nontrivial solution to (1.10). Furthermore, by (4.3), (4.4) and (4.7), we arrive at

$$
\|u_n^j\|^2 = \|v_n^j\|^2 = \|u_n^{j-1}\|^2 - \|u^{j-1}\|^2 + o_n(1) = \cdots = \|u_n\|^2 - \sum_{i=0}^{j-1}\|u^i\|^2 + o_n(1),
$$

and

$$
\begin{aligned}
J_{\varepsilon,\infty}(u_n^j) &= J_{\varepsilon,\infty}(v_n^j) = J_{\varepsilon,\infty}(u_n^{j-1}) - J_{\varepsilon,\infty}(u^{j-1}) + o_n(1) \\
&= \cdots = J_\varepsilon(u_n) - J_\varepsilon(u^0) - \sum_{i=1}^{j-1} J_{\varepsilon,\infty}(u^i).
\end{aligned}
$$

Since

$$
\begin{aligned}
0 = J'_{\varepsilon,\infty}(u^j)u^j &= \varepsilon^{2s}\|u^j\|^2 - \varepsilon^{-2s}\int_{\mathbb{R}^3}\phi_{u^j}|u^j|^{2_s^*-1}\,\mathrm{d}x - \int_{\mathbb{R}^3}|u^j|^{2_s^*}\,\mathrm{d}x \\
&\ge \varepsilon^{2s}\|u^j\|^2 - \varepsilon^{-2s}S^{-2_s^*}\|u^j\|^{2(2_s^*-1)} - S^{-\frac{2_s^*}{2}}\|u^j\|^{2_s^*},
\end{aligned}
$$

we conclude

$$
\|u^j\|^2 \ge \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}\varepsilon^{3-2s},
$$

and the iteration must stop at some index $k \ge 0$. □

Corollary 4.2. *If $\{u_n\}_{n\in\mathbb{N}}$ is a nonnegative $(\mathrm{PS})_c$-sequence for J_ε with $c \in (m_\varepsilon, 2m_\varepsilon)$, then, for each $\varepsilon > 0$, $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact in $D^{s,2}(\mathbb{R}^3)$.*

Proof. It follows by Lemma 4.1, that there exist a number $k \in \mathbb{N}$, a solution u^0 of (1.1) and solutions $u^1, \cdots, u^k$ of (1.10), fulfilling

$$\|u_n\|^2 \to \sum_{j=0}^{k} \|u^j\|^2 \quad \text{and} \quad J_\varepsilon(u_n) \to J_\varepsilon(u^0) + \sum_{j=1}^{k} J_{\varepsilon,\infty}(u^j) = c + o_n(1),$$

as $n \to \infty$. By Lemma 3.4, if u^0 is a nontrivial solution of (1.1), then $J_\varepsilon(u^0) > m_\varepsilon$. Noticing that, for each nontrivial solution u^j of (1.10), whether u^j is positive or negative, then we have

$$J_{\varepsilon,\infty}(u^j) = \varepsilon^3 \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left[12 + (1-\sqrt{5})(3-2s)\right]}{6(3+2s)} S^{\frac{3}{2s}} = m_\varepsilon.$$

If u^j is a sign-changing solution of (1.10), we can easily deduce that

$$J_{\varepsilon,\infty}(u^j) \geq 2m_\varepsilon,$$

see Zhang-Zhang [55]. Since

$$m_\varepsilon < c < 2m_\varepsilon,$$

we must have $k = 0$, which implies that $u_n \to u^0$ in $D^{s,2}(\mathbb{R}^3)$. □

5. Proof of Theorem 1.1

In this section, we focus our attention in showing the multiplicity of high energy semiclassical states by means of the the Lusternik-Schnirelmann category theory. We begin with some preparations. For small $\mu > 0$, we may take $\rho = \rho(\mu) > 0$ such that $M_\mu \subset B_\rho(0)$. Set

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define $\beta\colon \mathcal{N}_\varepsilon \to \mathbb{R}^3$ and $\gamma\colon \mathcal{N}_\varepsilon \to \mathbb{R}^+$ by

$$\beta(u) = \frac{1}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x,$$

and

$$\gamma(u) = \frac{1}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(u)| |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x.$$

It is not difficulty to see that, for any fixed $\widetilde{U}_{\delta,z} \in D^{s,2}(\mathbb{R}^3)$, there exists a unique $t_{\delta,z}$ in $(0, +\infty)$ such that

$$\Phi_{\delta,z}(x) := t_{\delta,z} \varepsilon^{\frac{3-2s}{2}} \widetilde{U}_{\delta,z}(x) \in \mathcal{N}_\varepsilon.$$

We define the set

$$\Gamma = \Gamma(\rho, \delta_1, \delta_1) := \{(x, \delta) \in \mathbb{R}^3 \times \mathbb{R} \,:\, |x| < \rho/2, \delta_1 < \delta < \delta_2\}. \tag{5.1}$$

A direct calculation shows that, for any fixed $z \in \mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} V(x) \widetilde{U}_{\delta,z}^2(x) \, \mathrm{d}x \to 0 \quad \text{as } \delta \to 0.$$

Thus, for each $\varepsilon > 0$ and any fixed $z \in \mathbb{R}^3$, we derive that $\lim_{\delta \to 0} t_{\delta,z} = 1$. So, for each $\varepsilon > 0$, there exist $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ with $\delta_1 < \delta_2$ and $\delta_1, \delta_2 \to 0$ as $\varepsilon \to 0$, satisfying the inequality

$$\begin{aligned}&\sup\{J_\varepsilon(\Phi_{\delta,z}) : (z,\delta)\in\Lambda\}\\&<\varepsilon^3\left(\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left[12+(1-\sqrt{5})(3-2s)\right]}{6(3+2s)}S^{\frac{3}{2s}}+h(\varepsilon)\right),\end{aligned}\tag{5.2}$$

where $h(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Lemma 5.1. *There holds* $\lim_{\delta\to 0}\gamma(\Phi_{\delta,z})=0$ *uniformly for* $|z|\le\frac{\rho}{2}$.

Proof. Let $\xi>0$ be such that $0<2\xi<\rho$. By $t_{\delta,z}=1+o_\delta(1)$, we have

$$\begin{aligned}\gamma(\Phi_{\delta,z})&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}|\chi(x)-\beta(\Phi_{\delta,z})||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x+o_\delta(1)\\&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3\setminus B_\xi(z)}|\chi(x)-\beta(\Phi_{\delta,z})||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x\\&\quad+\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{B_\xi(z)}|\chi(x)-\beta(\Phi_{\delta,z})||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x+o_\delta(1)\\&:=I_1+I_2+o_\delta(1).\end{aligned}$$

By Lemma 4.1 of He-Zhao-Zou [26], we derive

$$\begin{aligned}I_1&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3\setminus B_\xi(0)}|\chi(x+z)-\beta(\Phi_{\delta,0})||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,0}|^2\,\mathrm{d}x\\&\le C\rho\int_{\mathbb{R}^3\setminus B_\xi(0)}|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,0}|^2\,\mathrm{d}x\to 0\end{aligned}$$

as $\delta\to 0$.

For I_2, by Remark 3.1 we infer to

$$\begin{aligned}\beta(\Phi_{\delta,z})&=\beta(\varepsilon^{\frac{3-2s}{2}}\widetilde{U}_{\delta,z})+o_\delta(1)\\&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}(x)|^2\,\mathrm{d}x+o_\delta(1)\\&=z+\frac{\delta^{3+2s}}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}[\chi(\delta x+z)-z]|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{1,0}|^2\,\mathrm{d}x+o_\delta(1)=z+o_\delta(1).\end{aligned}\tag{5.3}$$

As a result, we get

$$\begin{aligned}
&\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} I_2\\
&\leq \int_{B_\xi(z)} |\chi(x)-\chi(z)||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x + \int_{B_\xi(z)} |\chi(z)-\beta(\Phi_{\delta,z})||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x\\
&\leq 2\int_{B_\xi(z)} |x-z||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x + 2\xi\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}\\
&\quad + \int_{B_\xi(z)} |\chi(z)-z||(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\delta,z}|^2\,\mathrm{d}x + o_\delta(1)\\
&\leq 4\xi\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}} + o_\delta(1),
\end{aligned}$$

where we have used Lemma 2 of Chabrowski-Yang [14], which states that

$$\chi(x)-\chi(z) \leq 2|x-z| + 2\xi,\ x\in B_\xi(z).$$

As $\xi>0$ can be arbitrarily small, we infer that $\lim_{\delta\to 0}\gamma(\Phi_{\delta,z})=0$, uniformly for $|z|\leq\rho/2$. □

Now, we define a set $\widetilde{\mathcal{N}}_\varepsilon\subset\mathcal{N}_\varepsilon$ by

$$\widetilde{\mathcal{N}}_\varepsilon = \left\{u\in\mathcal{N}_\varepsilon\,:\,\varepsilon^3 c^* < J_\varepsilon(u) < \varepsilon^3(c^*+h(\varepsilon)),\ (\beta(u),\gamma(u))\in\Gamma\right\},$$

where Γ is given as (5.1) satisfying (5.2) and the number c^* is defined as

$$c^* := \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left[12+(1-\sqrt{5})(3-2s)\right]}{6(3+2s)} S^{\frac{3}{2s}}.$$

According to Lemma 5.1, we can adjust $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that $\widetilde{\mathcal{N}}_\varepsilon\neq\emptyset$ for $\varepsilon>0$ small enough.

Lemma 5.2. *There holds* $\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}}_\varepsilon}\operatorname{dist}(\beta(u),M_\mu)=0$ *for any* $\mu>0$.

Proof. Let $u_n\in\widetilde{\mathcal{N}}_{\varepsilon_n}$ be so as to

$$\operatorname{dist}(\beta(u_n),M_\mu) = \sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}}\operatorname{dist}(\beta(u),M_\mu) + o_n(1),$$

and $\varepsilon_n\to 0$ as $n\to\infty$. It is sufficient to obtain a sequence $z_n\in M_\mu$ such that

$$\beta(u_n) = z_n + o_n(1). \tag{5.4}$$

In view of $u_n\in\mathcal{N}_{\varepsilon_n}$, we have

$$\int_{\mathbb{R}^3}\left(\varepsilon_n^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\,\mathrm{d}x = \varepsilon_n^{-2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x + \int_{\mathbb{R}^3}|u_n|^{2_s^*}\,\mathrm{d}x. \tag{5.5}$$

For further discussion, we set

$$\begin{aligned}
\ell_n &:= \int_{\mathbb{R}^3}\left(\varepsilon_n^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\,\mathrm{d}x,\\
a_n &:= \varepsilon_n^{-2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x,\quad b_n := \int_{\mathbb{R}^3}|u_n|^{2_s^*}\,\mathrm{d}x.
\end{aligned}$$

Multiplying both sides of the second equation of (1.1) (with ε replaced by ε_n) by $|u_n|$ and integrating by parts, using Young's inequality, we infer, for $\varepsilon_n > 0$ small enough,

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^{2_s^*}\,\mathrm{d}x &= \varepsilon_n^{2s}\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}\phi_{u_n}(-\Delta)^{\frac{s}{2}}|u_n|\,\mathrm{d}x\\ &\le \frac{\varepsilon_n^{2s}\tau^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}|u_n||^2\,\mathrm{d}x + \frac{\varepsilon_n^{2s}}{2\tau^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_{u_n}|^2\,\mathrm{d}x\\ &\le \frac{\tau^2}{2}\int_{\mathbb{R}^3}\varepsilon_n^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x + \frac{1}{2\tau^2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x\\ &\le \frac{\tau^2}{2}\int_{\mathbb{R}^3}\left(\varepsilon_n^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\mathrm{d}x + \frac{\varepsilon_n^{-2s}}{2\tau^2}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x.\end{aligned}$$

From the definition of ℓ_n, a_n and b_n, it follows that

$$b_n \le \frac{1}{2\tau^2}a_n + \frac{\tau^2}{2}\ell_n. \tag{5.6}$$

Choosing $\tau^2 = \frac{\sqrt{5}-1}{2}$ and using (5.5) as well as (5.6), we obtain that $a_n \ge \frac{3-\sqrt{5}}{2}\ell_n$. It follows from (5.5) and (5.6) that

$$\begin{aligned}J_{\varepsilon_n}(u_n) &= J_{\varepsilon_n}(u_n) - \frac{1}{2}J'_{\varepsilon_n}(u_n)u_n\\ &= \frac{2s}{3+2s}a_n + \frac{s}{3}b_n\\ &= \frac{s(3-2s)}{3(3+2s)}a_n + \frac{s}{3}\ell_n\\ &\ge \frac{s(15-2s-\sqrt{5}(3-2s))}{6(3+2s)}\ell_n.\end{aligned} \tag{5.7}$$

On the other hand, by the estimate (3.13), we have

$$\begin{aligned}\ell_n &= \int_{\mathbb{R}^3}\left(\varepsilon_n^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\mathrm{d}x\\ &> \varepsilon_n^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x\\ &\ge \varepsilon_n^3\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}.\end{aligned} \tag{5.8}$$

Then from (5.7) and (5.8) we obtain

$$\begin{aligned}J_{\varepsilon_n}(u_n) &\ge \frac{s(15-2s-\sqrt{5}(3-2s))}{6(3+2s)}\ell_n\\ &> \varepsilon_n^3\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s(15-2s-\sqrt{5}(3-2s))}{6(3+2s)}S^{\frac{3}{2s}} := \varepsilon_n^3 c^*.\end{aligned}$$

Consequently, we have

$$\varepsilon_n^3 c^* < J_{\varepsilon_n}(u_n) - \frac{1}{2}J'_{\varepsilon_n}(u_n)u_n < \varepsilon_n^3(c^* + h(\varepsilon_n)). \tag{5.9}$$

Letting $w_n := \varepsilon_n^{\frac{2s-3}{2}}u_n$, we have from (5.7) and (5.9) that

$$\begin{aligned}\varepsilon_n^3 c^* &< J_{\varepsilon_n}(u_n) - \frac{1}{2}J'_{\varepsilon_n}(u_n)u_n\\ &= \frac{s(3-2s)}{3(3+2s)}\varepsilon_n^{-2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}\,\mathrm{d}x + \frac{s}{3}\int_{\mathbb{R}^3}\left(\varepsilon_n^{2s}|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)|u_n|^2\right)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_n^3 \left[\frac{s(3-2s)}{3(3+2s)} \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2_s^*-1}\,\mathrm{d}x + \frac{s}{3} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2\,\mathrm{d}x + \frac{s}{3}\varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x)|w_n|^2\,\mathrm{d}x \right] \\
&< \varepsilon_n^3 (c^* + h(\varepsilon_n)),
\end{aligned}
$$

from which we conclude that

$$
\varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) w_n^2\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.
$$

Again by $u_n \in \mathcal{N}_{\varepsilon_n}$, we obtain

$$
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2\,\mathrm{d}x + \varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) w_n^2\,\mathrm{d}x = \int_{\mathbb{R}^3} \phi_{w_n} |w_n|^{2_s^*-1}\,\mathrm{d}x + \int_{\mathbb{R}^3} |w_n|^{2_s^*}\,\mathrm{d}x,
$$

that is, $\{w_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence for J_1. It derives from Lemmas 3.4 and 4.1 with $\varepsilon = 1$ that there exist a number $k \in \mathbb{N}$, sequences of points $x_n^1, \cdots, x_n^k \in \mathbb{R}^3$ and radii $r_n^1, \cdots, r_n^k$ such that:

(1) $w_n^0 \equiv w_n \rightharpoonup w^0$ in $D^{s,2}(\mathbb{R}^3)$;
(2) $w_n^j \equiv (w_n^{j-1} - w^{j-1})_{r_n^j, x_n^j} \rightharpoonup w^j$ in $D^{s,2}(\mathbb{R}^3)$, $j = 1, 2, \cdots, k$;
(3) $\|w_n\|^2 \to \sum_{j=0}^k \|w^j\|^2$;
(4) $J_1(w_n) \to J_1(w^0) + \sum_{j=1}^k J_{1,\infty}(w^j)$,

as $n \to \infty$, where w^0 is a solution of equation (1.1) with $\varepsilon = 1$ and $w^j, 1 \le j \le k$, are the nontrivial solutions of equation (3.2). Assume that $w^0 \neq 0$, then by Lemma 3.4, we have

$$
J_1(w^0) > c^* := \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s(15 - 2s - \sqrt{5}(3-2s))}{6(3+2s)} S^{\frac{3}{2s}},
$$

which is a contradiction to conclusion (4) above, as $J_{1,\infty}(w^j) \ge c^*$ and $J_1(w_n) \to c^*$. Consequently, $w^0 = 0$ and we must have $k = 1$ and w^1 is a ground state solution of (3.2) with $J_{1,\infty}(w^1) = c^*$. Hence, there exist $\delta_1 > 0$ and $z_1 \in \mathbb{R}^3$ such that $w^1 = \widetilde{U}_{\delta_1, z_1}$ and there exist $(r_n^1, x_n^1) \in \mathbb{R}+ \times \mathbb{R}^3$ such that

$$
\|(w_n)_{r_n^1, x_n^1} - w^1\| \to 0.
$$

Thus, there exist a sequence of points $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and a sequence of $\{\sigma_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that $\|q_n\| := \|w_n - \widetilde{U}_{\sigma_n, z_n}\| \to 0$ with $z_n = x_n^1 + r_n^1 z_1$ and $\sigma_n = r_n^1 \delta_1$. We assert that

$$
\sigma_n \to 0 \quad \text{and} \quad \{z_n\}_{n\in\mathbb{N}} \text{ is bounded.} \tag{5.10}
$$

In fact, setting $\Psi_{\sigma_n,z_n}=\varepsilon_n^{\frac{3-2s}{2}}\widetilde{U}_{\sigma_n,z_n}$, as $q_n\to 0$ in $D^{s,2}(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}w_n|^2\,\mathrm{d}x-\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\,\mathrm{d}x\right|\\
&=\left|\int_{\mathbb{R}^3}\chi(x)\big(|(-\Delta)^{\frac{s}{2}}w_n|^2-|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\big)\,\mathrm{d}x\right|\\
&\leq\int_{\mathbb{R}^3}|\chi(x)|\times\left||(-\Delta)^{\frac{s}{2}}w_n|^2-|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\right|\mathrm{d}x\\
&\leq\rho\int_{\mathbb{R}^3}\Big(|(-\Delta)^{\frac{s}{2}}w_n|+|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|\Big)\left||(-\Delta)^{\frac{s}{2}}w_n|-|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|\right|\,\mathrm{d}x\\
&\leq\rho\left\{\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}w_n|+|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|)^2\,\mathrm{d}x\right\}^{\frac{1}{2}}\\
&\quad\times\left\{\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}w_n|-|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|)^2\,\mathrm{d}x\right\}^{\frac{1}{2}}\\
&\leq\sqrt{2}\rho\sqrt{\|w_n\|^2+\|\widetilde{U}_{\sigma_n,z_n}\|^2}\times\left\{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}w_n-(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\,\mathrm{d}x\right\}^{\frac{1}{2}}\\
&\leq C\|w_n-\widetilde{U}_{\sigma_n,z_n}\|=C\|q_n\|\to 0
\end{aligned}\tag{5.11}$$

as $n\to\infty$. Then, by (5.11), we have

$$\begin{aligned}
\beta(u_n)&=\frac{1}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}u_n|^2\,\mathrm{d}x\\
&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}w_n|^2\,\mathrm{d}x\\
&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\,\mathrm{d}x+o_n(1)\\
&=\frac{1}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x+o_n(1)\\
&=\beta(\Psi_{\sigma_n,z_n})+o_n(1).
\end{aligned}\tag{5.12}$$

By $u_n\in\widetilde{\mathcal{N}}_{\varepsilon_n}$, it can be assumed that

$$\beta(\Psi_{\sigma_n,z_n})\subset B_{\rho/2}(0).\tag{5.13}$$

If $\sigma_n\to\infty$, then for each $R>0$, by Proposition 2.2 of Di Nezza-Palatucci-Valdinoci [18] and Lemma 4.1 of He-Zhao-Zou [26], we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{\varepsilon_n^{3-2s}}\int_{B_R(0)}|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x&=\lim_{n\to\infty}\int_{B_R(0)}|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\,\mathrm{d}x\\
&\leq\lim_{n\to\infty}\kappa\int_{B_R(0)}|\nabla U_{\sigma_n,z_n}|^2\,\mathrm{d}x=0.
\end{aligned}$$

Combining this fact, the definition of the map γ and (5.13), we have

$$\begin{aligned}\gamma(\Psi_{\sigma_n,z_n}) &= \frac{1}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}|\chi(x)-\beta(\Psi_{\sigma_n,z_n})||(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x\\
&\geq \frac{1}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}|\chi(x)||(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x-|\beta(\Psi_{\sigma_n,z_n})|\\
&\geq \frac{\rho}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}}\int_{\mathbb{R}^3\setminus B_R(0)}|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x-\frac{\rho}{2}+o_n(1)\\
&= \frac{\rho}{\varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x-\frac{\rho}{2}+o_n(1)\\
&= \frac{\rho}{2}+o_n(1).\end{aligned}\tag{5.14}$$

By $\|h_n\| := \|u_n-\Psi_{\sigma_n,z_n}\|\to 0$, (5.12) and Hölder's inequality, we have, as $n\to\infty$,

$$\begin{aligned}&|\gamma(u_n)-\gamma(\Psi_{\sigma_n,z_n})|\\
&=\frac{1}{c_\varepsilon^*}\left|\int_{\mathbb{R}^3}\left(|\chi(x)-\beta(u_n)||(-\Delta)^{\frac{s}{2}}u_n|^2-|\chi(x)-\beta(\Psi_{\sigma_n,z_n})||(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\right)\mathrm{d}x\right|\\
&\leq\frac{1}{c_\varepsilon^*}\int_{\mathbb{R}^3}|\chi(x)-\beta(u_n)|\times\left||(-\Delta)^{\frac{s}{2}}u_n|^2-|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\right|\mathrm{d}x\\
&\quad+\frac{1}{c_\varepsilon^*}\int_{\mathbb{R}^3}\left||\chi(x)-\beta(u_n)|-|\chi(x)-\beta(\Psi_{\sigma_n,z_n})|\right|\times|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x\\
&\leq\frac{1}{c_\varepsilon^*}\int_{\mathbb{R}^3}|\chi(x)-\beta(u_n)|\times\left(|(-\Delta)^{\frac{s}{2}}u_n|+|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|\right)\left||(-\Delta)^{\frac{s}{2}}u_n|-|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|\right|\mathrm{d}x\\
&\quad+\frac{1}{c_\varepsilon^*}\int_{\mathbb{R}^3}|\beta(u_n)-\beta(\Psi_{\sigma_n,z_n})|\times|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x\\
&\leq\frac{\sqrt{2}\rho}{c_\varepsilon^*}\left\{\int_{\mathbb{R}^3}\left||(-\Delta)^{\frac{s}{2}}u_n|^2+|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\right|\right\}^{\frac{1}{2}}\times\left\{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}(u_n-\Psi_{\sigma_n,z_n})|^2\,\mathrm{d}x\right\}^{\frac{1}{2}}\\
&\quad+o_n(1)\times\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x\\
&\leq C\|h_n\|+o_n(1)\times\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\Psi_{\sigma_n,z_n}|^2\,\mathrm{d}x\to 0,\end{aligned}$$

where $c_\varepsilon^* := \varepsilon^{3-2s}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} S^{\frac{3}{2s}}$. Thus,

$$\gamma(u_n)=\gamma(\Psi_{\sigma_n,z_n})+o_n(1),$$

and so,

$$\gamma(u_n)>\frac{\rho}{2}+o_n(1).\tag{5.15}$$

But from $u_n\in\widetilde{\mathcal{N}}_{\varepsilon_n}$, we have

$$\delta_1(\varepsilon_n)<\gamma(u_n)<\delta_2(\varepsilon_n),\tag{5.16}$$

where $\delta_i(\varepsilon_n)\to 0, i=1,2$ as $n\to\infty$, a contradiction to (5.15). Thus, $\{\sigma_n\}_{n\in\mathbb{N}}$ is bounded, and there exists some $\bar{\sigma}\geq 0$ such that $\sigma_n\to\bar{\sigma}$, as $n\to\infty$. Now, if $\bar{\sigma}>0$, then we must have that $|z_n|\to\infty$. Otherwise, $\widetilde{U}_{\sigma_n,z_n}$ would converge strongly in $D^{s,2}(\mathbb{R}^3)$ and so it would w_n. Therefore, J_1 has a nontrivial minimizer on $\mathcal{N}_1$, which contradicts to Lemma 3.4. Thus, for each

$R>0$, by the fact that $\lim_{n\to\infty}|z_n|=\infty$ and due to Proposition 2.2 of Di Nezza-Palatucci-Valdinoci [18] and Lemma 4.1 of He-Zhao-Zou [26], we have that

$$\lim_{n\to\infty}\int_{B_R(0)}|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\,\mathrm{d}x=0.$$

Therefore, we can obtain the estimation similar to (5.14), a contradiction to (5.16). By a similar argument, we can show the boundedness of the sequence $\{z_n\}_{n\in\mathbb{N}}$. So, (5.10) holds true.

Based on the above analysis, we can assume that $z_n\to z^*$ and $\sigma_n\to 0$. By extracting subsequences of $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$, still denoted as $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$, such that $\frac{\sigma_{n_i}}{\varepsilon_{n_i}}=o_{n_i}(1)$ as $n_i\to\infty$, we may change $\{\sigma_{n_i}\}_{n\in\mathbb{N}}$ by $\{\varepsilon_{n_i}\}_{n\in\mathbb{N}}$ and relabel $\{\varepsilon_{n_i}\}_{n\in\mathbb{N}}$ by $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Define

$$v_n(x):=\varepsilon_n^{\frac{3-2s}{2}}w_n(\varepsilon_n x+z_n).$$

Then $v_n\to\widetilde{U}_{1,0}$ in $D^{s,2}(\mathbb{R}^3)$ and we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}V(\varepsilon_n x+z_n)v_n(x)^2\,\mathrm{d}x=\lim_{n\to\infty}\varepsilon_n^{-2s}\int_{\mathbb{R}^3}V(x)w_n(x)^2\,\mathrm{d}x=0,$$

which shows that $\int_{\mathbb{R}^3}V(z^*)\widetilde{U}_{1,0}^2(x)\,\mathrm{d}x=0$. Thus, $V(z^*)=0$ and $z^*\in M$, and so, $z_n\in M_\mu$ for large n. Returning to (5.12), we get

$$\begin{aligned}\beta(u_n)&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}\chi(x)|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{\sigma_n,z_n}|^2\,\mathrm{d}x+o_n(1)\\&=\frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}S^{\frac{3}{2s}}}\int_{\mathbb{R}^3}[\chi(\varepsilon_n x+z_n)-z_n]|(-\Delta)^{\frac{s}{2}}\widetilde{U}_{1,0}|^2\,\mathrm{d}x+z_n+o_n(1).\end{aligned}$$

Since $\varepsilon_n x+z_n\to z^*\in M$, we have that $\beta(u_n)=z_n+o_n(1)$ and the sequence $\{z_n\}_{n\in\mathbb{N}}$ is what we desire. Consequently, (5.4) follows and the proof is complete. □

Proof of Theorem 1.1. For any fixed $\mu>0$, let $\varepsilon=\varepsilon_\mu>0$ be small. Therefore, $\Phi\colon[\delta_1,\delta_2]\times M\to\widetilde{\mathcal{N}}_\varepsilon$ given by $\Phi(\delta,z)=\Phi_{\delta,z}$ is well defined and by Lemma 5.2, we see that $\beta(\widetilde{\mathcal{N}}_\varepsilon)\subset M_\mu$. By (5.3), we infer to

$$\beta(\Phi_{\delta,z})=z+o_\delta(1)\quad\text{uniformly in } z\in M.$$

For $\delta\in[\delta_1,\delta_2]$, we denote $\beta(\Phi_{\delta,z})=z+\vartheta(z)$ for $z\in M$, with $|\vartheta(z)|<\mu/2$ uniformly for $z\in M$. We introduce the map

$$\mathcal{H}(t,(\delta,z)):=(\delta,z+(1-t)\vartheta(z)).$$

It is easy to check that $\mathcal{H}\colon[0,1]\times[\delta_1,\delta_2]\times M\to[\delta_1,\delta_2]\times M_\mu$ is continuous. Clearly,

$$\mathcal{H}(0,(\delta,z))=(\delta,\beta(\Phi_{\delta,z})),\ \mathcal{H}(1,(\delta,z))=(\delta,z),$$

which implies the map

$$H(\delta,z):=(\delta,\beta(\Phi_{\delta,z}))\colon[\delta_1,\delta_2]\times M\to[\delta_1,\delta_2]\times M_\mu$$

is homotopic to the inclusion mapping $\mathrm{Id}\colon[\delta_1,\delta_2]\times M\to[\delta_1,\delta_2]\times M_\mu$. Thus, applying the Lusternik-Schnirelmann category theory [51], we have

$$\mathrm{cat}(\widetilde{\mathcal{N}}_\varepsilon)\geq\mathrm{cat}_{[\delta_1,\delta_2]\times M_\mu}([\delta_1,\delta_2]\times M)=\mathrm{cat}_{M_\mu}(M).$$

By Corollaries 3.5 and 4.2, the functional J_ε satisfies the $(\mathrm{PS})_c$-condition on $\widetilde{\mathcal{N}}_\varepsilon$. Hence, the Lusternik-Schnirelmann category theory of critical points shows that J_ε has at least $\mathrm{cat}_{M_\mu}(M)$ solutions. □

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