INVERSE PROBLEMS FOR DOUBLE-PHASE OBSTACLE PROBLEMS WITH VARIABLE EXPONENTS

SHENGDA ZENG, NIKOLAOS S. PAPAGEORGIOU, AND PATRICK WINKERT

Abstract. In the present paper, we are concerned with the study of a variable exponent double phase obstacle problem which involves a nonlinear and nonhomogeneous partial differential operator, a multivalued convection term, a general multivalued boundary condition and an obstacle constraint. Under the framework of anisotropic Musielak-Orlicz Sobolev spaces, we establish the nonemptiness, boundedness and closedness of the solution set of such problems by applying a surjectivity theorem for multivalued pseudomonotone operators and the variational characterization of the first eigenvalue of the Steklov eigenvalue problem for the p-Laplacian. In the second part, we consider a nonlinear inverse problem which is formulated by a regularized optimal control problem to identify the discontinuous parameters for the variable exponent double phase obstacle problem. We then introduce the parameter-to-solution map, study a continuous result of Kuratowski type and prove the solvability of the inverse problem.

1. Introduction

In this paper we investigate an inverse problem to an elliptic differential inclusion problem driven by the variable exponent double phase operator and involving a multivalued convection term, a multivalued boundary condition as well as an obstacle constraint. In order to formulate our problem, let us assume that $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with Lipschitz continuous boundary $\Gamma := \partial\Omega$ and suppose that Γ is divided into three mutually disjoint parts Γ_1, Γ_2, and Γ_3 where Γ_1 is supposed to have positive Lebesgue measure. We consider the following problem

$$\begin{aligned}
-D_{p(\cdot),q(\cdot)}u + g(x,u) + \mu(x)|u|^{q(x)-2}u &\in f(x,u,\nabla u) && \text{in} \quad \Omega,\\
u &= 0 && \text{on} \quad \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &= h(x) && \text{on} \quad \Gamma_2,\\
\frac{\partial u}{\partial \nu_a} &\in U(x,u) && \text{on} \quad \Gamma_3,\\
u(x) &\leq \Phi(x) && \text{in} \quad \Omega,
\end{aligned} \tag{1.1}$$

where $p,q\colon \overline{\Omega} \to (1,+\infty)$ are continuous functions with $p(x) < N$ and $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, $0 \leq \mu(\cdot) \in L^\infty(\Omega)$, $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ and $U\colon \Gamma_3 \times \mathbb{R} \to 2^{\mathbb{R}}$ are two given multivalued functions, $\Phi\colon \Omega \to \mathbb{R}$ is an obstacle function and the differential operator $D_{p(\cdot),q(\cdot)}$ is defined by

$$D_{p(\cdot),q(\cdot)}u := \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \quad \text{for all } u \in W^{1,\mathcal{H}}(\Omega), \tag{1.2}$$

while $W^{1,\mathcal{H}}(\Omega)$ stands for the anisotropic Musielak-Orlicz Sobolev space and

$$\frac{\partial u}{\partial \nu_a} := \left(a(x)|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nu,$$

with ν being the outward unit normal vector on Γ. Here, $a\colon \Omega \to (0,+\infty)$ and $h\colon \Gamma_2 \to \mathbb{R}$ are two given discontinuous functions.

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The current paper is devoted to study the variable exponent elliptic obstacle inclusion problem (1.1) from the following two perspectives:

- we apply a surjectivity theorem for multivalued pseudomonotone mappings, the theory of nonsmooth analysis and the variational characterization of the Steklov eigenvalue problem for the p-Laplacian, to examine the solvability of problem (1.1).
- a nonlinear inverse problem governed by the variable exponent elliptic obstacle problem (1.1) is introduced and a general framework for determining the existence of solutions to the inverse problem is established.

To the best of our knowledge, this is the first work for studying the identification of discontinuous parameters to nonlinear elliptic equation which combines the variable exponent double phase differential operator along with an obstacle constraint, a multivalued convection term and a multivalued mixed boundary condition.

The first interesting phenomena is the fact that the right-hand side of (1.1) is on the one hand multivalued which is motivated by several physical applications (see, for example, Panagiotopoulos [41, 42], Carl-Le [6] and the references therein) and on the other hand it depends on the gradient of the solution. Such right-hand sides are often called multivalued convection terms. This dependence makes the study of such problems quite complicated since standard variational tools cannot be applied due to the lack of a variational structure. Nevertheless, several works exist in this direction using different treatments as the frozen variable method or properties of corresponding eigenvalue problems. We refer, for example, to the papers of El Manouni-Marino-Winkert [14], Faraci-Motreanu-Puglisi [15], Faraci-Puglisi [16], Figueiredo-Madeira [18], Gasiński-Papageorgiou [21], Liu-Motreanu-Zeng [30], Liu-Papageorgiou [31], Marano-Winkert [33], Motreanu-Winkert [40], Papageorgiou-Rădulescu-Repovš [43] and Zeng-Papageorgiou [54], see also the references therein.

A second interesting phenomenon is the studying of inverse problems of parameter identification which is an important field in mathematics motivated by several applications. One interesting work in the direction of inverse problems of mixed quasi-variational inequalities has been done by Migórski-Khan-Zeng [37] who treated the problem

$$\langle T(a,u), v-u\rangle + \varphi(v) - \varphi(u) \geq \langle m, v-u\rangle \quad \text{for all } v \in K(u),$$

where $K\colon C \to 2^C$ is a multivalued mapping, $T\colon B \times V \to V^*$ is a nonlinear map, $\varphi\colon V \to \mathbb{R} \cup \{+\infty\}$ is a functional and $m \in V^*$, while V is a real reflexive Banach space, B is another Banach space and C is a nonempty, closed, convex subset of V. These abstract results are quite interesting and can be applied to several types of operators, for example the p-Laplacian in form of hemivariational inequalities, see also [36]. Without guarantee of completeness, we refer to the results of Clason-Khan-Sama-Tammer [8] for noncoercive variational problems, Gwinner [25] for variational inequalities of second kind, Gwinner-Jadamba-Khan-Sama [26] for an optimization setting and Migórski-Ochal [38] for nonlinear parabolic problems.

Finally, a third interesting phenomenon is the used weighted double phase operator with variable exponents given in (1.2). This operator was just studied in [12] and has several applications in Mechanics, Physics and Engineering Sciences. If $a \equiv 1$, the energy functional corresponding to (1.2) is given by

$$\omega \mapsto \int_\Omega \left(\frac{|\nabla \omega|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla \omega|^{q(x)}}{q(x)} \right) \mathrm{d}x, \tag{1.3}$$

where the integrand $H(x,\xi) = \frac{1}{p(x)}|\xi|^{p(x)} + \frac{\mu(x)}{q(x)}|\xi|^{q(x)}$ for all $(x,\xi) \in \Omega \times \mathbb{R}^N$ has unbalanced growth, that is,

$$b_1|\xi|^{p(x)} \leq H(x,\xi) \leq b_2 \left(1 + |\xi|^{q(x)}\right) \quad \text{for a. a. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N \text{ with } b_1, b_2 > 0.$$

The main feature of the functional (1.3) is the change of ellipticity on the set where the weight function is zero, that is, on the set $\{x \in \Omega : \mu(x) = 0\}$. This means, that the energy density of (1.3) exhibits ellipticity in the gradient of order $q(x)$ on the points x where $\mu(x)$ is positive and of order $p(x)$ on the points x where $\mu(x)$ vanishes. So the integrand H switches between two different phases of elliptic behaviours. Functionals of the form (1.3) have been initially introduced by Zhikov [55] in 1986 in order to describe models for strongly anisotropic materials and it also turned out its relevance in the study of duality theory as well as in the context of the Lavrentiev phenomenon, see Zhikov [56]. For example, in the elasticity theory, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents p and q, see Zhikov [57]. Note that functionals of type (1.3) have been considered concerning regularity of local minimizers by several authors. We mention the significant works of Baroni-Colombo-Mingione [2, 3], Byun-Oh [5], Colombo-Mingione [10, 11], Marcellini [34, 35] and Ragusa-Tachikawa [48].

Moreover, we refer to recent existence results for double phase equations with different right-hand sides and different treatments. We mention the works of Bahrouni-Rădulescu-Winkert [1], Biagi-Esposito-Vecchi [4], Colasuonno-Squassina [9], Fiscella [19], Farkas-Winkert [17], Gasiński-Papageorgiou [20], Gasiński-Winkert [22, 23, 24], Liu-Dai [29], Liu-Winkert [32], Papageorgiou-Vetro-Vetro [44], Perera-Squassina [46], Stegliński [50] and Zeng-Bai-Gasiński-Winkert [51, 52, 53].

The rest of the paper is organized as follows. In Section 2, we review some basic notation and necessary results for anisotropic Musielak-Orlicz Lebesgue and anisotropic Musielak-Orlicz Sobolev spaces, the p-Laplacian eigenvalue problem with Steklov boundary condition and the theory of pseudomonotone multivalued operators. In Section 3, we apply a surjectivity theorem for multivalued pseudomonotone operators to prove the nonemptiness and compactness of the solution set of problem (1.1). Section 4 is devoted to introduce the nonlinear inverse problem and develops a new existence result to such inverse problem.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries which will be applied in the next sections to derive the main results of the paper.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ such that Γ is decomposed into three mutually disjoint parts Γ_1, Γ_2 and Γ_3 such that Γ_1 has positive Lebesgue measure. By $M(\Omega)$ we denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$ and we identify two of such functions when they differ on a Lebesgue-null set. Let D be a nonempty subset of $\overline{\Omega}$. For any $r \in [1,\infty)$, we denote by $L^r(D) := L^r(D;\mathbb{R})$ and $L^r(D;\mathbb{R}^N)$ the usual Lebesgue spaces equipped with the norm $\|\cdot\|_{r,D}$ defined as

$$\|u\|_{r,D} := \left(\int_D |u|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \quad \text{for all } u \in L^r(D).$$

We set

$$L^r(D)_+ := \left\{u \in L^r(D) \,:\, u(x) \geq 0 \text{ for a. a. } x \in D\right\}.$$

The corresponding Sobolev space $W^{1,r}(\Omega)$ is endowed with the norm $\|\cdot\|_{1,r,\Omega}$ given by

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

Moreover, we recall that the r-Laplacian eigenvalue problem with Steklov boundary condition for $r \in (1,\infty)$ is given by

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u && \text{in } \Omega,\\ |u|^{r-2}u\cdot\nu &= \lambda|u|^{r-2}u && \text{on } \Gamma. \end{aligned} \tag{2.1}$$

It is known that (2.1) has a smallest eigenvalue $\lambda_{1,r}^S > 0$ which turns out to be isolated and simple. Furthermore, $\lambda_{1,r}^S > 0$ has the variational characterization given by

$$\lambda_{1,r}^S = \inf_{u \in W^{1,r}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{r,\Omega}^r + \|u\|_{r,\Omega}^r}{\|u\|_{r,\Gamma}^r}, \tag{2.2}$$

see Lê [28].

Next, we introduce a subset $C_+(\overline{\Omega})$ of $C(\overline{\Omega})$ defined by

$$C_+(\overline{\Omega}) := \left\{a \in C(\overline{\Omega}) \,:\, 1 < a(x) \text{ for all } x \in \overline{\Omega}\right\}.$$

For any $r \in C_+(\overline{\Omega})$, we define

$$r_- := \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_+ := \max_{x\in\overline{\Omega}} r(x)$$

and $r' \in C_+(\overline{\Omega})$ stands for the conjugate variable exponent to r, namely,

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Also, we denote by r^* and r_* the critical Sobolev variable exponents to $r \in C_+(\overline{\Omega})$ in the domain and on the boundary, respectively, given by

$$r^*(x) = \begin{cases} \frac{Nr(x)}{N-r(x)} & \text{if } r(x) < N,\\ +\infty & \text{if } r(x) \geq N, \end{cases} \quad \text{for all } x \in \overline{\Omega} \tag{2.3}$$

and

$$r_*(x) = \begin{cases} \frac{(N-1)r(x)}{N-r(x)} & \text{if } r(x) < N,\\ +\infty & \text{if } r(x) \geq N, \end{cases} \quad \text{for all } x \in \overline{\Omega}. \tag{2.4}$$

For $r \in C_+(\overline{\Omega})$ fixed, the variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ is defined by

$$L^{r(\cdot)}(\Omega) = \left\{u \in M(\Omega) \,:\, \int_\Omega |u|^{r(x)}\,\mathrm{d}x < +\infty\right\},$$

equipped with the Luxemburg norm

$$\|u\|_{r(\cdot)} := \inf\left\{\lambda > 0 \,:\, \int_\Omega \left(\frac{|u|}{\lambda}\right)^{r(x)} \mathrm{d}x \leq 1\right\}.$$

We know that $L^{r(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Moreover, the dual space of $L^{r(\cdot)}(\Omega)$ is $L^{r'(\cdot)}(\Omega)$ and the following Hölder type inequality holds

$$\int_\Omega |uv|\,\mathrm{d}x \leq \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(\cdot)}\|v\|_{r'(\cdot)} \leq 2\|u\|_{r(\cdot)}\|v\|_{r'(\cdot)}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$. Clearly, if $r_1, r_2 \in C_+(\overline{\Omega})$ are such that $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega).$$

For any $r \in C_+(\overline{\Omega})$, we consider the modular function $\rho_{r(\cdot)} \colon L^{r(\cdot)}(\Omega) \to \mathbb{R}$ given by

$$\rho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}\,\mathrm{d}x \quad \text{for all } u \in L^{r(\cdot)}(\Omega). \tag{2.5}$$

The following proposition states some important relations between the norm of $L^{r(\cdot)}(\Omega)$ and the modular function $\rho_{r(\cdot)}$ defined in (2.5).

Proposition 2.1. *If $r \in C_+(\overline{\Omega})$ and $u, u_n \in L^{r(\cdot)}(\Omega)$, then we have the following assertions:*

(i) $\|u\|_{r(\cdot)} = \lambda \iff \rho_{r(\cdot)}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$;

(ii) $\|u\|_{r(\cdot)} < 1$ *(resp.* $= 1, > 1$*)* $\iff \rho_{r(\cdot)}(u) < 1$ *(resp.* $= 1, > 1$*)*;

(iii) $\|u\|_{r(\cdot)} < 1 \quad \Longrightarrow \quad \|u\|_{r(\cdot)}^{r_+} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_-}$;
(iv) $\|u\|_{r(\cdot)} > 1 \quad \Longrightarrow \quad \|u\|_{r(\cdot)}^{r_-} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_+}$;
(v) $\|u_n\|_{r(\cdot)} \to 0 \quad \Longleftrightarrow \quad \rho_{r(\cdot)}(u_n) \to 0$;
(vi) $\|u_n\|_{r(\cdot)} \to +\infty \quad \Longleftrightarrow \quad \rho_{r(\cdot)}(u_n) \to +\infty$.

For $r \in C_+(\overline{\Omega})$, we denote by $W^{1,r(\cdot)}(\Omega)$ the variable exponent Sobolev space given by

$$W^{1,r(\cdot)}(\Omega) = \left\{ u \in L^{r(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{r(\cdot)}(\Omega) \right\}.$$

We know that $W^{1,r(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W^{1,r(\cdot)}(\Omega)$$

is a separable and reflexive Banach space, where $\|\nabla u\|_{r(\cdot)} := \|\, |\nabla u| \,\|_{r(\cdot)}$. We also consider the subspace $W_0^{1,r(\cdot)}(\Omega)$ of $W^{1,r(\cdot)}(\Omega)$ defined by

$$W_0^{1,r(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(\cdot)}}.$$

From Poincaré's inequality we know that we can endow the space $W_0^{1,r(\cdot)}(\Omega)$ with the equivalent norm

$$\|u\|_{1,r(\cdot),0} = \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

We suppose the following hypotheses on the weight function μ and the variable exponents p, q in problem (1.1) satisfy the following conditions:

(H1): $p, q \in C_+(\overline{\Omega})$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$ such that
 (i) $p(x) < N$ for all $x \in \overline{\Omega}$;
 (ii) $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

Now we introduce the nonlinear function $\mathcal{H}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ defined as

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x) t^{q(x)} \quad \text{for all } (x,t) \in \Omega \times [0,+\infty).$$

In addition, we denote by $\rho_{\mathcal{H}}(\cdot)$ the modular function given by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,u)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)} \right) \mathrm{d}x. \tag{2.6}$$

Then, $L^{\mathcal{H}}(\Omega)$ stands for the corresponding Musielak-Orlicz Lebesgue space related to the function $\mathcal{H}$ defined by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty \right\},$$

which is, equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{ \lambda > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \leq 1 \right\} \quad \text{for all } u \in L^{\mathcal{H}}(\Omega),$$

uniformly convex and so a reflexive Banach space, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [12, Proposition 2.12]. Similarly, we introduce the Musielak-Orlicz Sobolev spaces $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ given by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\},$$
$$W_0^{1,\mathcal{H}}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}}},$$

where the norm $\|\cdot\|_{1,\mathcal{H}}$ for both spaces is defined by

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W^{1,\mathcal{H}}(\Omega) \text{ resp. } W_0^{1,\mathcal{H}}(\Omega).$$

Furthermore, we introduce the seminormed space $L_\mu^{q(\cdot)}(\Omega)$ defined by

$$L_\mu^{q(\cdot)}(\Omega) := \left\{ u \in M(\Omega) \,:\, \int_\Omega \mu(x)|u|^{q(x)}\,\mathrm{d}x < +\infty \right\}$$

endowed with the seminorm

$$\|u\|_{q(\cdot),\mu} := \inf\left\{\lambda > 0 \,:\, \int_\Omega \mu(x)\left(\frac{|u|}{\lambda}\right)^{q(x)} \mathrm{d}x \le 1\right\} \quad \text{for all } u \in L^{q(\cdot)}_\mu(\Omega).$$

From Crespo-Blanco-Gasiński-Harjulehto-Winkert [12, Proposition 2.13] and Rădulescu-Repovš [47], we have the following proposition.

Proposition 2.2. *Let hypotheses* (H1) *be satisfied and let $\rho_\mathcal{H}$ be defined by* (2.6)*. Then, we have*

(i) *if $u \neq 0$, then $\|u\|_\mathcal{H} = \lambda$ if and only if $\rho_\mathcal{H}(\frac{u}{\lambda}) = 1$;*
(ii) *$\|u\|_\mathcal{H} < 1$ (resp. > 1, $= 1$) if and only if $\rho_\mathcal{H}(u) < 1$ (resp. > 1, $= 1$);*
(iii) *if $\|u\|_\mathcal{H} < 1$, then $\|u\|_\mathcal{H}^{q_+} \leqslant \rho_\mathcal{H}(u) \leqslant \|u\|_\mathcal{H}^{p_-}$;*
(iv) *if $\|u\|_\mathcal{H} > 1$, then $\|u\|_\mathcal{H}^{p_-} \leqslant \rho_\mathcal{H}(u) \leqslant \|u\|_\mathcal{H}^{q_+}$;*
(v) *$\|u\|_\mathcal{H} \to 0$ if and only if $\rho_\mathcal{H}(u) \to 0$;*
(vi) *$\|u\|_\mathcal{H} \to +\infty$ if and only if $\rho_\mathcal{H}(u) \to +\infty$.*

Next, we collect some useful embedding results for the spaces $L^\mathcal{H}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$. We refer to Crespo-Blanco-Gasiński-Harjulehto-Winkert [12, Proposition 2.16].

Proposition 2.3. *Let hypotheses* (H1) *be satisfied and let $p^*(\cdot)$ be the critical exponent to $p(\cdot)$ given in* (2.3) *with $s = p$. Then the following embeddings hold:*

(i) *$L^\mathcal{H}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow W^{1,r(\cdot)}_0(\Omega)$ are continuous for all $r \in C(\overline{\Omega})$ with $1 \le r(x) \le p(x)$ for all $x \in \overline{\Omega}$;*
(ii) *$W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ are compact for all $r \in C(\overline{\Omega})$ with $1 \le r(x) < p^*(x)$ for all $x \in \overline{\Omega}$;*
(iii) *$L^\mathcal{H}(\Omega) \hookrightarrow L^{q(\cdot)}_\mu(\Omega)$ is continuous;*
(iv) *$L^{q(\cdot)}(\Omega) \hookrightarrow L^\mathcal{H}(\Omega)$ is continuous.*

We now equip the space $W^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\|_{\varrho_\mathcal{H}} := \inf\left\{\lambda > 0 \,:\, \int_\Omega \left[\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\lambda}\right|^{q(x)} + \left|\frac{u}{\lambda}\right|^{p(x)} + \mu(x)\left|\frac{u}{\lambda}\right|^{q(x)}\right] \mathrm{d}x \le 1\right\},$$

where the modular $\varrho_\mathcal{H}$ is given by

$$\varrho_\mathcal{H}(u) := \int_\Omega \left(|\nabla u|^{p(x)} + \mu|\nabla u|^{q(x)}\right) \mathrm{d}x + \int_\Omega \left(|u|^{p(x)} + \mu|u|^{q(x)}\right) \mathrm{d}x \tag{2.7}$$

for all $u \in W^{1,\mathcal{H}}(\Omega)$. Moreover, let us introduce a subspace V of $W^{1,\mathcal{H}}(\Omega)$ defined by

$$V := \{u \in W^{1,\mathcal{H}}(\Omega) \,:\, u = 0 \text{ for a. a. } x \in \Gamma_1\}.$$

It is obvious that V equipped the norm $\|\cdot\|_{\varrho_\mathcal{H}}$ becomes a reflexive Banach space. In what follows, we denote by $\|\cdot\|_V := \|\cdot\|_{\varrho_\mathcal{H}}$ the norm of V. Clearly, if we replace the space $W^{1,\mathcal{H}}(\Omega)$ by V in Proposition 2.3, then the embeddings (ii) and (iii) remain valid.

The next proposition can be found in Crespo-Blanco-Gasiński-Harjulehto-Winkert [12, Proposition 2.14].

Proposition 2.4. *Let hypotheses* (H1) *be satisfied, let $u \in W^{1,\mathcal{H}}(\Omega)$ and let $\varrho_\mathcal{H}$ be defined by* (2.7)*. Then, we have*

(i) *if $u \neq 0$, then $\|u\|_{\varrho_\mathcal{H}} = \lambda$ if and only if $\varrho_\mathcal{H}(\frac{u}{\lambda}) = 1$;*
(ii) *$\|u\|_{\varrho_\mathcal{H}} < 1$ (resp. > 1, $= 1$) if and only if $\varrho_\mathcal{H}(u) < 1$ (resp. > 1, $= 1$);*
(iii) *if $\|u\|_{\varrho_\mathcal{H}} < 1$, then $\|u\|_{\varrho_\mathcal{H}}^{q_+} \leqslant \rho_{\varrho_\mathcal{H}}(u) \leqslant \|u\|_{\varrho_\mathcal{H}}^{p_-}$;*
(iv) *if $\|u\|_{\varrho_\mathcal{H}} > 1$, then $\|u\|_{\varrho_\mathcal{H}}^{p_-} \leqslant \rho_{\varrho_\mathcal{H}}(u) \leqslant \|u\|_{\varrho_\mathcal{H}}^{q_+}$;*
(v) *$\|u\|_{\varrho_\mathcal{H}} \to 0$ if and only if $\varrho_\mathcal{H}(u) \to 0$;*
(vi) *$\|u\|_{\varrho_\mathcal{H}} \to +\infty$ if and only if $\varrho_\mathcal{H}(u) \to +\infty$.*

Throughout the paper we denote by the symbols " $\xrightarrow{w}$ " and "$\to$" the weak and the strong convergence in various spaces, respectively. Moreover, for a Banach space $(X, \|\cdot\|_X)$ we denote its dual space by X^* and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. We write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{X^*\times X}$ if it is clear from the context.

Let $a \in L^\infty(\Omega)_+$ be such that $\inf_{x\in\Omega} a(x) > 0$ and consider the nonlinear map $F\colon V \to V^*$ given by

$$\begin{aligned}\langle F(u), v\rangle := &\int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x\\ &+\int_\Omega \left(|u|^{p(x)-2}u + \mu(x)|u|^{q(x)-2}u\right) v\,\mathrm{d}x,\end{aligned} \tag{2.8}$$

for $u, v \in V$. We have the following properties of F, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [12, Proposition 3.5].

Proposition 2.5. *Under hypotheses* (H1) *and $a \in L^\infty(\Omega)_+$ with $\inf_{x\in\Omega} a(x) > 0$, the operator F defined by* (2.8) *is bounded, continuous, monotone (hence maximal monotone) and of type* (S_+)*, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{and} \quad \limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \le 0,$$

imply $u_n \to u$ in V.

We recall the notions of pseudomonotonicity and generalized pseudomonotonicity, see Migórski-Ochal-Sofonea [39, Definition 3.57] or Carl-Le [6, Definitions 2.39 and 2.40].

Definition 2.6. *Let X be a reflexive real Banach space. The operator $A\colon X \to 2^{X^*}$ is called*

(a) *pseudomonotone if the following conditions hold:*
   (i) *the set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$;*
   (ii) *A is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*;*
   (iii) *if $\{u_n\}_{n\in\mathbb{N}} \subset X$ with $u_n \xrightarrow{w} u$ in X and $u_n^* \in A(u_n)$ are such that*
   $$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$
   then to each element $v \in X$, there exists $u^*(v) \in A(u)$ with
   $$\langle u^*(v), u - v\rangle_{X^*\times X} \le \liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

(b) *generalized pseudomonotone if the following holds: Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ and $\{u_n^*\}_{n\in\mathbb{N}} \subset X^*$ with $u_n^* \in A(u_n)$. If $u_n \xrightarrow{w} u$ in X and $u_n^* \xrightarrow{w} u^*$ in X^* and if*
$$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$
then the element u^* lies in $A(u)$ and
$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X} \text{ as } n \to \infty.$$

It is well known that every pseudomonotone operator is generalized pseudomonotone, see Carl-Le-Motreanu [7, Proposition 2.122]. The converse statement also holds under an additional boundedness hypothesis, see Carl-Le-Motreanu [7, Proposition 2.123].

Proposition 2.7. *Let X be a reflexive real Banach space and assume that $A\colon X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^*;*
(ii) *$A\colon X \to 2^{X^*}$ is bounded;*
(iii) *A is generalized pseudomonotone.*

Then the operator $A\colon X \to 2^{X^*}$ is pseudomonotone.

The following definition about Kuratowski limits can be found in Papageorgiou-Winkert [45, Definition 6.7.4].

Definition 2.8. *Let (X,τ) be a Hausdorff topological space and let $\{A_n\}_{n\in\mathbb{N}} \subset 2^X$ be a sequence of sets. The τ-Kuratowski lower limit of the sets A_n is defined by*

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x\in X \,:\, x=\tau\text{-}\lim_{n\to\infty} x_n,\, x_n\in A_n \text{ for all } \ n\geq 1 \right\},$$

and the τ-Kuratowski upper limit of the sets A_n is given by

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x\in X \,:\, x=\tau\text{-}\lim_{k\to\infty} x_{n_k},\, x_{n_k}\in A_{n_k},\, n_1<n_2<\ldots<n_k<\ldots \right\}.$$

If $A=\tau\text{-}\liminf_{n\to\infty} A_n=\tau\text{-}\limsup_{n\to\infty} A_n$, then A is called τ-Kuratowski limit of the sets A_n.

We end this section by recalling the following surjectivity theorem for multivalued mappings, see Le [27, Theorem 2.2].

Theorem 2.9. *Let X be a real reflexive Banach space, let $\mathcal{G}\colon D(\mathcal{G})\subset X\to 2^{X^*}$ be a maximal monotone operator, let $\mathcal{F}\colon D(\mathcal{F})=X\to 2^{X^*}$ be a bounded multivalued pseudomonotone operator, let $\mathcal{L}\in X^*$ and let $B_R(0):=\{u\in X\,:\,\|u\|_X<R\}$. Assume that there exist $u_0\in X$ and $R\geq \|u_0\|_X$ such that $D(\mathcal{G})\cap B_R(0)\neq\emptyset$ and*

$$\langle \xi+\eta-\mathcal{L}, u-u_0\rangle_{X^*\times X}>0 \tag{2.9}$$

for all $u\in D(\mathcal{G})$ with $\|u\|_X=R$, for all $\xi\in\mathcal{G}(u)$ and for all $\eta\in\mathcal{F}(u)$. Then the inclusion

$$\mathcal{F}(u)+\mathcal{G}(u)\ni\mathcal{L}$$

has a solution in $D(\mathcal{G})$.

We point out that if

$$\lim_{\substack{\|u\|_X\to+\infty\\ u\in D(\mathcal{G})}} \frac{\langle \xi+\eta, u-u_0\rangle_{X^*\times X}}{\|u\|_X}=+\infty, \tag{2.10}$$

is fulfilled, then (2.9) holds for some R large enough. We will use (2.10) in Section 3.

3. Obstacle double phase problems with variable exponents

In this section we are going to prove the existence of at least one nontrivial weak solution of problem (1.1) by using the variational characterization of the first eigenvalue of the Steklov eigenvalue problem for the p_--Laplacian. First we state the full assumptions on the data of problem (1.1).

(H2): The multivalued mapping $f\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to 2^{\mathbb{R}}$ has nonempty, compact and convex values such that $f(x,0,0)\neq\{0\}$ for a. a. $x\in\Omega$ and

(i) $x\mapsto f(x,s,\xi)$ has a measurable selection for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$;

(ii) $(s,\xi)\mapsto f(x,s,\xi)$ is upper semicontinuous for a. a. $x\in\Omega$;

(iii) there exist $0\leq\alpha_f(\cdot)\in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)$ and $a_f,b_f\geq 0$ such that

$$|\eta|\leq a_f|\xi|^{\frac{p(x)(r(x)-1)}{r(x)}}+b_f|s|^{r(x)-1}+\alpha_f(x)$$

for all $\eta\in f(x,s,\xi)$, for all $s\in\mathbb{R}$, for all $\xi\in\mathbb{R}^N$ and for a. a. $x\in\Omega$, where $r\in C_+(\overline{\Omega})$ is such that

$$r(x)<p^*(x)\quad \text{for all } x\in\overline{\Omega}$$

with p^* being the critical Sobolev variable exponent of p given in (2.3) with $s=p$;

(iv) there exist $\beta_f \in L^1_+(\Omega)$ and $c_f, d_f \geq 0$ satisfying

$$\eta s \leq c_f |\xi|^{p(x)} + d_f |s|^{p(x)} + \beta_f(x)$$

for all $\eta \in f(x,s,\xi)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$ and for a. a. $x \in \Omega$.

(H3): The function $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $x \mapsto g(x,s)$ is measurable for all $s \in \mathbb{R}$;

(ii) $s \mapsto g(x,s)$ is continuous for a. a. $x \in \Omega$;

(iii) there exist $a_g > 0$ and $b_g \in L^1(\Omega)$ such that

$$g(x,s)s \geq a_g |s|^{\varsigma(x)} - b_g(x)$$

for all $s \in \mathbb{R}$ and a. a. $x \in \Omega$, where $\varsigma \in C(\overline{\Omega})$ is such that

$$p(x) < \varsigma(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega};$$

(iv) for any $u, v \in L^{p^*(\cdot)}(\Omega)$, the function $x \mapsto g(x,u(x))v(x)$ belongs to $L^1(\Omega)$.

(H4): The function $\Phi \colon \Omega \to [0,\infty)$ is such that $\Phi \in M(\Omega)$.

(H5): $U \colon \Gamma_3 \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies the following conditions:

(i) $U(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. a. $x \in \Gamma_3$ and for all $s \in \mathbb{R}$;

(ii) $x \mapsto U(x,s)$ is measurable on Γ_3 for all $s \in \mathbb{R}$;

(iii) $s \mapsto U(x,s)$ is u.s.c. for a. a. $x \in \Gamma_3$;

(iv) there exist $0 \leq \alpha_U(\cdot) \in L^{\delta'(\cdot)}(\Gamma_3)$ and $a_U \geq 0$ such that

$$|U(x,s)| \leq \alpha_U(x) + a_U |s|^{\delta(x)-1}$$

for a. a. $x \in \Gamma_3$ and for all $s \in \mathbb{R}$, where $\delta \in C_+(\overline{\Omega})$ is such that

$$\delta(x) < p_*(x) \quad \text{for all } x \in \overline{\Omega}$$

with the critical exponent p_* of p on the boundary Γ given in (2.4);

(v) there exist $0 \leq \beta_U(\cdot) \in L^1(\Gamma_3)$ and $b_U \geq 0$ such that

$$\xi s \leq b_U |s|^{p_-} + \beta_U(x)$$

for all $\xi \in U(x,s)$, for all $s \in \mathbb{R}$ and for a. a. $x \in \Gamma_3$.

(H6): $a \in L^\infty(\Omega)$ is such that $\inf_{x\in\Omega} a(x) \geq c_\Lambda > 0$ and $h \in L^{p'(\cdot)}(\Gamma_2)$.

(H7): The inequality holds

$$c_\Lambda - c_f - b_U \left(\lambda^S_{1,p_-}\right)^{-1} > 0,$$

where λ^S_{1,p_-} is the first eigenvalue of the p_--Laplacian with Steklov boundary condition, see (2.1) and (2.2) for $r = p_-$.

Finally the obstacle set K is defined by

$$K = \{u \in V : u(x) \leq \Phi(x) \text{ for a. a. } x \in \Omega\}.$$

Note that under hypotheses (H4) it is clear that K is a nonempty, closed and convex subset of V.

We understand weak solutions of problem (1.1) as follows.

Definition 3.1. *We say that $u \in K$ is a weak solution of problem (1.1), if there exist functions $\eta \in L^{r'(\cdot)}(\Omega)$ and $\xi \in L^{\delta'(\cdot)}(\Gamma_3)$ such that $\eta(x) \in f(x,u(x),\nabla u(x))$ for a. a. $x \in \Omega$ as well as $\xi(x) \in U(x,u(x))$ for a. a. $x \in \Gamma_3$ and if*

$$\int_\Omega \Big(a(x)|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(v-u)\,\mathrm{d}x$$
$$+ \int_\Omega g(x,u)(v-u)\,\mathrm{d}x + \int_\Omega \mu(x)|u|^{q(x)-2}u(v-u)\,\mathrm{d}x$$

$$\geq \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_2} h(x)(v-u)\,\mathrm{d}\Gamma + \int_{\Gamma_3} \xi(x)(v-u)\,\mathrm{d}\Gamma$$

is satisfied for all $v \in K$.

The main result in the present section is given by the following theorem.

Theorem 3.2. *Assume that* (H1)–(H7) *are satisfied. Then, the solution set of problem* (1.1) *corresponding to $(a,h) \in L^\infty(\Omega) \times L^{p'(\cdot)}(\Omega)$, denoted by $\mathcal{S}(a,h)$, is nonempty, bounded and weakly closed (hence, weakly compact).*

Proof. **Part I Nonemptiness of $\mathcal{S}(a,h)$:** Let $F\colon V \to V^*$, $G\colon V \subset L^{\varsigma(\cdot)}(\Omega) \to L^{\varsigma'(\cdot)}(\Omega)$ and $L\colon L^{p(\cdot)}(\Omega) \to L^{p'(\cdot)}(\Omega)$ be nonlinear mappings defined by

$$\begin{aligned}
\langle Fu, v\rangle &:= \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v\,\mathrm{d}x\\
&\quad + \int_\Omega \left(|u|^{p(x)-2}u + \mu(x)|u|^{q(x)-2}u\right) v\,\mathrm{d}x,\\
\langle Gu, w\rangle_{L^{\varsigma'(\cdot)}(\Omega)\times L^{\varsigma(\cdot)}(\Omega)} &:= \int_\Omega g(x,u)w\,\mathrm{d}x,\\
\langle Ly, z\rangle_{L^{p'(\cdot)}(\Omega)\times L^{p(\cdot)}(\Omega)} &:= \int_\Omega |y|^{p(x)-2}yz\,\mathrm{d}x
\end{aligned}$$

for all $u,v \in V$, for all $w \in L^{\varsigma(\cdot)}(\Omega)$ and for all $y,z \in L^{p(\cdot)}(\Omega)$. Applying the Yankov-von Neumann-Aumann selection theorem (see Papageorgiou-Winkert [45, Theorem 2.7.25]), for any fixed $u \in V$, along with hypotheses (H2)(i), (ii), the multivalued function $x \mapsto f(x,u(x),\nabla u(x))$ has at least a measurable selection, that is, there exists a measurable function $\eta\colon \Omega \to \mathbb{R}$ such that $\eta(x) \in f(x,u(x),\nabla u(x))$ for a. a. $x \in \Omega$. From (H2)(iii) we find $M_1 > 0$ such that

$$\begin{aligned}
&\int_\Omega |\eta(x)|^{r(x)'}\,\mathrm{d}x\\
&\leq \int_\Omega \left(a_f|\nabla u|^{\frac{p(x)}{r(x)'}} + b_f|u|^{r(x)-1} + \alpha_f(x)\right)^{r(x)'}\,\mathrm{d}x\\
&\leq M_1 \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{r(x)} + \alpha_f(x)^{r(x)'}\right)\mathrm{d}x\\
&= M_1\left(\rho_{p(\cdot)}(|\nabla u|) + \rho_{r(\cdot)}(u) + \rho_{r'(\cdot)}(\alpha_f)\right)\\
&\leq M_1\left(\max\left\{\|\nabla u\|_{p(\cdot)}^{p_-}, \|\nabla u\|_{p(\cdot)}^{p_+}\right\} + \max\left\{\|u\|_{r(\cdot)}^{r_-}, \|u\|_{r(\cdot)}^{r_+}\right\} + \max\left\{\|\alpha_f\|_{r'(\cdot)}^{r'_-}, \|\alpha_f\|_{r'(\cdot)}^{r'_+}\right\}\right)\\
&< +\infty.
\end{aligned} \tag{3.1}$$

Note that we have used Proposition 2.1(iii), (iv) and the inequality

$$\int_\Omega c_2^{r(x)}\,\mathrm{d}x \leq \max\left\{|\Omega|c_2^{r_-}, |\Omega|c_2^{r_+}\right\} \quad \text{for any } c_2 > 0,$$

and the fact that the embeddings of V into $W^{1,p(\cdot)}(\Omega)$ and of V into $L^{r(\cdot)}(\Omega)$ are continuous. Therefore, we conclude that $\eta \in L^{r'(\cdot)}(\Omega)$. Using this consideration, we can introduce the Nemytskij operator $N_f\colon V \subset L^{r(\cdot)}(\Omega) \to 2^{L^{r'(\cdot)}(\Omega)}$ related to the multivalued mapping f given by

$$N_f(u) := \left\{\eta \in L^{r'(\cdot)}(\Omega) \,:\, \eta(x) \in f(x,u(x),\nabla u(x)) \text{ for a. a. } x \in \Omega\right\}$$

for all $u \in V$.

Let $u \in L^{\delta(\cdot)}(\Gamma_3)$ be fixed. Analogously, taking hypotheses (H5)(i)–(iii) into account, we find a measurable selection $\xi \colon \Gamma_3 \to \mathbb{R}$ of $x \mapsto U(x, u(x))$ and $M_2 > 0$ such that

$$\begin{aligned}\int_{\Gamma_3} |\xi(x)|^{\delta'(x)} \,\mathrm{d}\Gamma &\leq \int_{\Gamma_3} \left(\alpha_U(x) + a_U |u|^{\delta(x)-1}\right)^{\delta'(x)} \,\mathrm{d}\Gamma \\ &\leq M_2 \int_{\Gamma_3} \left(\alpha_U(x)^{\delta'(x)} + |u|^{\delta(x)}\right) \mathrm{d}\Gamma \\ &= M_2 \left(\rho_{\delta'(\cdot)}(\alpha_U) + \rho_{\delta(\cdot)}(u)\right) \\ &\leq M_2 \left(\max\left\{\|\alpha_U\|_{\delta'(\cdot)}^{\delta'_-}, \|\alpha_U\|_{\delta'(\cdot)}^{\delta'_+}\right\} + \max\left\{\|u\|_{\delta(\cdot)}^{\delta_-}, \|u\|_{\delta(\cdot)}^{\delta_+}\right\}\right).\end{aligned} \tag{3.2}$$

So, we consider the Nemytskij operator $N_U \colon L^{\delta(\cdot)}(\Gamma_3) \to 2^{L^{\delta'(\cdot)}(\Gamma_3)}$ corresponding to the multivalued mapping U defined by

$$N_U(u) := \left\{\eta \in L^{\delta'(\cdot)}(\Gamma_3) \,:\, \eta(x) \in U(x, u(x)) \text{ for a. a. } x \in \Gamma_3\right\}$$

for all $u \in L^{\delta(\cdot)}(\Gamma_3)$.

We denote by $\iota \colon V \to L^{r(\cdot)}(\Omega)$, $\omega \colon V \to L^{\varsigma(\cdot)}(\Omega)$ and $\beta \colon V \to L^{p(\cdot)}(\Omega)$ the embedding operators of V to $L^{r(\cdot)}(\Omega)$, of V to $L^{\varsigma(\cdot)}(\Omega)$ and of V to $L^{p(\cdot)}(\Omega)$, respectively. Moreover, we denote its adjoint operators by $\iota^* \colon L^{r'(\cdot)}(\Omega) \to V^*$, $\omega^* \colon L^{\varsigma'(\cdot)}(\Omega) \to V^*$ and $\beta^* \colon L^{p'(\cdot)}(\Omega) \to V^*$, respectively. Additionally, the trace operator of V into $L^{\delta(\cdot)}(\Gamma_3)$ is denoted by $\gamma \colon V \to L^{\delta(\cdot)}(\Gamma_3)$ and $\gamma^* \colon L^{\delta'(\cdot)}(\Gamma_3) \to V^*$ stands for its adjoint operator. Next, we consider the indicator function of set K given by

$$I_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Based on the considerations above we know that $u \in K$ is a weak solution of problem (1.1) if and only if it satisfies the following inclusion problem:

$$Fu + \omega^* Gu - \beta^* Lu - \iota^* N_f(u) - \gamma^* N_U(u) + \partial_c I_K(u) \ni h \quad \text{in } V^*,$$

with $\partial_c I_K$ being the convex subdifferential operator of I_K.

First, we see that F, G and L are bounded operators. Using this along with (3.1), (3.2) and hypotheses (H2) and (H5) guarantee that for every fixed $u \in V$ the set

$$H(u) := Fu + \omega^* Gu - \beta^* Lu - \iota^* N_f(u) - \gamma^* N_U(u)$$

is nonempty, bounded, closed and convex in V^*. Let us now prove the pseudomonotonicity of the operator H. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subset V$, $\{\zeta_n\}_{n\in\mathbb{N}} \subset V^*$ be sequences and let $(u, \zeta) \in V \times V^*$ be such that

$$\zeta_n \in H(u_n) \quad \text{for each } n \in \mathbb{N}, \quad \zeta_n \xrightarrow{w} \zeta \quad \text{and} \quad \limsup_{n\to\infty} \langle \zeta_n, u_n - u\rangle \leq 0. \tag{3.3}$$

Hence, for each $n \in \mathbb{N}$, we can find functions $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ such that

$$\zeta_n = Fu_n + \omega^* Gu_n - \beta^* Lu_n - \iota^* \eta_n - \gamma^* \xi_n \quad \text{for all } n \in \mathbb{N}.$$

By virtue of (3.1) and (3.2), it can be easily shown that the sequences $\{\eta_n\}_{n\in\mathbb{N}} \subset L^{r'(\cdot)}(\Omega)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset L^{\delta'(\cdot)}(\Gamma_3)$ are bounded. Furthermore, we can find functions $(\eta, \xi) \in L^{r'(\cdot)}(\Omega) \times L^{\delta'(\cdot)}(\Gamma_3)$ such that

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r'(\cdot)}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{\delta'(\cdot)}(\Gamma_3).$$

Due to the compactness of V into $L^{\varsigma(\cdot)}(\Omega)$, $L^{r(\cdot)}(\Omega)$, $L^{p(\cdot)}(\Omega)$, respectively, and the compactness of $\gamma\colon V \to L^{\delta(\cdot)}(\Gamma_3)$ we obtain

$$\begin{aligned}
\lim_{n\to\infty} \langle \omega^* G u_n, u_n - u\rangle &= \lim_{n\to\infty} \langle G u_n, \omega(u_n-u)\rangle_{L^{\varsigma'(\cdot)}(\Omega)\times L^{\varsigma(\cdot)}(\Omega)} = 0,\\
\lim_{n\to\infty} \langle \beta^* L u_n, u_n - u\rangle &= \lim_{n\to\infty} \langle L u_n, \beta(u_n-u)\rangle_{L^{p'(\cdot)}(\Omega)\times L^{p(\cdot)}(\Omega)} = 0,\\
\lim_{n\to\infty} \langle \iota^* \eta_n, u_n - u\rangle &= \lim_{n\to\infty} \langle \eta_n, \iota(u_n-u)\rangle_{L^{r'(\cdot)}(\Omega)\times L^{r(\cdot)}(\Omega)} = 0,\\
\lim_{n\to\infty} \langle \gamma^* \xi_n, u_n - u\rangle &= \lim_{n\to\infty} \langle \xi_n, \gamma(u_n-u)\rangle_{L^{\delta'(\cdot)}(\Gamma_3)\times L^{\delta(\cdot)}(\Gamma_3)} = 0.
\end{aligned}\tag{3.4}$$

Now, using (3.4) and (3.3) gives

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty} \langle \zeta_n, u_n - u\rangle\\
&\geq \limsup_{n\to\infty} \langle F u_n, u_n - u\rangle + \liminf_{n\to\infty} \langle \omega^* G u_n, u_n - u\rangle - \limsup_{n\to\infty} \langle \beta^* L u_n, u - u_n\rangle\\
&\quad + \liminf_{n\to\infty} \langle \iota^* \eta_n, u - u_n\rangle + \liminf_{n\to\infty} \langle \gamma^* \xi_n, u - u_n\rangle\\
&\geq \limsup_{n\to\infty} \langle F u_n, u_n - u\rangle.
\end{aligned}$$

Since F satisfies the (S_+)-property, we conclude that

$$u_n \to u \quad \text{in } V,$$

see Proposition 2.5. If we pass to a subsequence if necessary, we may suppose that

$$u_n(x) \to u(x) \quad \text{and} \quad \nabla u_n(x) \to \nabla u(x) \quad \text{for a. a. } x \in \Omega. \tag{3.5}$$

Taking Mazur's theorem into account, we are able to find a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ of convex combinations of $\{\eta_n\}_{n\in\mathbb{N}}$ such that

$$\chi_n \to \eta \quad \text{in } L^{r'(\cdot)}(\Omega).$$

Thus, we may assume that $\chi_n(x) \to \eta(x)$ for a. a. $x \in \Omega$. From the convexity of f we conclude that

$$\chi_n(x) \in f(x, u_n(x), \nabla u_n(x)) \quad \text{for a. a. } x \in \Omega.$$

Additionally, we apply the upper semicontinuity of f, hypotheses (H2)(i), (ii) and Denkowski-Migórski-Papageorgiou [13, Proposition 4.1.9] to obtain that the graph of $(s,\xi) \mapsto f(x,s,\xi)$ is closed for a. a. $x \in \Omega$. Also, (3.5) and $\chi_n(x) \to \eta(x)$ for a. a. $x \in \Omega$ turn out that

$$\eta(x) \in f(x, u(x), \nabla u(x)) \quad \text{for a. a. } x \in \Omega.$$

Hence $\eta \in N_f(u)$. Similarly, we can show that $\xi \in N_U(u)$. The continuity of F, G as well as L and the convergence properties in (3.3) imply that

$$\zeta_n = F u_n + \omega^* G u_n - \beta^* L u_n - \iota^* \eta_n - \gamma^* \xi_n \xrightarrow{w} F u + \omega^* G u - \beta^* L u - \iota^* \eta - \gamma^* \xi = \zeta \quad \text{in } V^*.$$

This shows that $\zeta \in H(u)$ and

$$\begin{aligned}
&\lim_{n\to\infty} \langle \zeta_n, u_n\rangle\\
&= \lim_{n\to\infty} \langle F u_n + \omega^* G u_n - \beta^* L u_n - \iota^* \eta_n - \gamma^* \xi_n, u_n\rangle\\
&= \lim_{n\to\infty} \langle F u_n + \omega^* G u_n - \beta^* L u_n, u_n\rangle - \lim_{n\to\infty} \langle \eta_n, \iota u_n\rangle_{L^{r'(\cdot)}(\Omega)\times L^{r(\cdot)}(\Omega)}\\
&\quad - \lim_{n\to\infty} \langle \xi_n, \gamma u_n\rangle_{L^{\delta'(\cdot)}(\Gamma_3)\times L^{\delta(\cdot)}(\Gamma_3)}\\
&= \langle F u + \omega^* G u - \beta^* L u - \iota^* \eta - \gamma^* \xi, u\rangle = \langle \zeta, u\rangle.
\end{aligned}$$

Therefore, H is a generalized pseudomonotone operator and due to Proposition 2.7 we infer that H is pseudomonotone as well.

Now we are going to prove that H is coercive. To this end, let $u \in V$ and $\zeta \in H(u)$ be arbitrary. Hence, there exist functions $\eta \in N_f(u)$ and $\xi \in N_U(u)$ satisfying $\zeta = Fu+\omega^* Gu-\beta^* Lu-\iota^*\eta-\gamma^*\xi$ and

$$\begin{aligned}
&\langle \zeta, u\rangle\\
&= \langle Fu, u\rangle + \langle \omega^* Gu - \beta^* Lu, u\rangle - \langle \eta, u\rangle_{L^{r'(\cdot)}(\Omega)\times L^{r(\cdot)}(\Omega)} - \langle \xi, u\rangle_{L^{\delta'(\cdot)}(\Gamma_3)\times L^{\delta(\cdot)}(\Gamma_3)}\\
&\geq c_\Lambda \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u|^{q(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|u|^{q(x)}\,\mathrm{d}x + \int_\Omega a_g|u|^{\varsigma(x)} - b_g(x)\,\mathrm{d}x\\
&\quad - \int_\Omega c_f|\nabla u|^{p(x)} + d_f|u|^{p(x)} + \beta_f(x)\,\mathrm{d}x - \int_{\Gamma_3} b_U|u|^{p_-} + \beta_U(x)\,\mathrm{d}\Gamma \qquad (3.6)\\
&\geq (c_\Lambda - c_f)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u|^{q(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|u|^{q(x)}\,\mathrm{d}x + \int_\Omega |u|^{p(x)}\,\mathrm{d}x\\
&\quad + \int_\Omega a_g|u|^{\varsigma(x)}\,\mathrm{d}x - \|b_g\|_{1,\Omega} - (d_f+1)\int_\Omega |u|^{p(x)}\,\mathrm{d}x - \|\beta_f\|_{1,\Omega} - b_U\|u\|_{p_-,\Gamma_3}^{p_-} - \|\beta_U\|_{1,\Gamma_3}.
\end{aligned}$$

We choose $\varepsilon = \frac{a_g}{2\left(\left(\lambda_{1,p}^S\right)^{-1} b_U + d_f + 1\right)}$ and recall that we have the inequality

$$b_U\|u\|_{p_-,\Gamma_3}^{p_-} \leq b_U\left(\lambda_{1,p_-}^S\right)^{-1}\left(\|\nabla u\|_{p_-,\Omega}^{p_-} + \|u\|_{p_-,\Omega}^{p_-}\right) \qquad (3.7)$$

from the p_--Laplacian eigenvalue problem with Steklov boundary condition, see (2.1) and (2.2). Moreover, since $\varsigma(x) > p(x)$ for all $x \in \overline{\Omega}$, it follows from Young's inequality that

$$\begin{aligned}
\int_\Omega |u|^{p(x)}\,\mathrm{d}x &\leq \varepsilon \int_\Omega |u|^{\varsigma(x)}\,\mathrm{d}x + c_1(\varepsilon)\\
\|u\|_{p_-,\Omega}^{p_-} &\leq \varepsilon \int_\Omega |u|^{\varsigma(x)}\,\mathrm{d}x + c_2(\varepsilon) \qquad (3.8)\\
\|\nabla u\|_{p_-,\Omega}^{p_-} &\leq \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + c_3
\end{aligned}$$

with some $c_1(\varepsilon), c_2(\varepsilon), c_3 > 0$. Then, from (3.6), (3.7) and (3.8) we can find a constant $c_4(\varepsilon) > 0$ such that

$$\begin{aligned}
&\langle \zeta, u\rangle\\
&\geq \left(c_\Lambda - c_f - b_U\left(\lambda_{1,p_-}^S\right)^{-1}\right)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u|^{q(x)}\,\mathrm{d}x\\
&\quad + \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right)\mathrm{d}x + \frac{a_g}{2}\int_\Omega |u|^{\varsigma(x)}\,\mathrm{d}x - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3}\\
&\geq \hat{M}_0\varrho_{\mathcal{H}}(u) + \frac{a_g}{2}\int_\Omega |u|^{\varsigma(x)}\,\mathrm{d}x - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c_4(\varepsilon) \qquad (3.9)\\
&\geq \hat{M}_0 \min\left\{\|u\|_V^{p_-}, \|u\|_V^{q_+}\right\} + \frac{a_g}{2}\min\left\{\|u\|_{\varsigma(\cdot)}^{\varsigma_-}, \|u\|_{\varsigma(\cdot)}^{\varsigma_+}\right\}\\
&\quad - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c_4(\varepsilon),
\end{aligned}$$

where we have used Proposition 2.4 and $\hat{M}_0 > 0$ is defined by

$$\hat{M}_0 := \min\left\{c_\Lambda - c_f - b_U\left(\lambda_{1,p_-}^S\right)^{-1}, 1\right\}.$$

Due to $c_\Lambda - c_f - b_U\left(\lambda_{1,p_-}^S\right)^{-1} > 0$ by (H7), we infer that H is coercive.

We know that I_K is a proper, convex and l.s.c. function and it holds

$$I_K(u) \geq \alpha_K\|u\|_V \quad \text{for all } u \in V \text{ with some } \alpha_K < 0.$$

Therefore, we have

$$\langle \kappa, u\rangle \geq I_K(u) - I_K(0) \geq \alpha_K\|u\|_V \quad \text{for all } \kappa \in \partial_c I_K(u) \text{ and for all } u \in K, \qquad (3.10)$$

since $0 \in K$. Combining (3.10) and (3.9) leads to

$$\begin{aligned}&\langle \zeta + \kappa - h, u\rangle\\ &\geq \hat{M}_0 \min\left\{\|u\|_V^{p_-}, \|u\|_V^{q_+}\right\} + \frac{a_g}{2}\min\left\{\|u\|_{\varsigma(\cdot)}^{\varsigma_-}, \|u\|_{\varsigma(\cdot)}^{\varsigma_+}\right\} - \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - c_4(\varepsilon)\\ &\quad - |\alpha_K|\|u\|_V - M_3\|h\|_{p'(\cdot),\Gamma_2}\|u\|_V\end{aligned}$$

for all $\zeta \in H(u)$ and for all $\kappa \in \partial_c I_K(u)$ with for some $M_3 > 0$. Thus, we see that (2.10) is fulfilled by taking $u_0 = 0$, $\mathcal{G} = \partial_c I_K$ and $\mathcal{F} = H$. Now we can apply Theorem 2.9 which ensures the existence of at least one nontrivial solution $u \in K$ of problem (1.1).

Part II Boundedness of $\mathcal{S}(a,h)$: Let us assume the assertion is not true, so we suppose that the set $\mathcal{S}(a,h)$ is unbounded. Then we are able to find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(a,h)$ satisfying $\|u_n\|_V \to +\infty$. Using the same treatment as in Part I, we have

$$\begin{aligned}0 \geq &\hat{M}_0 \min\left\{\|u_n\|_V^{p_-}, \|u_n\|_V^{q_+}\right\} + \frac{a_g}{2}\min\left\{\|u_n\|_{\varsigma(\cdot)}^{\varsigma_-}, \|u_n\|_{\varsigma(\cdot)}^{\varsigma_+}\right\}\\ &- \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - M_4\|h\|_{p'(\cdot),\Gamma_2}\|u_n\|_V - M_5\end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N}$ and for some $M_4, M_5 > 0$. Letting $n \to \infty$ in (3.11), this leads to a contradiction and so, the solution set $\mathcal{S}(a,h)$ is bounded in V.

Part III Weak closedness of $\mathcal{S}(a,h)$: Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(a,h)$ be such that

$$u_n \xrightarrow{w} u \quad \text{in } V$$

for some $u \in K$. Then, we can find functions $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ satisfying

$$\begin{aligned}&\langle Fu_n + \omega^* G u_n - \beta^* L u_n, v - u_n\rangle\\ &\geq \int_\Omega \eta_n(v-u_n)\,\mathrm{d}x + \int_{\Gamma_2} h(x)(v-u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_3} \xi_n(x)(v-u_n)\,\mathrm{d}\Gamma\end{aligned} \tag{3.12}$$

for all $v \in K$. From the boundedness of the operators N_f and N_U there exist functions $\eta \in L^{r'(\cdot)}(\Omega)$ and $\xi \in L^{\delta'(\cdot)}(\Gamma_3)$ satisfying

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r'(\cdot)}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{\delta'(\cdot)}(\Gamma_3).$$

Now we can choose $v = u$ in (3.12) and pass to the upper limit as $n \to \infty$. This yields

$$\begin{aligned}&\limsup_{n\to\infty}\, \langle Fu_n, u_n - u\rangle\\ &\leq \lim_{n\to\infty}\int_\Omega (\eta_n + |u_n|^{p(x)-2}u_n + g(x,u_n))(u-u_n)\,\mathrm{d}x + \lim_{n\to\infty}\int_{\Gamma_2} h(x)(u-u_n)\,\mathrm{d}\Gamma\\ &\quad + \lim_{n\to\infty}\int_{\Gamma_3}\xi_n(u-u_n)\,\mathrm{d}\Gamma\\ &\leq 0.\end{aligned}$$

From Proposition 2.5 we conclude that $u_n \to u$ in V. Then, by the upper semicontinuity of f and U, we have $\eta \in N_f(u)$ and $\xi \in N_U(u)$. Passing to the upper limit as $n \to \infty$ in (3.12), we obtain $u \in \mathcal{S}(a,h)$. Thus, $\mathcal{S}(a,h)$ is weakly closed. □

4. Inverse problem for variable exponents double phase obstacle system

In this section we study and solve a nonlinear inverse problem which is formulated by a regularized optimal control problem to identify the discontinuous parameters in problem (1.1).

In order to formulate the problem, we first recall the notion of total variation and bounded variation functions. To this end, for any fixed $g \in L^1(\Omega)$, we denote by $\mathrm{TV}(g)$ the total variation of function g defined by

$$\mathrm{TV}(g) := \sup_{\varphi\in C^1(\Omega;\mathbb{R}^N)}\left\{\int_\Omega g(x)\ \mathrm{div}\,\varphi(x)\,\mathrm{d}x \ :\ |\varphi(x)| \leq 1 \text{ for all } x \in \Omega\right\}.$$

Furthermore, $\mathrm{BV}(\Omega)$ stands for the function space of all integrable functions with bounded variation given by

$$\mathrm{BV}(\Omega) := \left\{ g \in L^1(\Omega) \, : \, \mathrm{TV}(g) < +\infty \right\},$$

equipped with the norm

$$\|g\|_{\mathrm{BV}(\Omega)} := \|g\|_{1,\Omega} + \mathrm{TV}(g) \quad \text{for all } g \in \mathrm{BV}(\Omega).$$

We know that $(\mathrm{BV}(\Omega), \|\cdot\|_{\mathrm{BV}(\Omega)})$ is a Banach space.

Moreover, let H be a nonempty, closed and convex subset of $L^{p'(\cdot)}(\Gamma_2)$ and denote by Λ the set of admissible parameters for the anisotropic double phase differential operator given in (1.2) defined by

$$\Lambda := \{ a \in \mathrm{BV}(\Omega) \, : \, 0 < c_\Lambda \leq a(x) \leq d_\Lambda \text{ for a. a. } x \in \Omega \},$$

where c_Λ and d_Λ are given positive constants. It is clear that Λ is a closed and convex subset of $\mathrm{BV}(\Omega)$ and $L^\infty(\Omega)$.

Now, let $\kappa > 0$ and $\tau > 0$ be two given regularization parameters and let $z \in L^{p(\cdot)}(\Omega;\mathbb{R}^N)$ be the known or measured datum. We study the inverse problem formulated in the following regularized optimal control setting:

Problem 4.1. *Find $a^* \in \Lambda$ and $h^* \in H$ such that*

$$\inf_{a\in\Lambda \text{ and } h\in H} C(a,h) = C(a^*,h^*), \tag{4.1}$$

where the cost functional $C\colon \Lambda \times H \to \mathbb{R}$ is defined by

$$C(a,h) := \min_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)} \,\mathrm{d}x + \kappa \, \mathrm{TV}(a) + \tau \int_{\Gamma_2} |h|^{p'(x)} \,\mathrm{d}\Gamma. \tag{4.2}$$

Here, $\mathcal{S}(a,h)$ stands for the solution set of problem (1.1) related to $a \in L^\infty(\Omega)$ and $h \in L^{p'(\cdot)}(\Gamma_2)$.

Our main result in this section reads as follows.

Theorem 4.2. *Assume that* (H1)–(H7) *are satisfied. Then the solution set of* Problem 4.1 *is nonempty and weakly compact.*

Proof. We are going to show the proof within four steps.

Step I: The functional C defined by (4.2) is well-defined.

We point out that it is enough to prove that for any fixed $(a,h) \in \Lambda \times H$ the optimal problem

$$\min_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)} \,\mathrm{d}x$$

is solvable. For this purpose, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(a,h)$ be a minimizing sequence of the following problem

$$\inf_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)} \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)} \,\mathrm{d}x.$$

First, we observe that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V due to Theorem 3.2. This permits us to find a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in V$. Since $\mathcal{S}(a,h)$ is weakly closed, we infer that $u^* \in \mathcal{S}(a,h)$. By the weak lower semicontinuity of the function

$$u \mapsto \int_\Omega |\nabla u - z|^{p(x)} \,\mathrm{d}x$$

(in fact, it is convex and lower semicontinuous), we obtain

$$\begin{aligned}\inf_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)} \,\mathrm{d}x &= \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)} \,\mathrm{d}x \\ &\geq \int_\Omega |\nabla u^* - z|^{p(x)} \,\mathrm{d}x\end{aligned}$$

$$\geq \inf_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x.$$

From this we see that for each $(a,h)\in\Lambda\times H$ we can find $u^*\in\mathcal{S}(a,h)$ such that

$$\inf_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x = \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x,$$

which implies that C is well-defined.

Note that, for any fixed $(a,h)\in\Lambda\times H$ and $u\in\mathcal{S}(a,h)$, by (3.11), we have

$$\begin{aligned}0 \geq &\hat{M}_0 \min\left\{\|u\|_V^{p_-}, \|u\|_V^{q_+}\right\} + \frac{a_g}{2}\min\left\{\|u\|_{\varsigma(\cdot)}^{\varsigma_-}, \|u\|_{\varsigma(\cdot)}^{\varsigma_+}\right\}\\ &- \|b_g\|_{1,\Omega} - \|\beta_f\|_{1,\Omega} - \|\beta_U\|_{1,\Gamma_3} - M_6\|h\|_{p'(\cdot),\Gamma_2}\|u\|_V - M_7\end{aligned}$$

for some $M_6, M_7>0$. This shows that $\mathcal{S}$ maps bounded sets of $\Lambda\times H\subset \mathrm{BV}(\Omega)\times L^{p'(\cdot)}(\Gamma_2)$ into bounded sets of K.

Step II: If $\{(a_n,h_n)\}_{n\in\mathbb{N}}\subset\Lambda\times H$ is such that $\{a_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $a_n\to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in H for some $(a,h)\in L^1(\Omega)\times H$, then $a\in\Lambda$ and

$$\emptyset \neq w\text{–}\limsup_{n\to\infty} \mathcal{S}(a_n,h_n) \subset \mathcal{S}(a,h). \tag{4.3}$$

To this end, let $\{(a_n,h_n)\}_{n\in\mathbb{N}}\subset\Lambda\times H$ be a sequence such that $a_n\to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in H for some $(a,h)\in L^1(\Omega)\times H$. From the definition of Λ, it is not difficult to see that $(a,h)\in\Lambda\times H$. Because $\{a_n\}\subset \mathrm{BV}(\Omega)\cap L^\infty(\Omega)$ is bounded and $\mathcal{S}$ is a bounded map, we know that $\cup_{n\geq 1}\mathcal{S}(a_n,h_n)$ is bounded in K as well. Moreover, from the reflexivity of V we conclude that the set $w\text{–}\limsup_{n\to\infty}\mathcal{S}(a_n,h_n)$ is nonempty.

Let $u\in w\text{–}\limsup_{n\to\infty}\mathcal{S}(a_n,h_n)$ be arbitrary. Passing to a subsequence if necessary, we are able to find a sequence $\{u_n\}_{n\in\mathbb{N}}\subset K$ satisfying

$$u_n\in\mathcal{S}(a_n,h_n) \quad\text{and}\quad u_n \xrightarrow{w} u \quad \text{in } V.$$

So, for every $n\in\mathbb{N}$, there exist functions $\eta_n\in N_f(u_n)$ and $\xi_n\in N_U(u_n)$ such that

$$\begin{aligned}&\int_\Omega \left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)\,\mathrm{d}x\\ &\quad+\int_\Omega \mu(x)|u_n|^{q(x)-2}u_n(v-u_n)\,\mathrm{d}x + \int_\Omega g(x,u_n)(v-u_n)\,\mathrm{d}x\\ &\geq \int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x + \int_{\Gamma_2} h_n(x)(v-u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_3}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma\end{aligned} \tag{4.4}$$

for all $v\in K$. If we choose $v=u$ in (4.4), then we obtain

$$\begin{aligned}&\int_\Omega \left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(u-u_n)\,\mathrm{d}x\\ &\quad+\int_\Omega \mu(x)|u_n|^{q(x)-2}u_n(u-u_n)\,\mathrm{d}x + \int_\Omega g(x,u_n)(u-u_n)\,\mathrm{d}x\\ &\geq \int_\Omega \eta_n(x)(u-u_n)\,\mathrm{d}x + \int_{\Gamma_2} h_n(x)(u-u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_3}\xi_n(x)(u-u_n)\,\mathrm{d}\Gamma.\end{aligned} \tag{4.5}$$

From assumptions (H2)(iii) and (H5)(iv) we know that the sequences $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are bounded in $L^{r'(\cdot)}(\Omega)$ and $L^{\delta'(\cdot)}(\Gamma_3)$, respectively. Recall that V is compactly embedded into

$L^{\varsigma(\cdot)}(\Omega)$, $L^{r(\cdot)}(\Omega)$, $L^{p(\cdot)}(\Gamma_2)$ and $L^{\delta(\cdot)}(\Gamma_3)$, respectively. From this we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \mu(x)|u_n|^{q(x)-2}u_n(u_n-u)\,\mathrm{d}x=0,\\
&\lim_{n\to\infty}\int_\Omega g(x,u_n)(u-u_n)\,\mathrm{d}x=0,\\
&\lim_{n\to\infty}\int_\Omega \eta_n(x)(u-u_n)\,\mathrm{d}x=0,\\
&\lim_{n\to\infty}\int_{\Gamma_2} h_n(x)(u-u_n)\,\mathrm{d}\Gamma=0,\\
&\lim_{n\to\infty}\int_{\Gamma_3} \xi_n(x)(u-u_n)\,\mathrm{d}\Gamma=0.
\end{aligned} \tag{4.6}$$

On the other hand, from Hölder's inequality, we get

$$\begin{aligned}
&\int_\Omega \Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\geq -\int_\Omega |a_n(x)-a(x)||\nabla u|^{p(x)-1}|\nabla(u_n-u)|\,\mathrm{d}x\\
&\geq -\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\Big\||a_n(\cdot)-a(\cdot)||\nabla u|\Big\|_{\frac{p(\cdot)}{p(\cdot)-1},\Omega}\|\nabla(u-u_n)\|_{p(\cdot),\Omega}\\
&\geq -\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\|\nabla(u-u_n)\|_{p(\cdot),\Omega}\times\min\Bigg\{\left(\int_\Omega |a_n(x)-a(x)|^{\frac{p(x)}{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\left(\frac{p}{p-1}\right)_-},\\
&\qquad\left(\int_\Omega |a_n(x)-a(x)|^{\frac{p(x)}{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\left(\frac{p}{p-1}\right)_+}\Bigg\}
\end{aligned}$$

where the last inequality is obtained by using Proposition 2.1. Since $a_n\to a$ in $L^1(\Omega)$, we may assume that $a_n(x)\to a(x)$ for a. a. $x\in\Omega$. The boundedness of $\{u_n\}_{n\in\mathbb{N}}\subset V$ and $\{a_n\}_{n\in\mathbb{N}}\subset L^\infty(\Omega)$ along with Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty}\left(\int_\Omega |a_n(x)-a(x)|^{\frac{p(x)}{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\left(\frac{p}{p-1}\right)_\pm}\times\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\|\nabla(u-u_n)\|_{p(\cdot),\Omega}=0.$$

Hence, we have

$$\lim_{n\to\infty}\int_\Omega \Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\geq 0. \tag{4.7}$$

Note that $u_n\xrightarrow{w}u$ in V. This implies

$$\lim_{n\to\infty}\left[\int_\Omega \Big(a(x)|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(u-u_n)\,\mathrm{d}x\right]=0. \tag{4.8}$$

From the monotonicity of $s \mapsto |s|^{q(x)-2}s$, we have

$$\begin{aligned}
&\int_\Omega \left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&= \int_\Omega a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \left((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \left(\mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\geq \int_\Omega a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \left((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \left(\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x.
\end{aligned}\tag{4.9}$$

Letting go to the limes superior in (4.5) as $n\to\infty$ and using (4.6), (4.7), (4.8) and (4.9) results in

$$\limsup_{n\to\infty}\int_\Omega a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x \leq 0.$$

The latter combined with the nonnegativity of $(|s|^{p(x)-2}s - |t|^{p(x)-2}t)(s-t)$ for all $s,t\in\mathbb{R}^N$ implies

$$\lim_{n\to\infty}\int_\Omega a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x = 0. \tag{4.10}$$

Next, we recall the well-known inequalities of Simon [49, formula (2.2)], namely

$$M_s|\xi-\eta|^s \leq \left(|\xi|^{s-2}\xi - |\eta|^{s-2}\eta\right)\cdot(\xi-\eta), \quad \text{if } \ s\geq 2, \tag{4.11}$$

$$\mathcal{M}_s|\xi-\eta|^2 \leq \left(|\xi|^{s-2}\xi - |\eta|^{s-2}\eta\right)\cdot(\xi-\eta)\left(|\xi|^s+|\eta|^s\right)^{\frac{2-s}{s}}, \quad \text{if } \ 1<s<2, \tag{4.12}$$

for all ξ, $\eta\in\mathbb{R}^N$, where the constants M_s, $\mathcal{M}_s>0$ are independent of ξ, $\eta\in\mathbb{R}^N$ given by

$$M_s = 5^{\frac{2-s}{2}} \quad \text{and} \quad \mathcal{M}_s = (s-1)2^{\frac{(s-1)(s-2)}{s}}.$$

We set

$$c_p := \min_{x\in\overline{\Omega}} 5^{\frac{2-p(x)}{2}} \quad \text{and} \quad C_p := \min_{x\in\overline{\Omega}}(p(x)-1)2^{\frac{(p(x)-1)(p(x)-2)}{p(x)}}.$$

For $p\in C_+(\overline{\Omega})$ we split the domain Ω into two mutually disjoint parts $\Omega_{p\geq 2}$ and $\Omega_{p<2}$, that is, $\Omega=\Omega_{p\geq 2}\cup\Omega_{p<2}$ and $\Omega_{p\geq 2}\cap\Omega_{p<2}=\emptyset$, where $\Omega_{p\geq 2}$ and $\Omega_{p<2}$ are given by

$$\Omega_{p\geq 2} := \{x\in\Omega \,:\, p(x)\geq 2\} \quad \text{and} \quad \Omega_{p<2} := \{x\in\Omega \,:\, p(x)<2\}.$$

In the domain $\Omega_{p\geq 2}$, we can use (4.11) to get

$$\begin{aligned}
&\int_{\Omega_{p\geq 2}} a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\geq \int_{\Omega_{p\geq 2}} a_n(x)M_{p(x)}|\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x\\
&\geq c_\Lambda c_p\rho_{p(\cdot),\Omega_{p\geq 2}}(|\nabla u_n-\nabla u|).
\end{aligned}\tag{4.13}$$

We set

$$\Omega_n=\{x\in\Omega\,:\,\nabla u_n\neq 0\}\cup\{x\in\Omega\,:\,\nabla u\neq 0\}\quad\text{and}\quad\Sigma_n=\{x\in\Omega\,:\,\nabla u=\nabla u_n=0\}.$$

Then, we have $\Omega=\Omega_n\cup\Sigma_n$ and $\Omega_n\cap\Sigma_n=\emptyset$. Using the absolute continuity of the Lebesgue integral gives

$$\int_{\Sigma_n}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x=0.$$

This implies

$$\begin{aligned}&\int_{\Omega}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\&=\int_{\Omega_n}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\&\quad+\int_{\Sigma_n}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\&=\int_{\Omega_n}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x.\end{aligned}$$

Concerning the part $\Omega_{p<2}$, it follows from (4.12) that

$$\begin{aligned}&\int_{\Omega_{p<2}}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\&=\int_{\Omega_n\cap\Omega_{p<2}}a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\\&\qquad\qquad\cdot\nabla(u_n-u)\frac{\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{2-p(x)}{p(x)}}}{\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{2-p(x)}{p(x)}}}\,\mathrm{d}x\\&\geq\int_{\Omega_n\cap\Omega_{p<2}}\mathcal{M}_{p(x)}a_n(x)\,|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x\\&\geq C_p\int_{\Omega_n\cap\Omega_{p<2}}a_n(x)\,|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x\\&\geq c_\Lambda C_p\int_{\Omega_n\cap\Omega_{p<2}}|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x.\end{aligned}$$

Since $1<p(x)<2$, we have $\frac{2}{p(x)}>1$. From Hölder's inequality, we obtain

$$\begin{aligned}&\int_{\Omega_n\cap\Omega_{p<2}}|\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x\\&=\int_{\Omega_n\cap\Omega_{p<2}}|\nabla u_n-\nabla u|^{2\times\frac{p(x)}{2}}\,\mathrm{d}x\\&=\int_{\Omega_n\cap\Omega_{p<2}}\left(|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\right)^{\frac{p(x)}{2}}\\&\qquad\times\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)^{\frac{2-p(x)}{2}}\,\mathrm{d}x\\&\leq\left[\frac{1}{\left(\frac{2}{p}\right)_-}+\frac{1}{\left(\frac{2}{2-p}\right)_-}\right]\|l_1\|_{\frac{2}{p(\cdot)},\Omega_n\cap\Omega_{p<2}}\|l_2\|_{\frac{2}{2-p(\cdot)},\Omega_n\cap\Omega_{p<2}},\end{aligned}$$

where functions l_1, l_2 are given by

$$l_1(x) = \left(|\nabla u_n - \nabla u|^2 \left(|\nabla u_n|^{p(x)} + |\nabla u|^{p(x)} \right)^{\frac{p(x)-2}{p(x)}} \right)^{\frac{p(x)}{2}},$$

$$l_2(x) = \left(|\nabla u_n|^{p(x)} + |\nabla u|^{p(x)} \right)^{\frac{2-p(x)}{2}},$$

respectively. Comparing the norm and the modular, see Proposition 2.1, we obtain

$$\begin{aligned}
&\|l_1\|_{\frac{2}{p(\cdot)},\Omega_n\cap\Omega_{p<2}} \|l_2\|_{\frac{2}{2-p(\cdot)},\Omega_n\cap\Omega_{p<2}} \\
&\leq \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{p}\right)_+}} \right\} \\
&\quad \times \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\}.
\end{aligned}$$

From the last two inequalities we infer

$$\begin{aligned}
&\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n - \nabla u|^{p(x)} \,\mathrm{d}x \\
&\leq \left[\frac{1}{\left(\frac{2}{p}\right)_-} + \frac{1}{\left(\frac{2}{2-p}\right)_-} \right] \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{p}\right)_+}} \right\} \\
&\quad \times \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\},
\end{aligned}$$

that is,

$$\begin{aligned}
&\left[\frac{1}{\left(\frac{2}{p}\right)_-} + \frac{1}{\left(\frac{2}{2-p}\right)_-} \right]^{-1} \int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n - \nabla u|^{p(x)} \,\mathrm{d}x \\
&\times \left(\max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\} \right)^{-1} \\
&\leq \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{p}\right)_+}} \right\}.
\end{aligned}$$

Let $\overline{M}_0 > 0$ be such that

$$\begin{aligned}
\overline{M}_0 \geq \max&\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}} \,\mathrm{d}x \right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\} \\
&\times \left[\frac{1}{\left(\frac{2}{p}\right)_-} + \frac{1}{\left(\frac{2}{2-p}\right)_-} \right]
\end{aligned}$$

for all $n \in \mathbb{N}$ thanks to the boundedness of $\{u_n\}_{n\in\mathbb{N}}$. From (4.10) it follows that the limit superior of $\int_\Omega l_1(x)^{\frac{2}{p(x)}}\,\mathrm{d}x$ is strictly smaller than one. Therefore, we have

$$
\begin{aligned}
&\max\left\{\left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\,\mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\,\mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_+}}\right\}\\
&=\left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\,\mathrm{d}x\right)^{\frac{p_-}{2}}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\Omega_{p<2}} a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\geq c_\Lambda C_p\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x\\
&\geq c_\Lambda C_p\left(\overline{M}_0^{-1}\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\frac{p_-}{2}}\\
&= c_\Lambda C_p\left(\overline{M}_0^{-1}\int_{\Omega_{p<2}} |\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\frac{p_-}{2}}.
\end{aligned}
$$

Inserting the inequality above and (4.13) into (4.10) gives

$$
\rho_{p(\cdot)}(|\nabla u_n-\nabla u|)=\int_\Omega |\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x\to 0.
$$

The latter combined with Proposition 2.1 implies

$$
u_n\to u \quad \text{in } W^{1,p(\cdot)}(\Omega).
$$

In addition, the boundedness of $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ as well as the reflexivity of $L^{r'(\cdot)}(\Omega)$ and $L^{\delta'(\cdot)}(\Gamma_3)$ point out that there exist subsequences of $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$, not relabeled, and functions $\eta\in L^{r'(\cdot)}(\Omega)$ and $\xi\in L^{\delta'(\cdot)}(\Gamma_3)$ satisfying

$$
\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r(\cdot)'}(\Omega) \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } L^{\delta(\cdot)'}(\Gamma_3).
$$

Since $u_n\to u$ in $W^{1,p(\cdot)}$ we may assume that $\nabla u_n(x)\to\nabla u(x)$ and $u_n(x)\to u(x)$ for a. a. $x\in\Omega$. Using the same arguments as in the proof of Theorem 3.2, we conclude that $\eta\in N_f(u)$ and $\xi\in N_U(u)$. Using Lebesgue's dominated convergence theorem leads to

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega\left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n+\mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)\,\mathrm{d}x\\
&=\int_\Omega\lim_{n\to\infty}\left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n+\mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)\,\mathrm{d}x\\
&=\int_\Omega\left(a(x)|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x,
\end{aligned}
$$

due to the boundedness of $\{a_n\}_{n\in\mathbb{N}}\subset L^\infty(\Omega)$ and $\{u_n\}_{n\in\mathbb{N}}\subset V$. Letting $n\to\infty$ in equality (4.4), from the convergence results above, we obtain that

$$
\begin{aligned}
&\int_\Omega\left(a(x)|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\mu(x)|u|^{q(x)-2}u(v-u)\,\mathrm{d}x+\int_\Omega g(x,u)(v-u)\,\mathrm{d}x
\end{aligned}
$$

$$\geq \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_2} h(x)(v-u)\,\mathrm{d}\Gamma + \int_{\Gamma_3} \xi(x)(v-u)\,\mathrm{d}\Gamma$$

for all $v \in K$. This implies that $u \in K$ is a solution of problem (1.1) related to $(a,h) \in \Lambda \times H$. Thus, $u \in \mathcal{S}(a,h)$ and so we have $\emptyset \neq w\text{--}\limsup_{n\to\infty} \mathcal{S}(a_n,h_n) \subset \mathcal{S}(a,h)$ which shows (4.3).

Step III: If $\{(a_n,h_n)\}_{n\in\mathbb{N}} \subset \Lambda \times H$ is such that $\{a_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, $a_n \to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in $L^{p'(\cdot)}(\Gamma_2)$ for some $(a,h) \in L^1(\Omega) \times H$, then

$$C(a,h) \leq \liminf_{n\to\infty} C(a_n,h_n). \tag{4.14}$$

Let $\{(a_n,h_n)\}_{n\in\mathbb{N}} \subset \Lambda \times H$ be such that $a_n \to a$ in $L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in $L^{p'(\cdot)}(\Gamma_2)$ for some $(a,h) \in L^1(\Omega) \times H$. Using Step II, it follows that $a \in \Lambda$. Now, $\{u_n\}_{n\in\mathbb{N}} \subset K$ be such that

$$u_n \in \mathcal{S}(a_n,h_n) \quad \text{and} \quad \inf_{u\in\mathcal{S}(a_n,h_n)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x = \int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x \tag{4.15}$$

for each $n \in \mathbb{N}$. Since $\cup_{n\geq 1}\mathcal{S}(a_n,h_n)$ is bounded, passing to a subsequence if necessary, we have $u_n \xrightarrow{w} u^*$ in V for some $u^* \in K$. Hence, $u^* \in w\text{--}\limsup_{n\to\infty} \mathcal{S}(a_n,h_n)$. Again from Step II, we have $u^* \in \mathcal{S}(a,h)$. Therefore, the lower semicontinuity of the function $L^1(\Omega) \ni a \mapsto \mathrm{TV}(a) \in \mathbb{R}$ and the weak lower semicontinuity of $V \ni u \mapsto \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x \in \mathbb{R}$ as well as $L^{p'(\cdot)}(\Gamma_2) \ni h \mapsto \int_\Omega |h(x)|^{p'(x)}\,\mathrm{d}\Gamma \in \mathbb{R}$ imply that

$$\begin{aligned}
&\liminf_{n\to\infty} C(a_n,h_n)\\
&= \liminf_{n\to\infty} \left[\int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x + \kappa\,\mathrm{TV}(a_n) + \tau \int_{\Gamma_2} |h_n(x)|^{p'(x)}\,\mathrm{d}\Gamma\right]\\
&\geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x + \liminf_{n\to\infty} \kappa\,\mathrm{TV}(a_n) + \liminf_{n\to\infty} \tau \int_{\Gamma_2} |h_n(x)|^{p'(x)}\,\mathrm{d}\Gamma\\
&\geq \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x + \kappa\,\mathrm{TV}(a) + \tau \int_{\Gamma_2} |h(x)|^{p'(x)}\,\mathrm{d}\Gamma\\
&\geq \inf_{u\in\mathcal{S}(a,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x + \kappa\,\mathrm{TV}(a) + \tau \int_{\Gamma_2} |h(x)|^{p'(x)}\,\mathrm{d}\Gamma\\
&= C(a,h).
\end{aligned}$$

This shows (4.14).

Step IV: The solution set of Problem 4.1 is nonempty and weakly compact.

First we observe that C is bounded from below by definition. Now, let $\{(a_n,h_n)\}_{n\in\mathbb{N}} \subset \Lambda \times H$ be a minimizing sequence of problem (4.1), that is,

$$\inf_{a\in\Lambda \text{ and } h\in H} C(a,h) = \lim_{n\to\infty} C(a_n,h_n). \tag{4.16}$$

This shows the boundedness of the sequences $\{a_n\}_{n\in\mathbb{N}} \subset \Lambda$ and $\{h_n\}_{n\in\mathbb{N}} \subset L^{p'(\cdot)}(\Gamma_2)$ in $\mathrm{BV}(\Omega)$ and $L^{p'(\cdot)}(\Gamma_2)$, respectively. Passing to a subsequence if necessary, we can assume that

$$a_n \to a^* \quad \text{in } L^1(\Omega) \quad \text{and} \quad h_n \xrightarrow{w} h^* \quad \text{in } L^{p'(\cdot)}(\Gamma_2) \tag{4.17}$$

for some $(a^*,h^*) \in \Lambda \times L^{p'(\cdot)}(\Gamma_2)$, where we have used the closedness of Λ in $L^1(\Omega)$ and the compactness of the embedding of $\mathrm{BV}(\Omega)$ to $L^1(\Omega)$.

Next, let us consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset K$ satisfying (4.15). From the convergence properties in (4.17) along with the boundedness of $\mathcal{S}$ we infer that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V. Thus, we find a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in K$.

Clearly, $u^* \in \mathcal{S}(a^*, h^*)$ because of Step II. Using these observations we get

$$\begin{aligned}
&\liminf_{n\to\infty} C(a_n, h_n)\\
&= \liminf_{n\to\infty} \left[\int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x + \kappa\,\mathrm{TV}(a_n) + \tau \int_{\Gamma_2} |h_n(x)|^{p'(x)}\,\mathrm{d}\Gamma\right]\\
&\geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x + \liminf_{n\to\infty} \kappa\,\mathrm{TV}(a_n) + \liminf_{n\to\infty} \tau \int_{\Gamma_2} |h_n(x)|^{p'(x)}\,\mathrm{d}\Gamma\\
&\geq \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x + \kappa\,\mathrm{TV}(a^*) + \tau \int_{\Gamma_2} |h^*(x)|^{p'(x)}\,\mathrm{d}\Gamma\\
&\geq \inf_{u^*\in\mathcal{S}(a^*,h^*)} \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x + \kappa\,\mathrm{TV}(a^*) + \tau \int_{\Gamma_2} |h^*(x)|^{p'(x)}\,\mathrm{d}\Gamma\\
&= C(a^*, h^*).
\end{aligned} \tag{4.18}$$

Combining (4.18) with (4.16) proves that $(a^*, h^*) \in \Lambda \times H$ is a solution to Problem 4.1.

In the last part we have to show that the solution set of Problem (4.1) is weakly compact. For this purpose, let $\{(a_n, h_n)\}_{n\in\mathbb{N}}$ be a sequence of solutions to Problem 4.1. First, it is easy to see that $\{a_n\}_{n\in\mathbb{N}} \subset \Lambda$ is bounded in $\mathrm{BV}(\Omega)$ and $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $L^{p'(\cdot)}(\Gamma_2)$. Therefore, we may assume that (4.17) holds with some $(a^*, h^*) \in \Lambda \times L^{p'(\cdot)}(\Gamma_2)$. Likewise, we can find a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that (4.15) is satisfied such that $u_n \xrightarrow{w} u^*$ in V for some $u^* \in \mathcal{S}(a^*, h^*)$. As before, we can show that (4.18) is fulfilled which means that $(a^*, h^*) \in \Lambda \times H$ is a solution to Problem 4.1. But this means that the solution set of Problem 4.1 is weakly compact and so the proof is complete. □

Remark 4.3. *We point out that our results in this section also hold if the functional* (4.2) *is replaced by the following regularized cost functional*

$$C(a,h) = \min_{u\in\mathcal{S}(a,h)} \left(\int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{p_-}} + \kappa\ \mathrm{TV}(a) + \tau \left(\int_{\Gamma_2} |h|^{p(x)'}\,\mathrm{d}\Gamma\right)^{\frac{1}{p'_-}}.$$

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(S. Zeng) Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, Guangxi, P.R. China, and Jagiellonian University in Krakow, Faculty of Mathematics and Computer Science, ul. Lojasiewicza 6, 30-348 Krakow, Poland.
Email address: zengshengda@163.com

(N. S. Papageorgiou) National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
Email address: npapg@math.ntua.gr

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
Email address: winkert@math.tu-berlin.de