ON CRITICAL DOUBLE PHASE KIRCHHOFF PROBLEMS WITH SINGULAR NONLINEARITY

RAKESH ARORA, ALESSIO FISCELLA, TUHINA MUKHERJEE, AND PATRICK WINKERT

Abstract. The paper deals with the following double phase problem

$$\begin{aligned} -m\left[\int_\Omega\left(\frac{|\nabla u|^p}{p}+a(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+a(x)|\nabla u|^{q-2}\nabla u\right)&=\lambda u^{-\gamma}+u^{p^*-1} &&\text{in } \Omega,\\ u&>0 &&\text{in } \Omega,\\ u&=0 &&\text{on } \partial\Omega, \end{aligned}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $N\geq 2$, m represents a Kirchhoff coefficient, $1<p<q<p^*$ with $p^*=Np/(N-p)$ being the critical Sobolev exponent to p, a bounded weight $a(\cdot)\geq 0$, $\lambda>0$ and $\gamma\in(0,1)$. By the Nehari manifold approach, we establish the existence of at least one weak solution.

1. Introduction

In this paper, we combine the effects of a nonlocal Kirchhoff coefficient and a double phase operator with a singular term and a critical Sobolev nonlinearity. Precisely, we study the problem

$$\begin{aligned} -m\left[\int_\Omega\left(\frac{|\nabla u|^p}{p}+a(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\mathcal{L}^a_{p,q}(u)&=\lambda u^{-\gamma}+u^{p^*-1} &&\text{in } \Omega,\\ u&>0 &&\text{in } \Omega,\\ u&=0 &&\text{on } \partial\Omega, \end{aligned}\tag{P_λ}$$

where along the paper, and without further mentioning, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, dimension $N\geq 2$, $\lambda>0$ is a real parameter and exponent $\gamma\in(0,1)$. The main operator $\mathcal{L}^a_{p,q}$ is the so-called double phase operator given by

$$\mathcal{L}^a_{p,q}(u):=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+a(x)|\nabla u|^{q-2}\nabla u\right),\quad u\in W^{1,\mathcal{H}}_0(\Omega),\tag{1.1}$$

with $W^{1,\mathcal{H}}_0(\Omega)$ being the homogeneous Musielak-Orlicz Sobolev space where we assume that

(h_1) $1<p<N$, $p<q<p^*$ and $0\leq a(\cdot)\in L^\infty(\Omega)$ with p^* being the critical Sobolev exponent to p given by

$$p^*=\frac{Np}{N-p}.\tag{1.2}$$

While the nonlocal term m in (P_λ) denotes a Kirchhoff coefficient satisfying

(h_2) $m\colon[0,\infty)\to[0,\infty)$ is a continuous function defined by

$$m(t)=a_0+b_0t^{\theta-1}\quad\text{for all } t\geq 0,$$

where $a_0\geq 0$, $b_0>0$ with $\theta\in[1,p^*/q)$.

Problem (P_λ) is said to be of double phase type because of the presence of two different elliptic growths p and q. The study of double phase problems and related functionals originates from the

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seminal paper by Zhikov [25], where he introduced for the first time in literature the related energy functional to (1.1) defined by

$$\omega \mapsto \int_\Omega \left(|\nabla \omega|^p + a(x)|\nabla \omega|^q\right) \mathrm{d}x. \tag{1.3}$$

This kind of functional has been used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. Indeed, the modulating coefficient $a(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents p and q. From the mathematical point of view, the behavior of (1.3) is related to the sets on which the weight function $a(\cdot)$ vanishes or not. In this direction, Zhikov found other mathematical applications for (1.3) in the study of duality theory and of the Lavrentiev gap phenomenon, as shown in [26, 27]. Also, (1.3) belongs to the class of the integral functionals with nonstandard growth condition, according to Marcellini's terminology [22, 23]. Following this line of research, Mingione et al. provide famous results in the regularity theory of local minimizers of (1.3), see, for example, the works of Baroni-Colombo-Mingione [4, 5] and Colombo-Mingione [9, 10].

Starting from [25], several authors studied existence and multiplicity results for nonlinear problems driven by (1.1) with the help of different variational techniques. In particular, Fiscella-Pinamonti [18] introduced two different double phase problems of Kirchhoff type, with the same variational structure set in $W_0^{1,\mathcal{H}}(\Omega)$. By the mountain pass and fountain theorems, existence and multiplicity results are provided in [18]. Following this direction, in [17] Fiscella-Marino-Pinamonti-Verzellesi consider some classes of Kirchhoff type problems on a double phase setting but with nonlinear boundary conditions. Combining variational methods, truncation arguments and topological tools, different multiplicity results are established. Recently, the authors [2] were able to study a Kirchhoff problem like (P_λ), but involving a subcritical term. By a suitable Nehari manifold decomposition, the existence of two different solutions are provided in [2]. We also mention the works of Cammaroto-Vilasi [7], Isernia-Repovš [20] and Ambrosio-Isernia [1] for Kirchhoff type problems driven by the $p(\cdot)$-Laplacian or the (p,q)-Laplacian.

The main novelty, as well as the main difficulty, of problem (P_λ) is the presence of a critical Sobolev nonlinearity. Indeed, in order to overcome the lack of compactness at critical levels arising from the presence of the critical term in (P_λ), the same fibering analysis used in [2] cannot work. For this, we exploit other variational tools inspired by more recent situations as in [14]. For this, Farkas-Fiscella-Winkert [14] used a suitable convergence analysis of gradients in order to handle the critical Sobolev nonlinearity of problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) &= \lambda|u|^{\vartheta-2}u + |u|^{p^*-2}u \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Following this direction, we mention [15, 16] concerning existence results for critical double phase problems involving a singular term and defined on Minkowski spaces in terms of Finsler manifolds, that is driven by the Finsler double phase operator

$$\mathcal{L}_{p,q}^{F,a}(u) := \operatorname{div}\left(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\right),$$

where $(\mathbb{R}^N, F)$ stands for a Minkowski space. While, Crespo-Blanco-Papageorgiou-Winkert [12] consider a nonhomogeneous singular Neumann double phase problem with critical growth on the boundary, given by

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) + \alpha(x)u^{p-1} &= \zeta(x)u^{-\gamma} + \lambda u^{q_1-1} \quad && \text{in } \Omega,\\ \left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nu &= -\beta(x)u^{p_*-1} && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

By the fibering approach introduced by Drábek-Pohozaev [13] along with a Nehari manifold decomposition, the existence of at least two solutions of (1.4) is obtained in [12].

Inspired by the above papers, we solve problem (P_λ) by a variational approach. Indeed, a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is said to be a weak solution of problem (P_λ) if $u^{-\gamma}\varphi \in L^1(\Omega)$, $u > 0$ a.e. in Ω and

$$m(\phi_{\mathcal{H}}(\nabla u))\left\langle \mathcal{L}_{p,q}^{a}(u), \varphi\right\rangle = \lambda \int_\Omega u^{-\gamma}\varphi \,\mathrm{d}x + \int_\Omega u^{p^*-1}\varphi \,\mathrm{d}x$$

is satisfied for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$ In particular, the weak solutions of (P_λ) are the critical points of the energy functional $J_\lambda \colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \left[a_0 \phi_{\mathcal{H}}(\nabla u) + \frac{b_0}{\theta} \phi_{\mathcal{H}}^\theta(\nabla u)\right] - \frac{\lambda}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, \mathrm{d}x - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, \mathrm{d}x,$$

for any $u \in W_0^{1,\mathcal{H}}(\Omega)$, where

$$\phi_{\mathcal{H}}(u) = \int_\Omega \left(\frac{|u|^p}{p} + a(x) \frac{|u|^q}{q} \right) \mathrm{d}x.$$

Hence, the main result reads as follows.

Theorem 1.1. *Let hypotheses* (h_1)*-*(h_2) *be satisfied. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*]$ *problem* (P_λ) *has at least one weak solution* u_λ *such that* $J_\lambda(u_\lambda) < 0$.

The proof of Theorem 1.1 is based on a suitable minimization argument on the Nehari manifold. For this, we extract a minimizing sequence whose energy values converge to a negative number. However, in order to verify that the sequence actually converges to a solution of (P_λ) we need a truncation argument combined with a delicate gradient analysis, inspired by [14].

The paper is organized as follows. In Section 2, we recall the main properties of Musielak-Orlicz Sobolev spaces $W_0^{1,\mathcal{H}}(\Omega)$ and state the main embeddings concerning these spaces. Section 3 gives a detailed analysis of the fibering map, presents the main properties of suitable subsets of the Nehari manifold and finally shows the existence of a weak solution of problem (P_λ).

2. Preliminaries

In this section, we will present the main properties and embedding results for Musielak-Orlicz Sobolev spaces. First, we denote by $L^r(\Omega) = L^r(\Omega; \mathbb{R})$ and $L^r(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces with the norm $\| \cdot \|_r$ and the corresponding Sobolev space $W_0^{1,r}(\Omega)$ is equipped with the norm $\|\nabla \cdot \|_r$, for $1 \leq r \leq \infty$.

Suppose hypothesis (h_1) and consider the nonlinear function $\mathcal{H} \colon \Omega \times [0, \infty) \to [0, \infty)$ defined by

$$\mathcal{H}(x,t) = t^p + a(x) t^q.$$

The Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ is given by

$$L^{\mathcal{H}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \varrho_{\mathcal{H}}(u) < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 \,\middle|\, \varrho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1 \right\},$$

where the modular function is given by

$$\varrho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|) \, \mathrm{d}x = \int_\Omega \left(|u|^p + a(x) |u|^q \right) \mathrm{d}x.$$

Next, we recall the relation between the norm $\| \cdot \|_{\mathcal{H}}$ and the modular function $\varrho_{\mathcal{H}}$, see Liu-Dai [21, Proposition 2.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Proposition 2.13].

Proposition 2.1. *Let* (h_1) *be satisfied,* $u \in L^{\mathcal{H}}(\Omega)$ *and* $c > 0$*. Then the following hold:*

(i) *If* $u \neq 0$*, then* $\|u\|_{\mathcal{H}} = c$ *if and only if* $\varrho_{\mathcal{H}}(\frac{u}{c}) = 1$*;*
(ii) $\|u\|_{\mathcal{H}} < 1$ *(resp.* > 1, $= 1$*) if and only if* $\varrho_{\mathcal{H}}(u) < 1$ *(resp.* > 1, $= 1$*);*
(iii) *If* $\|u\|_{\mathcal{H}} < 1$*, then* $\|u\|_{\mathcal{H}}^q \leq \varrho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^p$*;*
(iv) *If* $\|u\|_{\mathcal{H}} > 1$*, then* $\|u\|_{\mathcal{H}}^p \leq \varrho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^q$*;*
(v) $\|u\|_{\mathcal{H}} \to 0$ *if and only if* $\varrho_{\mathcal{H}}(u) \to 0$*;*
(vi) $\|u\|_{\mathcal{H}} \to \infty$ *if and only if* $\varrho_{\mathcal{H}}(u) \to \infty$*.*

Moreover, we define the weighted space

$$L^q_a(\Omega)=\left\{u:\Omega\to\mathbb{R}\,\Big|\, u \text{ is measurable and } \int_\Omega a(x)|u|^q\,\mathrm{d}x<\infty\right\}$$

endowed with the seminorm

$$\|u\|_{q,a}=\left(\int_\Omega a(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

The corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega)=\left\{u\in L^{\mathcal{H}}(\Omega)\,:\,|\nabla u|\in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}}=\|\nabla u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}}=\|\,|\nabla u|\,\|_{\mathcal{H}}$. In addition, we denote by $W^{1,\mathcal{H}}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. Thanks to hypothesis (h_1), we know that

$$\|u\|=\|\nabla u\|_{\mathcal{H}},$$

is an equivalent norm in $W^{1,\mathcal{H}}_0(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Proposition 2.16(ii)]. Furthermore, it is known that $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are uniformly convex and so reflexive Banach spaces, see Colasuonno-Squassina [8, Proposition 2.14] or Harjulehto-Hästö [19, Theorem 6.1.4].

Finally, we recall some useful embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$, see Colasuonno-Squassina [8, Proposition 2.15] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Propositions 2.17 and 2.19].

Proposition 2.2. *Let* (h_1) *be satisfied and let* p^* *be the critical exponent to* p *given in* (1.2). *Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega)\hookrightarrow L^r(\Omega)$ *and* $W^{1,\mathcal{H}}_0(\Omega)\hookrightarrow W^{1,r}_0(\Omega)$ *are continuous for all* $r\in[1,p]$;
(ii) $W^{1,\mathcal{H}}_0(\Omega)\hookrightarrow L^r(\Omega)$ *is continuous for all* $r\in[1,p^*]$ *and compact for all* $r\in[1,p^*)$;
(iii) $L^{\mathcal{H}}(\Omega)\hookrightarrow L^q_a(\Omega)$ *is continuous;*
(iv) $L^q(\Omega)\hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

3. Proof the main result

In order to solve problem (P_λ), we apply a minimization argument for J_λ on a suitable subset of $W^{1,\mathcal{H}}_0(\Omega)$. For this, we define the fibering function $\psi_u\colon[0,\infty)\to\mathbb{R}$ defined by

$$\psi_u(t)=J_\lambda(tu)\quad\text{for all } t\geq 0,$$

which gives

$$\psi_u(t)=\left[a_0\phi_{\mathcal{H}}(t\nabla u)+\frac{b_0}{\theta}\phi^\theta_{\mathcal{H}}(t\nabla u)\right]-\lambda\frac{t^{1-\gamma}}{1-\gamma}\int_\Omega|u|^{1-\gamma}\,\mathrm{d}x-\frac{t^{p^*}}{p^*}\int_\Omega|u|^{p^*}\,\mathrm{d}x.$$

It is easy to see that $\psi_u\in C^\infty((0,\infty))$. In particular, we have for $t>0$

$$\psi_u'(t)=\left[a_0+b_0\phi^{\theta-1}_{\mathcal{H}}(t\nabla u)\right]\left(t^{p-1}\|\nabla u\|_p^p+t^{q-1}\|\nabla u\|^q_{q,a}\right)-\lambda t^{-\gamma}\int_\Omega|u|^{1-\gamma}\,\mathrm{d}x-t^{p^*-1}\int_\Omega|u|^{p^*}\,\mathrm{d}x$$

and

$$\begin{aligned}\psi_u''(t)&=\left[a_0+b_0\phi^{\theta-1}_{\mathcal{H}}(t\nabla u)\right]\left[(p-1)t^{p-2}\|\nabla u\|_p^p+(q-1)t^{q-2}\|\nabla u\|^q_{q,a}\right]\\&\quad+b_0(\theta-1)\phi^{\theta-2}_{\mathcal{H}}(t\nabla u)\left(t^{p-1}\|\nabla u\|_p^p+t^{q-1}\|\nabla u\|^q_{q,a}\right)^2\\&\quad+\lambda\gamma t^{-\gamma-1}\int_\Omega|u|^{1-\gamma}\,\mathrm{d}x-(p^*-1)t^{p^*-2}\int_\Omega|u|^{p^*}\,\mathrm{d}x.\end{aligned}$$

Thus, we can introduce the Nehari manifold related to our problem which is defined by

$$\mathcal{N}_\lambda = \left\{ u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\} \,:\, \psi_u'(1) = 0 \right\}.$$

In particular, we have $u \in \mathcal{N}_\lambda$ if and only if

$$\left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u)\right] \left(\|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q\right) = \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x + \int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

Also $tu \in \mathcal{N}_\lambda$ if and only if $\psi_{tu}'(1) = 0$. Observe that $\mathcal{N}_\lambda$ contains all weak solutions of (P_λ). Moreover, we define the following subsets of $\mathcal{N}_\lambda$

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda \,:\, \psi_u''(1) > 0\} \quad \text{and} \quad \mathcal{N}_\lambda^\circ = \{u \in \mathcal{N}_\lambda \,:\, \psi_u''(1) = 0\}.$$

In contrast to [2] we are not going to study the set $\mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda \,:\, \psi_u''(1) < 0\}$. The next Lemma can be shown as in [2, Lemmas 3.1 and 3.2] replacing r by p^*.

Lemma 3.1. *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied.*

(i) *The functional $J_\lambda\big|_{\mathcal{N}_\lambda}$ is coercive and bounded from below for any $\lambda > 0$.*
(ii) *There exists $\Lambda_1 > 0$ such that $\mathcal{N}_\lambda^\circ = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.*

Let S be the best Sobolev constant in $W_0^{1,p}(\Omega)$ defined as

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_{p^*}^p}. \tag{3.1}$$

Note that we can write $\psi_u'(t)$ in the form

$$\psi_u'(t) = t^{-\gamma}\left(\sigma_u(t) - \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x\right), \quad t > 0, \tag{3.2}$$

where

$$\sigma_u(t) = \left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left(t^{p-1+\gamma}\|\nabla u\|_p^p + t^{q-1+\gamma}\|\nabla u\|_{q,a}^q\right) - t^{p^*-1+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

From this definition we see that $tu \in \mathcal{N}_\lambda$ if and only if

$$\sigma_u(t) = \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x. \tag{3.3}$$

The next Lemma shows that $\mathcal{N}_\lambda^+$ is nonempty whenever λ is sufficiently small.

Lemma 3.2. *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied and let $u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$. Then there exist $\Lambda_2 > 0$ and unique $t_1^u < t_{\max}^u < t_2^u$ such that*

$$0 < \sigma_u'(t_1^u) = (t_1^u)^\gamma \psi_u''(t_1^u), \quad 0 > \sigma_u'(t_2^u) = (t_2^u)^\gamma \psi_u''(t_2^u) \quad \text{and} \quad \sigma_u(t_{\max}^u) = \max_{t>0}\sigma_u(t)$$

whenever $\lambda \in (0, \Lambda_2)$. In particular, $t_1^u u \in \mathcal{N}_\lambda^+$ for $\lambda \in (0, \Lambda_2)$.

Proof. For $u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ the equation

$$\begin{aligned}
0 = \sigma_u'(t) &= \left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1+\gamma)t^{p-2+\gamma}\|\nabla u\|_p^p + (q-1+\gamma)t^{q-2+\gamma}\|\nabla u\|_{q,a}^q\right] \\
&\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-1+\gamma}\|\nabla u\|_p^p + t^{q-1+\gamma}\|\nabla u\|_{q,a}^q\right)\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right) \\
&\quad - (p^*-1+\gamma)t^{p^*-2+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x
\end{aligned}$$

can be equivalently written as

$$\begin{aligned}
&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1+\gamma)t^{p-p^*}\|\nabla u\|_p^p + (q-1+\gamma)t^{q-p^*}\|\nabla u\|_{q,a}^q\right] \\
&\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-p^*+1}\|\nabla u\|_p^p + t^{q-p^*+1}\|\nabla u\|_{q,a}^q\right)\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right) \\
&= (p^*-1+\gamma)\int_\Omega |u|^{p^*}\,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

From $p^* > q\theta$ and $\theta \geq 1$ we see that

$$\begin{aligned} p(\theta-1)+p-p^* &< \min\{p(\theta-1)+q-p^*, q(\theta-1)+p-p^*\} \\ &\leq \max\{p(\theta-1)+q-p^*, q(\theta-1)+p-p^*\} \\ &< q(\theta-1)+q-p^* = q\theta-p^* < 0. \end{aligned} \tag{3.5}$$

We denote the left-hand side of (3.4) by

$$\begin{aligned} T_u(t) &= \left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1+\gamma)t^{p-p^*}\|\nabla u\|_p^p + (q-1+\gamma)t^{q-p^*}\|\nabla u\|_{q,a}^q\right] \\ &\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-p^*+1}\|\nabla u\|_p^p + t^{q-p^*+1}\|\nabla u\|_{q,a}^q\right)\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right). \end{aligned}$$

Then, from (3.5) and $0<\gamma<1<p<q<p^*$, we know that

$$\text{(i)} \quad \lim_{t\to 0^+} T_u(t) = \infty, \quad \text{(ii)} \quad \lim_{t\to\infty} T_u(t) = 0, \quad \text{(iii)} \quad T_u'(t) < 0 \ \text{ for all } t>0.$$

From the intermediate value theorem along with (i) and (ii) we can find $t_{\max}^u > 0$ such that (3.4) holds. In addition, (iii) implies that $t_{\max}^u$ is unique due to the injectivity of T_u. Moreover, if we consider $\sigma_u'(t) > 0$, then in place of (3.4) we get

$$T_u(t) > (p^*-1+\gamma)\int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

Since T_u is strictly decreasing, this holds for all $t < t_{\max}^u$. The same can be said for $\sigma_u'(t) < 0$ and $t > t_{\max}^u$. Hence, σ_u is injective in $(0, t_{\max}^u)$ and in $(t_{\max}^u, \infty)$. Furthermore,

$$\sigma_u(t_{\max}^u) = \max_{t>0}\sigma_u(t)$$

with the global maximum $t_{\max}^u > 0$ of σ_u. Moreover, we have

$$\lim_{t\to 0^+}\sigma_u(t) = 0 \quad \text{and} \quad \lim_{t\to\infty}\sigma_u(t) = -\infty.$$

Applying the estimate $p\phi_{\mathcal{H}}(\nabla u) \geq \|\nabla u\|_p^p$ we obtain

$$\sigma_u'(t) \geq \frac{b_0}{p^{\theta-1}}(p\theta-1+\gamma)t^{p\theta-2+\gamma}\|\nabla u\|_p^{p\theta} - (p^*-1+\gamma)t^{p^*-2+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x, \tag{3.6}$$

which by using Hölder's inequality and (3.1) results in

$$t_{\max}^u \geq \frac{1}{\|\nabla u\|_p}\left(\frac{b_0(p\theta-1+\gamma)S^{\frac{p^*}{p}}}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{1}{p^*-p\theta}} := t_0^u. \tag{3.7}$$

Note that σ_u is increasing on $(0, t_{\max}^u)$. Hence from $p\phi_{\mathcal{H}}(\nabla u) \geq \|\nabla u\|_p^p$, $p<q$, Hölder's inequality, (3.1) and the representation of t_0^u in (3.7) we have

$$\begin{aligned} \sigma_u(t_{\max}^u) \geq \sigma_u(t_0^u) &\geq \frac{b_0}{p^{\theta-1}}(t_0^u)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta} - (t_0^u)^{p^*-1+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x \\ &\geq (t_0^u)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta}\left(\frac{b_0}{p^{\theta-1}} - (t_0^u)^{p^*-p\theta}S^{\frac{-p^*}{p}}\|\nabla u\|_p^{p^*-p\theta}\right) \\ &\geq \left(\frac{p^*-p\theta}{p^*-1+\gamma}\right)\frac{b_0}{p^{\theta-1}}(t_0^u)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta} > \left(\frac{p^*-q\theta}{p^*-1+\gamma}\right)\frac{b_0}{p^{\theta-1}}(t_0^u)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta} \\ &= \left(\frac{p^*-q\theta}{p^*-1+\gamma}\right)\|\nabla u\|_p^{1-\gamma}\frac{b_0}{p^{\theta-1}}\left(\frac{b_0(p\theta-1+\gamma)S^{\frac{p^*}{p}}}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{p\theta-1+\gamma}{p^*-p\theta}} \\ &\geq \Lambda_2\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x, \end{aligned}$$

where Λ_2 is given by

$$\Lambda_2 = \frac{b_0}{p^{\theta-1}}\left(\frac{p^*-q\theta}{p^*-1+\gamma}\right)\left(\frac{b_0(p\theta-1+\gamma)S}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{p\theta-1+\gamma}{p^*-p\theta}} \frac{S^{\frac{1-\gamma}{p}}}{|\Omega|^{\frac{p^*+\gamma-1}{p^*}}}.$$

From the considerations above we conclude that

$$\sigma_u(t^u_{\max}) > \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x$$

whenever $\lambda\in(0,\Lambda_2)$. Since σ_u is injective in $(0,t^u_{\max})$ and in $(t^u_{\max},\infty)$, we can find unique $t^u_1, t^u_2 > 0$ such that

$$\sigma_u(t^u_1) = \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x = \sigma_u(t^u_2) \quad \text{with} \quad \sigma'_u(t^u_2) < 0 < \sigma'_u(t^u_1).$$

Due to (3.3) we have $t^u_1 u\in\mathcal{N}_\lambda$. Then, from the representation in (3.2), we observe that

$$\sigma'_u(t) = t^\gamma\psi''_u(t) + \gamma t^{\gamma-1}\psi'_u(t).$$

Finally, since $\psi'_u(t^u_1) = \psi'_u(t^u_2) = 0$ and $\sigma'_u(t^u_2) < 0 < \sigma'_u(t^u_1)$ we derive that

$$0 < \sigma'_u(t^u_1) = (t^u_1)^\gamma\psi''_u(t^u_1) \quad \text{and} \quad 0 > \sigma'_u(t^u_2) = (t^u_2)^\gamma\psi''_u(t^u_2).$$

This shows, in particular, that $t^u_1 u\in\mathcal{N}^+_\lambda$ for $\lambda\in(0,\Lambda_2)$. ☐

Next we show that the modular $\varrho_{\mathcal{H}}(\nabla\cdot)$ is upper bounded with respect to the elements of $\mathcal{N}^+_\lambda$. The proof is similar to that in [2, Proposition 3.4] and so we omitted it.

Lemma 3.3. *Let hypotheses* (h_1)-(h_2) *be satisfied. Then there exist* $\Lambda_3 > 0$ *and constant* $D_1 = D_1(\lambda) > 0$ *such that*

$$\varrho_{\mathcal{H}}(\nabla u) = \|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q < D_1$$

for every $u\in\mathcal{N}^+_\lambda$ *and for every* $\lambda\in(0,\Lambda_3)$.

By Lemma 3.1(ii), we observe that $\mathcal{N}^+_\lambda$ is closed in $W^{1,\mathcal{H}}_0(\Omega)$ for $\lambda > 0$ small enough. We define

$$\Theta^+_\lambda = \inf_{u\in\mathcal{N}^+_\lambda} J_\lambda(u).$$

The next proposition shows that $\Theta^+_\lambda < 0$. We refer to [2, Proposition 4.1] for its proof.

Proposition 3.4. *Let hypotheses* (h_1)-(h_2) *be satisfied and let* $\lambda\in(0,\min\{\Lambda_1,\Lambda_2\})$, *with* Λ_1, Λ_2 *given in Lemmas 3.1(ii) and 3.2. Then* $\Theta^+_\lambda < 0$.

Based on the implicit function theorem in its version stated in Berger [6, p. 115] we can proof the following Lemma which proof is similar to the one in [2, Lemma 4.2].

Lemma 3.5. *Let hypotheses* (h_1)-(h_2) *be satisfied and let* $\lambda > 0$. *Let us consider* $u\in\mathcal{N}^+_\lambda$. *Then there exist* $\varepsilon > 0$ *and a continuous function* $\zeta\colon B_\varepsilon(0)\to(0,\infty)$ *such that*

$$\zeta(0) = 1 \quad \text{and} \quad \zeta(v)(u+v)\in\mathcal{N}^+_\lambda \quad \text{for all } v\in B_\varepsilon(0),$$

where $B_\varepsilon(0) := \{v\in W^{1,\mathcal{H}}_0(\Omega) : \|v\| < \varepsilon\}$.

Now, we set $\Lambda^* := \min\{\Lambda_1,\Lambda_2,\Lambda_3\}$ with Λ_1, Λ_2 and $\Lambda_3 > 0$ given in Lemmas 3.1(ii), 3.2 and 3.3. Let $\lambda\in(0,\Lambda^*)$. Applying Ekeland's variational principle, we obtain a sequence $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{N}^+_\lambda$ satisfying

$$\theta^+_\lambda < J_\lambda(u_n) < \theta^+_\lambda + \frac{1}{n}, \tag{3.8}$$

$$J_\lambda(u) \geq J_\lambda(u_n) + \frac{\|u-u_n\|}{n} \tag{3.9}$$

for any $u \in \mathcal{N}_\lambda^+$. By Lemma 3.1(i), we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Hence, by Proposition 2.2(ii) along with the reflexivity of $W_0^{1,\mathcal{H}}(\Omega)$, there exist a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and an element $u_\lambda \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in } W_0^{1,\mathcal{H}}(\Omega), \quad u_n \to u_\lambda \quad \text{in } L^s(\Omega) \quad \text{and} \quad u_n \to u_\lambda \quad \text{a.e. in } \Omega \tag{3.10}$$

for any $s \in [1,p^*)$. By the coercivity given in Lemma 3.1(i), we can assume that there exist $E_1,\ E_2 \geq 0$ such that

$$\lim_{n\to\infty} \|u_n\|_p^p = E_1 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{q,a}^q = E_2. \tag{3.11}$$

We get the following technical results.

Lemma 3.6. *Let hypotheses* (h_1)-(h_2) *be satisfied, let* $\lambda \in (0,\Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)-(3.9). *Then* $u_\lambda \neq 0$.

Proof. Let us assume by contradiction that $u_\lambda = 0$. Then $\psi_{u_n}'(1) = 0$ implies

$$\left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right] \left(\|u_n\|_p^p + \|u_n\|_{q,a}^q\right) - \lambda \int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - \int_\Omega |u_n|^{p^*}\,\mathrm{d}x = 0.$$

Using (3.10), (3.11) and letting $n \to \infty$, we get

$$\left[a_0 + b_0 \left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\right](E_1+E_2) - d^{p^*} = 0, \tag{3.12}$$

where we set

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p^*}\,\mathrm{d}x =: d^{p^*} \geq 0.$$

Moreover by (3.8) we have

$$\lim_{n\to\infty} J_\lambda(u_n) = \Theta_\lambda^+ < 0,$$

which implies that

$$\left[a_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right) + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta}\right] - \frac{d^{p^*}}{p^*} < 0. \tag{3.13}$$

Recall that $E_1,\ E_2 \geq 0$. Then, taking the value of d^{p^*} from (3.12) into (3.13), we derive that

$$\left[a_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right) + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta}\right] - \left[a_0 + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\right]\frac{E_1+E_2}{p^*} < 0.$$

This implies

$$a_0\left[\frac{E_1}{p} + \frac{E_2}{q} - \frac{E_1+E_2}{p^*}\right] + b_0\left[\frac{1}{\theta}\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta} - \left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\frac{E_1+E_2}{p^*}\right] < 0$$

and so

$$a_0\left[E_1\left(\frac{1}{p} - \frac{1}{p^*}\right) + E_2\left(\frac{1}{q} - \frac{1}{p^*}\right)\right] + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\left[E_1\left(\frac{1}{p\theta} - \frac{1}{p^*}\right) + E_2\left(\frac{1}{q\theta} - \frac{1}{p^*}\right)\right] < 0,$$

which is a contradiction because of $p < q \leq q\theta < p^*$. $\square$

Lemma 3.7. *Let hypotheses* (h_1)-(h_2) *be satisfied, let* $\lambda \in (0,\Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)-(3.9). *Then* $\liminf_{n\to\infty} \psi_{u_n}''(1) > 0$, *that is,*

$$\begin{aligned}\liminf_{n\to\infty}\Bigg\{ &\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,a}^q\right] \\ &+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 - (p^*-1+\gamma)\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\Bigg\} > 0.\end{aligned}$$

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$, we have $\psi'_{u_n}(1) = 0$ and $\psi''_{u_n}(1) > 0$, that is,

$$\begin{aligned}&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,a}^q\right]\\&\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 - (p^*-1+\gamma)\int_\Omega |u_n|^{p^*}\,\mathrm{d}x > 0\end{aligned}$$

and

$$\begin{aligned}&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p + (q-p^*)\|\nabla u_n\|_{q,a}^q\right]\\&\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^*-1+\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x > 0.\end{aligned} \tag{3.14}$$

Thus, in order to prove the lemma, it is enough to show that

$$\begin{aligned}\liminf_{n\to\infty}\Big\{&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p + (q-p^*)\|\nabla u_n\|_{q,a}^q\right]\\&+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^*-1+\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x\Big\} > 0.\end{aligned}$$

By contradicting (3.14), let us assume that

$$\begin{aligned}\liminf_{n\to\infty}\Big\{&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p + (q-p^*)\|\nabla u_n\|_{q,a}^q\right]\\&+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^*-1+\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x\Big\} = 0.\end{aligned} \tag{3.15}$$

By Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x = \int_\Omega |u_\lambda|^{1-\gamma}\mathrm{d}x. \tag{3.16}$$

Using (3.16) in (3.15), we get

$$\begin{aligned}\liminf_{n\to\infty}\Big\{&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p + (q-p^*)\|\nabla u_n\|_{q,a}^q\right]\\&+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2\Big\} = -\lambda(p^*-1+\gamma)\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x,\end{aligned}$$

which yields, by applying (3.11),

$$\begin{aligned}-\lambda\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x &= \left[a_0 + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\right]\frac{[(p-p^*)E_1 + (q-p^*)E_2]}{(p^*-1+\gamma)}\\&\quad + \frac{b_0(\theta-1)}{(p^*-1+\gamma)}\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-2}(E_1+E_2)^2.\end{aligned} \tag{3.17}$$

From this, due to $p < q < p^*$, we have

$$\begin{aligned}-\lambda\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x &\le b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\left[\frac{(q-p^*)(E_1+E_2)}{(p^*-1+\gamma)} + \frac{b_0(\theta-1)q(E_1+E_2)}{(p^*+\gamma-1)}\right]\\&= \frac{b_0(q\theta-p^*)(E_1+E_2)}{(p^*+\gamma-1)}\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}.\end{aligned} \tag{3.18}$$

Considering $\psi'_{u_n}(1) = 0$ and (3.16), we have

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p^*}\,\mathrm{d}x = \left[a_0 + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\right][E_1+E_2] - \lambda\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x.$$

From this and (3.17), we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\\
&=\left[a_0+b_0\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}\right]\left[\left(\frac{p+\gamma-1}{p^*+\gamma-1}\right)E_1+\left(\frac{q+\gamma-1}{p^*+\gamma-1}\right)E_2\right]\\
&\quad+\frac{b_0(\theta-1)}{p^*-1+\gamma}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-2}(E_1+E_2)^2\\
&\geq \frac{b_0(p+\gamma-1)}{p^*+\gamma-1}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}(E_1+E_2)+\frac{b_0(p\theta-p)}{p^*-1+\gamma}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}(E_1+E_2)\\
&=\frac{b_0(p\theta+\gamma-1)}{p^*+\gamma-1}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}(E_1+E_2)\\
&\geq \frac{b_0(p\theta+\gamma-1)}{p^{\theta-1}(p^*+\gamma-1)}E_1^\theta.
\end{aligned}
\tag{3.19}
$$

For any fixed $w\in W_0^{1,\mathcal{H}}(\Omega)\setminus\{0\}$, we know that there exists a unique $t_{\max}>0$ such that $\sigma_w'(t_{\max})=0$. From this and (3.6), we conclude that

$$
t_{\max}\geq\left(\frac{b_0(p\theta+\gamma-1)\|\nabla w\|_p^{p\theta}}{p^{\theta-1}(p^*-1+\gamma)\int_\Omega |w|^{p^*}\,\mathrm{d}x}\right)^{\frac{1}{p^*-p\theta}} := t_{00} \tag{3.20}
$$

It is easy to verify that $t_{\max}\geq t_{00}\geq t_0^w$ as defined in (3.7) and from the proof of Lemma 3.2, we know that $\Lambda_2>0$ is chosen in such a way that

$$
\frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}(t_0^w)^{p\theta+\gamma-1}\|\nabla w\|_p^{p\theta}\geq\Lambda_2\int_\Omega |w|^{1-\gamma}\mathrm{d}x.
$$

We define

$$
S(w):=\frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}(t_{00})^{p\theta+\gamma-1}\|\nabla w\|_p^{p\theta}-\Lambda_2\int_\Omega |w|^{1-\gamma}\mathrm{d}x\geq 0 \quad \text{for all } w\in W_0^{1,\mathcal{H}}(\Omega), \tag{3.21}
$$

with t_{00} given in (3.20). Taking $w=u_n$ in (3.21) and then passing to the limit as $n\to\infty$ we get

$$
\lim_{n\to\infty}S(u_n)\geq 0.
$$

On the other hand, by Lemma 3.6 and (3.11), we have that at least one of E_1 and E_2 is not zero. Let us assume, without any loss of generality, that $E_1>0$, $E_2\geq 0$. Then by (3.18), (3.19), (3.20) along with $q\theta<p^*$ and $\lambda\in(0,\Lambda_2)$, we obtain

$$
\begin{aligned}
\lim_{n\to\infty}S(u_n)&\leq\frac{\frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}\left(\frac{b_0(p\theta+\gamma-1)E_1^\theta}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{(p\theta-1+\gamma)}{p^*-p\theta}}E_1^\theta}{\left(\frac{b_0(p\theta+\gamma-1)}{p^{\theta-1}(p^*+\gamma-1)}E_1^\theta\right)^{\frac{p\theta+\gamma-1}{p^*-p\theta}}}+\frac{\Lambda_2}{\lambda}\frac{b_0(q\theta-p^*)(E_1+E_2)}{(p^*+\gamma-1)}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}\\
&<\frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}E_1^\theta+\frac{b_0(q\theta-p^*)E_1^\theta}{p^{\theta-1}(p^*+\gamma-1)}=0.
\end{aligned}
$$

This proves the assertion of the lemma. □

Let $h\in W_0^{1,\mathcal{H}}(\Omega)$ be nonnegative. From Lemma 3.5 there exists a sequence of maps $\{\zeta_n\}_{n\in\mathbb{N}}$ such that $\zeta_n(0)=1$ and $\zeta_n(th)(u_n+th)\in\mathcal{N}_\lambda^+$ for sufficiently small $t>0$ and for each $n\in\mathbb{N}$. From this and $u_n\in\mathcal{N}_\lambda$, we have the equations

$$
\left[a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q\right)-\lambda\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x-\int_\Omega |u_n|^{p^*}\,\mathrm{d}x=0 \tag{3.22}
$$

and

$$\begin{aligned} &\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n)\right]\left(\zeta_n^p(th)\|\nabla w_n\|_p^p + \zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\right) \\ &- \lambda\zeta_n^{1-\gamma}(th)\int_\Omega |w_n|^{1-\gamma}\,\mathrm{d}x - \zeta_n^{p^*}(th)\int_\Omega |w_n|^{p^*}\,\mathrm{d}x = 0 \end{aligned} \tag{3.23}$$

where $w_n = u_n + th$.

Lemma 3.8. *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied, let* $\lambda \in (0, \Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)-(3.9). *For any nonnegative function* $h \in W_0^{1,\mathcal{H}}(\Omega)$, *the sequence* $\{\langle\zeta_n'(0), h\rangle\}_{n\in\mathbb{N}}$ *is uniformly bounded.*

Proof. Subtracting (3.22) from (3.23), we get

$$\begin{aligned} &(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[(\|\nabla w_n\|_p^p - \|\nabla u_n\|_p^p) + (\|\nabla w_n\|_{q,a}^q - \|\nabla u_n\|_{q,a}^q) + (\zeta_n^p(th) - 1)\|\nabla w_n\|_p^p\right. \\ &\quad \left.+ (\zeta_n^q(th) - 1)\|\nabla w_n\|_{q,a}^q\right] \\ &\quad + b_0\left[\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n) - \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left(\zeta_n^p(th)\|\nabla w_n\|_p^p + \zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\right) \\ &\quad - \lambda\left(\zeta_n^{1-\gamma}(th) - 1\right)\int_\Omega |w_n|^{1-\gamma}\,\mathrm{d}x - \lambda\int_\Omega \left(|w_n|^{1-\gamma} - |u_n|^{1-\gamma}\right)\,\mathrm{d}x \\ &\quad - \left(\zeta_n^{p^*}(th) - 1\right)\int_\Omega |w_n|^{p^*}\,\mathrm{d}x - \int_\Omega \left(|w_n|^{p^*} - |u_n|^{p^*}\right)\,\mathrm{d}x = 0. \end{aligned} \tag{3.24}$$

For notational convenience, we set

$$\langle u_n, h\rangle_p = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x \quad \text{and} \quad \langle u_n, h\rangle_{q,a} = \int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x.$$

We have the following limits

$$\begin{aligned} \lim_{t\to 0}\frac{\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n) - \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)}{t} &= (\zeta_n'(0), h)(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q) \\ &\quad + (\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a}), \\ \lim_{t\to 0}\frac{\|\nabla w_n\|_p^p - \|\nabla u_n\|_p^p}{t} &= p\langle u_n, h\rangle_p, \\ \lim_{t\to 0}\frac{\|\nabla w_n\|_{q,a}^q - \|\nabla u_n\|_{q,a}^q}{t} &= q\langle u_n, h\rangle_{q,a}, \\ \lim_{t\to 0}\left(|w_n|^{p^*} - |u_n|^{p^*}\right)\,\mathrm{d}x &= p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x, \\ \lim_{t\to 0}\frac{\zeta^s(th) - 1}{t} &= s\langle\zeta_n'(0), h\rangle \quad \text{for any } s > 1. \end{aligned} \tag{3.25}$$

Taking into account

$$\int_\Omega \left(|w_n|^{1-\gamma} - |u_n|^{1-\gamma}\right)\,\mathrm{d}x \geq 0$$

since h is nonnegative, dividing both sides of (3.24) by $t > 0$ and then passing the limit as $t \to 0^+$, we obtain

$$\begin{aligned} 0 \leq\; &(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left(p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla h\,\mathrm{d}x + q\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\nabla h\,\mathrm{d}x\right. \\ &\left. + p\,\langle\zeta_n'(0), h\rangle\|\nabla u_n\|_p^p + q\langle\zeta_n'(0), h\rangle\|\nabla u_n\|_{q,a}^q\right) \\ &+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)\langle\zeta_n'(0), h\rangle\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q\right)^2 \\ &- \lambda(1-\gamma)\langle\zeta_n'(0), h\rangle\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - p^*\langle\zeta_n'(0), h\rangle\int_\Omega |u_n|^{p^*}\,\mathrm{d}x - p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x. \end{aligned}$$

This implies

$$
\begin{aligned}
0 \leq \langle \zeta_n'(0), h\rangle \bigg[& \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right) \left[p \|\nabla u_n\|_p^p + q\|\nabla u_n\|_{q,a}^q\right] \\
& + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q\right)^2 - \lambda(1-\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - p^*\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\bigg] \\
& + \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\left(p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + q\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot \nabla h\,\mathrm{d}x\right) \\
& - p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x.
\end{aligned}
$$

Therefore, using the fact that $u_n \in \mathcal{N}_\lambda$, we have

$$
\begin{aligned}
0 \leq \langle \zeta_n'(0), h\rangle \Big\{ & \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\left[(p+\gamma-1)\|\nabla u_n\|_p^p + (q+\gamma-1)\|\nabla u_n\|_{q,a}^q\right] \\
& + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 - (p^*+\gamma-1)\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\Big\} \\
& + \bigg[(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\left(p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + q\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right) \\
& \qquad - p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x\bigg].
\end{aligned}
$$

Now using Lemma 3.7 and taking into account the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$, we infer that $\{\langle\zeta_n'(0), h\rangle\}_{n\in\mathbb{N}}$ is bounded below for any nonnegative $h \in W_0^{1,\mathcal{H}}(\Omega)$.

It remains to show that $\{\langle\zeta_n'(0), h\rangle\}_{n\in\mathbb{N}}$ is bounded above for any nonnegative $h \in W_0^{1,\mathcal{H}}(\Omega)$. Assume by contradiction that $\limsup_{n\to\infty}\langle\zeta_n'(0), h\rangle = \infty$. Thus, without loss of generality, we can consider $\zeta_n(th) > \zeta_n(0) = 1$ for $n \in \mathbb{N}$ large enough . It is easy to see that

$$
|\zeta_n(th) - 1|\|u_n\| + \zeta_n(th)\|th\| \geq \|(\zeta_n(th)-1)u_n + th\zeta_n(th)\| = \|\zeta_n(th)w_n - u_n\|.
$$

Applying this in (3.9) with $u = \zeta_n(th)w_n$, we get

$$
\begin{aligned}
& |\zeta_n(th) - 1|\frac{\|u_n\|}{n} + \zeta_n(th)\frac{\|th\|}{n} \\
& \geq J_\lambda(u_n) - J_\lambda(\zeta_n(th)w_n) \\
& = a_0\left[\phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n)\right] + \frac{b_0}{\theta}\left[\phi_{\mathcal{H}}^\theta(\nabla u_n) - \phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n)\right] \\
& \quad - \frac{\lambda}{1-\gamma}\int_\Omega\left[|u_n|^{1-\gamma} - |\zeta_n(th)w_n|^{1-\gamma}\right]\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega\left[|u_n|^{p^*} - |\zeta_n(th)w_n|^{p^*}\right]\,\mathrm{d}x.
\end{aligned}
$$

Using (3.22) and (3.23) in the inequality above, we obtain

$$
\begin{aligned}
& |\zeta_n(th) - 1|\frac{\|u_n\|}{n} + \zeta_n(th)\frac{\|th\|}{n} \\
& = a_0\left[\phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n) - \frac{1}{1-\gamma}\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q - \zeta_n^p(th)\|\nabla w_n\|_p^p - \zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\right)\right] \\
& \quad + b_0\bigg[\frac{\phi_{\mathcal{H}}^\theta(\nabla u_n) - \phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n)}{\theta} - \frac{\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)}{1-\gamma}\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q\right) \\
& \qquad + \frac{\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n)}{1-\gamma}\left(\zeta_n^p(th)\|\nabla w_n\|_p^p + \zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\right)\bigg] \\
& \quad - \left(\frac{1}{1-\gamma} - \frac{1}{p^*}\right)\int_\Omega\left[|\zeta_n(th)w_n|^{p^*} - |u_n|^{p^*}\right]\,\mathrm{d}x.
\end{aligned}
$$

Now dividing the above inequality by $t>0$, passing to the limit as $t\to 0^+$ and using (3.25), we have

$$\begin{aligned}
\frac{\|h\|}{n} &\geq a_0\bigg[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}-\langle \zeta_n'(0),h\rangle\left(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q\right)\\
&\qquad +\frac{1}{1-\gamma}\Big\{\langle \zeta_n'(0),h\rangle\left(p\|\nabla u_n\|_p^p+q\|\nabla u_n\|_{q,a}^q\right)+p\langle u_n,h\rangle_p+q\langle u_n,h\rangle_{q,a}\Big\}\bigg]\\
&\quad +b_0\bigg[\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\langle \zeta_n'(0),h\rangle\left(p\|\nabla u_n\|_p^p+q\|\nabla u_n\|_{q,a}^q\right)\\
&\qquad +\frac{1}{1-\gamma}\Big\{\langle \zeta_n'(0),h\rangle(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2\\
&\qquad\quad +\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\langle \zeta_n'(0),h\rangle\left(p\|\nabla u_n\|_p^p+q\|\nabla u_n\|_{q,a}^q\right)+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\left(p\langle u_n,h\rangle_p+q\langle u_n,h\rangle_{q,a}\right)\Big\}\bigg]\\
&\quad -\left(\frac{p^*-1+\gamma}{1-\gamma}\right)\left[\langle \zeta_n'(0),h\rangle\int_\Omega |u_n|^{p^*}\,\mathrm{d}x+\int_\Omega |u_n|^{p^*-2}u_nh\,\mathrm{d}x\right]\\
&=\frac{\langle \zeta_n'(0),h\rangle}{1-\gamma}\bigg[\left(a_0+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\Big\{(p-1+\gamma)\|\nabla u_n\|_p^p+(q-1+\gamma)\|\nabla u_n\|_{q,a}^q\Big\}\\
&\qquad +b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2-(p^*-1+\gamma)\int_\Omega|u_n|^{p^*}\,\mathrm{d}x-\frac{(1-\gamma)\|u_n\|}{n}\bigg]\\
&\quad +\frac{a_0}{1-\gamma}\Big[(p-\gamma+1)\langle u_n,h\rangle_p+(q-\gamma+1)\langle u_n,h\rangle_{q,a}\Big]+\frac{b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)}{1-\gamma}\Big[p\langle u_n,h\rangle_p+q\langle u_n,h\rangle_{q,a}\Big]\\
&\quad -\left(\frac{p^*-1+\gamma}{1-\gamma}\right)\int_\Omega |u_n|^{p^*-2}u_nh\,\mathrm{d}x,
\end{aligned}$$

which gives a contradiction if we take the limits $n\to\infty$ on both sides, considering $\limsup_{n\to\infty}\langle \zeta_n'(0),h\rangle=\infty$, since by Lemma 3.7 and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, there exists some $M_1>0$ such that

$$\begin{aligned}
&\bigg[\left(a_0+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\Big\{(p-1+\gamma)\|\nabla u_n\|_p^p+(q-1+\gamma)\|\nabla u_n\|_{q,a}^q\Big\}\\
&\quad +b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2-(p^*-1+\gamma)\int_\Omega|u_n|^{p^*}\,\mathrm{d}x-\frac{(1-\gamma)\|u_n\|}{n}\bigg]>M_1
\end{aligned}$$

for $n\in\mathbb{N}$ large enough. Thus $\{\langle \zeta_n'(0),h\rangle\}_{n\in\mathbb{N}}$ must be bounded above. □

Since $J_\lambda(u_n)=J_\lambda(|u_n|)$, without loss of generality, we may assume that $u_n\geq 0$ a. e. in Ω and so, $u_\lambda\geq 0$ a. e. in Ω. With this assumption, we state our next result.

Lemma 3.9. *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied, let* $\lambda\in(0,\Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)-(3.9). *For any* $h\in W_0^{1,\mathcal{H}}(\Omega)$ *and* $n\in\mathbb{N}$, $u_n^{-\gamma}h\in L^1(\Omega)$ *and as* $n\to\infty$

$$\begin{aligned}
&\left(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\left[\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x+\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right]\\
&\quad -\lambda\int_\Omega u_n^{-\gamma}h\,\mathrm{d}x-\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x=o_n(1).
\end{aligned}\tag{3.26}$$

Proof. Let $h\in W_0^{1,\mathcal{H}}(\Omega)$ be nonnegative and recall the following estimate from the proof of Lemma 3.8

$$\begin{aligned}
&|\zeta_n(th)-1|\frac{\|u_n\|}{n}+\zeta_n(th)\frac{\|th\|}{n}\\
&\geq a_0\left[\phi_{\mathcal{H}}(\nabla u_n)-\phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n)\right]+\frac{b_0}{\theta}\left[\phi_{\mathcal{H}}^\theta(\nabla u_n)-\phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n)\right]\\
&\quad -\frac{\lambda}{1-\gamma}\int_\Omega\left[|u_n|^{1-\gamma}-|\zeta_n(th)w_n|^{1-\gamma}\right]\mathrm{d}x-\frac{1}{p^*}\int_\Omega\left[|u_n|^{p^*}-|\zeta_n(th)w_n|^{p^*}\right]\mathrm{d}x\\
&=a_0\left[(\phi_{\mathcal{H}}(\nabla u_n)-\phi_{\mathcal{H}}(\nabla w_n))+(\phi_{\mathcal{H}}(\nabla w_n)-\phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n))\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{b_0}{\theta}\left[(\phi_{\mathcal{H}}^{\theta}(\nabla u_n)-\phi_{\mathcal{H}}^{\theta}(\nabla w_n))+(\phi_{\mathcal{H}}^{\theta}(\nabla w_n)-\phi_{\mathcal{H}}^{\theta}(\zeta_n(th)\nabla w_n))\right]\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[|u_n|^{1-\gamma}-|w_n|^{1-\gamma}\right]\mathrm{d}x-\frac{\lambda}{1-\gamma}\int_\Omega\left[|w_n|^{1-\gamma}-|\zeta_n(th)w_n|^{1-\gamma}\right]\mathrm{d}x\\
&-\frac{1}{p^*}\int_\Omega\left[|u_n|^{p^*}-|w_n|^{p^*}\right]\mathrm{d}x-\frac{1}{p^*}\int_\Omega\left[|w_n|^{p^*}-|\zeta_n(th)w_n|^{p^*}\right]\mathrm{d}x.
\end{aligned}
$$

Dividing the above equation with $t>0$ and then passing to limit as $t\to 0^+$, we get

$$
\begin{aligned}
&|\langle\zeta_n'(0),h\rangle|\frac{\|u_n\|}{n}+\frac{\|h\|}{n}\\
&\geq-(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}+\langle\zeta_n'(0),h\rangle(\|u_n\|_p^p+\|u_n\|_{q,a}^q)\right]\\
&\quad-\frac{\lambda}{1-\gamma}\liminf_{t\to0^+}\int_\Omega\frac{\left[|u_n|^{1-\gamma}-|w_n|^{1-\gamma}\right]}{t}\mathrm{d}x+\lambda\langle\zeta_n'(0),h\rangle\int_\Omega|u_n|^{1-\gamma}\mathrm{d}x\\
&\quad+\langle\zeta_n'(0),h\rangle\int_\Omega|u_n|^{p^*}\mathrm{d}x+\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x\\
&=-\langle\zeta_n'(0),h\rangle\left[(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[(\|u_n\|_p^p+\|u_n\|_{q,a}^q)\right]-\lambda\int_\Omega|u_n|^{1-\gamma}\mathrm{d}x-\int_\Omega|u_n|^{p^*}\mathrm{d}x\right]\\
&\quad-(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}\right]\\
&\quad-\frac{\lambda}{1-\gamma}\liminf_{t\to0^+}\int_\Omega\frac{\left[|u_n|^{1-\gamma}-|w_n|^{1-\gamma}\right]}{t}\mathrm{d}x+\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x\\
&=-(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}\right]\\
&\quad-\frac{\lambda}{1-\gamma}\liminf_{t\to0^+}\int_\Omega\frac{\left[|u_n|^{1-\gamma}-|w_n|^{1-\gamma}\right]}{t}\mathrm{d}x+\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x,
\end{aligned}
$$

where we used $u_n\in\mathcal{N}_\lambda$ that is $\psi_{u_n}'(1)=0$. This implies

$$
\begin{aligned}
\frac{\lambda}{1-\gamma}\liminf_{t\to0^+}\int_\Omega\frac{\left[|u_n+th|^{1-\gamma}-|u_n|^{1-\gamma}\right]}{t}\mathrm{d}x&\leq(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}\right]\\
&\quad-\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x+|\langle\zeta_n'(0),h\rangle|\frac{\|u_n\|}{n}+\frac{\|h\|}{n}.
\end{aligned}\tag{3.27}
$$

Observe that $|u_n+th|^{1-\gamma}-|u_n|^{1-\gamma}\geq 0$, so we can use Fatou's lemma in (3.27) to obtain

$$
\begin{aligned}
\lambda\int_\Omega u_n^{-\gamma}h\,\mathrm{d}x&\leq(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}\right]\\
&\quad-\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x+|\langle\zeta_n'(0),h\rangle|\frac{\|u_n\|}{n}+\frac{\|h\|}{n}.
\end{aligned}
$$

Recall that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Then, passing to the limit as $n\to\infty$ in the above estimate, we obtain

$$
\begin{aligned}
&(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x+\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right]\\
&-\lambda\int_\Omega u_n^{-\gamma}h\,\mathrm{d}x-\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x\geq o_n(1),
\end{aligned}\tag{3.28}
$$

for each nonnegative $h\in W_0^{1,\mathcal{H}}(\Omega)$, where we used the uniform boundedness from Lemma 3.8.

We aim to establish that (3.28) holds true for any arbitrary $h\in W_0^{1,\mathcal{H}}(\Omega)$. For this, we replace h in (3.28) by $(u_n+\varepsilon h)^+$ with $\varepsilon>0$ and $h\in W_0^{1,\mathcal{H}}(\Omega)$. Renaming as $h_\varepsilon=u_n+\varepsilon h$ and using (3.28), we get

$$
o_n(1)\leq(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h_\varepsilon^+\,\mathrm{d}x+\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h_\varepsilon^+\,\mathrm{d}x\right]
$$

$$
\begin{aligned}
&- \lambda \int_\Omega u_n^{-\gamma} h_\varepsilon^+ \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h_\varepsilon^+ \,\mathrm{d}x \\
&= (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x \right] \\
&\quad + (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x \right] \\
&\quad - \lambda \int_\Omega u_n^{-\gamma} (h_\varepsilon + h_\varepsilon^-) \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} (h_\varepsilon + h_\varepsilon^-) \,\mathrm{d}x \\
&= \left[(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[(\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] - \lambda \int_\Omega |u_n|^{1-\gamma} \,\mathrm{d}x - \int_\Omega |u_n|^{p^*} \,\mathrm{d}x \right] \\
&\quad + \varepsilon \Bigg\{ (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \right] \\
&\qquad - \lambda \int_\Omega u_n^{-\gamma} h \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h \,\mathrm{d}x \Bigg\} - \lambda \int_\Omega u_n^{-\gamma} h_\varepsilon^- \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h_\varepsilon^- \,\mathrm{d}x \\
&\quad + (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x \right].
\end{aligned}
$$

We define $\Omega_\varepsilon = \{x \in \Omega : u_n + \varepsilon h \le 0\}$. Using $u_n \in \mathcal{N}_\lambda$ and $\int_\Omega u_n^{-\gamma} h_\varepsilon^- \,\mathrm{d}x \ge 0$ in the above estimate, we get

$$
\begin{aligned}
o_n(1) \le \varepsilon \Bigg\{ &(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \right] \\
&- \lambda \int_\Omega u_n^{-\gamma} h \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h \,\mathrm{d}x \Bigg\} + \int_{\Omega_\varepsilon} u_n^{p^*-1} h_\varepsilon \,\mathrm{d}x \\
&- (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x + \int_{\Omega_\varepsilon} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x \right].
\end{aligned} \tag{3.29}
$$

Note that

$$
\begin{aligned}
- \int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x &= - \int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n + \varepsilon h) \,\mathrm{d}x \\
&= - \int_{\Omega_\varepsilon} |\nabla u_n|^p \,\mathrm{d}x - \varepsilon \int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \\
&\le -\varepsilon \int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x
\end{aligned}
$$

and similarly,

$$
- \int_{\Omega_\varepsilon} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x \le -\varepsilon \int_{\Omega_\varepsilon} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x.
$$

Moreover, applying Hölder's inequality and $u_n \le -\varepsilon h$ in Ω_ε, we have

$$
\begin{aligned}
\left| \int_{\Omega_\varepsilon} u_n^{p^*-1} h_\varepsilon \,\mathrm{d}x \right| &\le \left| \int_{\Omega_\varepsilon} u_n^{p^*} \,\mathrm{d}x \right| + \epsilon \left| \int_{\Omega_\varepsilon} u_n^{p^*-1} |h| \,\mathrm{d}x \right| \\
&\le \varepsilon^{p^*} \int_{\Omega_\varepsilon} |h|^{p^*} \,\mathrm{d}x + \varepsilon \left(\int_{\Omega_\varepsilon} u_n^{p^*} \,\mathrm{d}x \right)^{\frac{p^*-1}{p^*}} \left(\int_{\Omega_\varepsilon} |h|^{p^*} \,\mathrm{d}x \right)^{\frac{1}{p^*}}.
\end{aligned}
$$

Putting all these in (3.29), we infer that

$$\begin{aligned} o_n(1) \leq \varepsilon \Bigg\{ & \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \right] \\ & - \lambda \int_\Omega u_n^{-\gamma} h \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h \,\mathrm{d}x \Bigg\} + \varepsilon^{p^*} \int_{\Omega_\varepsilon} |h|^{p^*} \,\mathrm{d}x \\ & + \varepsilon \left(\int_{\Omega_\varepsilon} u_n^{p^*} \,\mathrm{d}x \right)^{\frac{p^*-1}{p^*}} \left(\int_{\Omega_\varepsilon} |h|^{p^*} \,\mathrm{d}x \right)^{\frac{1}{p^*}} \\ & - \varepsilon \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right) \left[\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x + \int_{\Omega_\varepsilon} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \right]. \end{aligned} \tag{3.30}$$

Since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$ and by the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$, if we divide (3.30) by $\varepsilon > 0$ and then pass to the limit as $\varepsilon \to 0^+$, we obtain

$$\begin{aligned} & \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \right] \\ & - \lambda \int_\Omega u_n^{-\gamma} h \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h \,\mathrm{d}x \geq o_n(1), \end{aligned} \tag{3.31}$$

as $n \to \infty$. By the arbitrariness of $h \in W_0^{1,\mathcal{H}}(\Omega)$, (3.31) actually implies (3.26) which completes the proof. ☐

Now, we prove the compactness property of the energy functional J_λ in a suitable range of λ. For this purpose, we set for any $\lambda > 0$

$$c_\lambda := \alpha_2 - \alpha_1 \lambda^{\frac{p^*}{p^*-1+\gamma}}$$

where

$$\alpha_0 := \left(\frac{1}{q\theta} - \frac{1}{p^*} \right), \qquad \alpha_1 := \frac{(p^*-1+\gamma)|\Omega|}{p^*} \left(\frac{q\theta - 1 + \gamma}{q\theta(1-\gamma)} \right)^{\frac{p^*}{p^*-1+\gamma}} \left(\frac{1-\gamma}{p^* \alpha_0} \right)^{\frac{1-\gamma}{p^*-1+\gamma}} \tag{3.32}$$

and

$$\alpha_2 := \alpha_0 \left(\frac{S b_0}{p^{\theta-1}} \right)^{\frac{p^*}{p^*-p}} \left(S^{p^*} \left(\frac{b_0}{p^{\theta-1}} \right)^p \right)^{\frac{(\theta-1)p^*}{(p^*-p\theta)(p^*-p)}} \tag{3.33}$$

Also, for any $k \in \mathbb{N}$, let T_k be the truncation defined by

$$T_k(t) := \begin{cases} t & \text{if } |t| \leq k, \\ k \dfrac{t}{|t|} & \text{if } |t| > k. \end{cases}$$

Proposition 3.10. *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied, let* $\lambda \in (0, \Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)-(3.9) *and*

$$J_\lambda(u_n) \to c < c_\lambda \quad \text{as } n \to \infty. \tag{3.34}$$

Then $\{u_n\}_{n\in\mathbb{N}}$ *possesses a strongly convergent subsequence in* $W_0^{1,\mathcal{H}}(\Omega)$.

Proof. Fixing $k \in \mathbb{N}$ and taking $h = T_k(u_n - u_\lambda) \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function in (3.26), we get

$$\begin{aligned} o_n(1) = & \left(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right) \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n - u_\lambda) \,\mathrm{d}x \right. \\ & \left. + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \nabla T_k(u_n - u_\lambda) \,\mathrm{d}x \right] \\ & - \lambda \int_\Omega u_n^{-\gamma}\, T_k(u_n - u_\lambda) \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} T_k(u_n - u_\lambda) \,\mathrm{d}x := I - J - K \quad \text{as } n \to \infty. \end{aligned} \tag{3.35}$$

Using Young's inequality, Proposition 2.1 (iii)-(iv), Proposition 2.2 (ii) and boundedness of the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{T_k(u_n-u_\lambda)\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$, we obtain

$$\begin{aligned}
|J| &\le |I| + |K| + o_n(1)\\
&\le \left(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\int_\Omega |\nabla u_n|^{p-1}|\nabla T_k(u_n-u_\lambda)|\,\mathrm{d}x + \int_\Omega a(x)|\nabla u_n|^{q-1}|\nabla T_k(u_n-u_\lambda)|\,\mathrm{d}x\\
&\quad + \int_\Omega |u_n|^{p^*-1}|T_k(u_n-u_\lambda)|\,\mathrm{d}x + o_n(1)\\
&\le \left(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right)\left(\rho_{\mathcal{H}}(\nabla u_n) + \rho_{\mathcal{H}}(\nabla T_k(u_n-u_\lambda))\right) + k\int_\Omega u_n^{p^*-1}\,\mathrm{d}x + o_n(1) \le C(1+k)
\end{aligned} \tag{3.36}$$

with a constant C independent of n and k, that is, the sequence $\{u_n^{-\gamma}\, T_k(u_n-u_\lambda)\}_{n\in\mathbb{N}}$ is uniformly integrable. Then, using (3.10) and Vitali's convergence theorem, we get

$$\int_\Omega u_n^{-\gamma}\, T_k(u_n-u_\lambda)\,\mathrm{d}x \to 0.$$

By Hölder's inequality, we observe that

$$[L^{\mathcal{H}}(\Omega)]^N \ni g \longmapsto \int_\Omega \left(|\nabla u_\lambda|^{p-2} + a(x)|\nabla u_\lambda|^{q-2}\right)\nabla u_\lambda\cdot g\,\mathrm{d}x$$

is a bounded linear functional. From (3.10), we see that $\nabla T_k(u_n-u_\lambda)\rightharpoonup 0$ in $[L^{\mathcal{H}}(\Omega)]^N$, so we can get

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_\lambda|^{p-2} + a(x)|\nabla u_\lambda|^{q-2}\right)\nabla u_n\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x = 0. \tag{3.37}$$

Let $\phi_{\mathcal{H}}(\nabla u_n)\to\beta := \frac{E_1}{p}+\frac{E_2}{q}$ as $n\to\infty$, where E_1 and E_2 are defined in (3.11). Thus, by using (3.36)-(3.37) in (3.35) and the fact that $a_0\ge 0$, $b_0>0, \beta>0$, we get

$$\begin{aligned}
(a_0+b_0\beta^{\theta-1})\limsup_{n\to\infty}\Bigg[&\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x\\
&+\int_\Omega a(x)\left(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u_\lambda|^{q-2}\nabla u_\lambda\right)\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x\Bigg]\\
&= \limsup_{n\to\infty}\int_\Omega u_n^{p^*-1}T_k(u_n-u_\lambda)\,\mathrm{d}x \le Ck.
\end{aligned}$$

By Simon's inequalities, see [24, formula (2.2)], we rewrite the above estimate as

$$\limsup_{n\to\infty}\left[\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x\right] \le \frac{Ck}{(a_0+b_0\beta^{\theta-1})}. \tag{3.38}$$

Set

$$s_n(x) = \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot\nabla(u_n-u_\lambda).$$

Note that $s_n(x)\ge 0$ a. e. in Ω. We divide the set Ω by

$$E_n^k = \{x\in\Omega : |u_n(x)-u_\lambda(x)|\le k\} \quad\text{and}\quad F_n^k = \{x\in\Omega : |u_n(x)-u_\lambda(x)| > k\},$$

where $k,n\in\mathbb{N}$ are fixed. Let $\eta\in(0,1)$. Then, from the definition of T_k, Hölder's inequality, (3.38) and the fact that $\lim_{n\to\infty}|F_n^k| = 0$, we get

$$\begin{aligned}
\limsup_{n\to\infty}\int_\Omega s_n^\eta\,\mathrm{d}x &\le \limsup_{n\to\infty}\left(\int_{E_n^k} s_n\,\mathrm{d}x\right)^\eta |E_n^k|^{1-\eta} + \limsup_{n\to\infty}\left(\int_{F_n^k} s_n\,\mathrm{d}x\right)^\eta |F_n^k|^{1-\eta}\,\mathrm{d}x.\\
&\le \left(\frac{Ck}{(a_0+b_0\beta^{\theta-1})}\right)^\eta |\Omega|^{1-\eta}.
\end{aligned}$$

Letting $k\to 0^+$, we obtain that $s_n^\eta\to 0$ in $L^1(\Omega)$. Thus, we may assume that $s_n\to 0$ a. e. in Ω (up to a subsequence) which along with Simon's inequalities [24, formula (2.2)] gives that

$$\nabla u_n \to \nabla u_\lambda \quad \text{a. e. in } \Omega. \tag{3.39}$$

Let M be the nodal set of the weight function $a(\cdot)$ given by

$$M := \{x \in \Omega : a(x) = 0\}.$$

Since, the sequences $\{|\nabla u_n|^{p-2}\nabla u_n\}_{n\in\mathbb{N}}$ and $\{|\nabla u_n|^{q-2}\nabla u_n\}_{n\in\mathbb{N}}$ are bounded in $L^{p'}(\Omega)$ and $L^{q'}(\Omega \setminus M, a(x)\,\mathrm{d}x)$, respectively, then by using (3.39) and [3, Proposition A.8], we conclude that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_\lambda = \|\nabla u_\lambda\|_p^p$$

and

$$\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla u_\lambda = \int_{\Omega\setminus M} a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla u_\lambda = \|\nabla u_\lambda\|_{q,a}^q.$$

Furthermore, using (3.10), (3.39) and the Brezis-Lieb Lemma, we obtain

$$\begin{aligned} &\rho_{\mathcal{H}}(\nabla u_n) - \rho_{\mathcal{H}}(\nabla u_n - \nabla u_\lambda) = \rho_{\mathcal{H}}(\nabla u_\lambda) + o_n(1),\\ &\|u_n\|_{p^*}^{p^*} - \|u_n - u_\lambda\|_{p^*}^{p^*} = \|u_\lambda\|_{p^*}^{p^*} + o_n(1) \end{aligned} \tag{3.40}$$

as $n \to \infty$. Let $\|u_n - u_\lambda\|_{p^*} \to \ell$ for some $\ell \geq 0$. Now, by taking $u_n - u_\lambda$ as a test function in (3.26), we get

$$\begin{aligned} &o_n(1)\\ &= (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_\lambda)\,\mathrm{d}x + \int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(u_n - u_\lambda)\,\mathrm{d}x\right]\\ &\quad - \lambda \int_\Omega u_n^{-\gamma}\,(u_n - u_\lambda)\,\mathrm{d}x - \int_\Omega u_n^{p^*-1}(u_n - u_\lambda)\,\mathrm{d}x\\ &= (a_0 + b_0\beta^{\theta-1})\left[\rho_{\mathcal{H}}(\nabla u_n) - \rho_{\mathcal{H}}(\nabla u_\lambda) + o_n(1)\right] - \|u_n\|_{p^*}^{p^*} + \|u_\lambda\|_{p^*}^{p^*} + o_n(1) \end{aligned}$$

as $n \to \infty$. Hence, by (3.10) and (3.40) it follows that

$$(a_0 + b_0\beta^{\theta-1})\left[\rho_{\mathcal{H}}(\nabla u_n) - \rho_{\mathcal{H}}(\nabla u_\lambda)\right] = \ell^{p^*} + o_n(1) \quad \text{as } n \to \infty, \tag{3.41}$$

which further gives

$$(a_0 + b_0\beta^{\theta-1}) \lim_{n\to\infty} \left(\|\nabla u_n - \nabla u_\lambda\|_p^p + \|\nabla u_n - \nabla u_\lambda\|_{q,a}^q\right) \leq \ell^{p^*}. \tag{3.42}$$

Now, we claim that $\ell = 0$. Assume by contradiction that $\ell > 0$. By (3.1) and (3.42), we have

$$Sa_0\ell^p \leq S(a_0 + b_0\beta^{\theta-1})\ell^p \leq (a_0 + b_0\beta^{\theta-1}) \lim_{n\to\infty} \|\nabla u_n - \nabla u_\lambda\|_p^p \leq \ell^{p^*}. \tag{3.43}$$

Note that (3.42) implies that

$$(a_0 + b_0\beta^{\theta-1})(E_1 + E_2 - \|\nabla u_\lambda\|_p^p - \|\nabla u_\lambda\|_{q,a}^q) \leq \ell^{p^*}. \tag{3.44}$$

Using (3.43) in (3.44), we get

$$\begin{aligned} \left(\ell^{p^*}\right)^{\frac{p^*-p}{p}} &\geq (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \left(E_1 + E_2 - \|\nabla u_\lambda\|_p^p - \|\nabla u_\lambda\|_{q,a}^q\right)^{\frac{p^*-p}{p}}\\ &= (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \lim_{n\to\infty} \left(\|\nabla u_n - \nabla u_\lambda\|_p^p + \|\nabla u_n - \nabla u_\lambda\|_{q,a}^q\right)^{\frac{p^*-p}{p}}\\ &\geq (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \lim_{n\to\infty} \left(\|\nabla u_n - \nabla u_\lambda\|_p^p\right)^{\frac{p^*-p}{p}} \geq S^{\frac{p^*-p}{p}} (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \ell^{p^*-p}\\ &\geq S^{\frac{p^*}{p}} (a_0 + b_0\beta^{\theta-1})^{\frac{p^*}{p}}. \end{aligned} \tag{3.45}$$

From (3.45) and (3.1), we obtain

$$E_1^{\frac{p^*-p}{p}} \geq \left(E_1 - \|\nabla u_\lambda\|_p^p\right)^{\frac{p^*-p}{p}} = \left(\lim_{n\to\infty} \|\nabla u_n - \nabla u_\lambda\|_p^p\right)^{\frac{p^*-p}{p}} \geq S^{\frac{p^*-p}{p}} \ell^{p^*-p} \geq S^{\frac{p^*}{p}} (a_0 + b_0\beta^{\theta-1}).$$

This gives

$$E_1 \geq S^{\frac{p^*}{p^*-p}} (a_0 + b_0 \beta^{\theta-1})^{\frac{p}{p^*-p}} \geq S^{\frac{p^*}{p^*-p}} \left(\frac{b_0}{p^{\theta-1}} \right)^{\frac{p}{p^*-p}} E_1^{\frac{(\theta-1)p}{p^*-p}}$$

and so we have

$$E_1 \geq \left[S^{p^*} \left(\frac{b_0}{p^{\theta-1}} \right)^p \right]^{\frac{1}{p^*-p\theta}}. \tag{3.46}$$

Combining (3.45) and (3.46), we obtain

$$\begin{aligned} \ell^{p^*} &\geq S^{\frac{p^*}{p^*-p}} (a_0 + b_0 \beta^{\theta-1})^{\frac{p^*}{p^*-p}} \geq \left(\frac{S b_0}{p^{\theta-1}} \right)^{\frac{p^*}{p^*-p}} E_1^{\frac{(\theta-1)p^*}{p^*-p}} \\ &\geq \left(\frac{S b_0}{p^{\theta-1}} \right)^{\frac{p^*}{p^*-p}} \left[S^{p^*} \left(\frac{b_0}{p^{\theta-1}} \right)^p \right]^{\frac{(\theta-1)p^*}{(p^*-p\theta)(p^*-p)}}. \end{aligned} \tag{3.47}$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{q\theta} \langle J'_\lambda(u_n), u_n \rangle &= a_0 \phi_{\mathcal{H}}(\nabla u_n) + \frac{b_0}{\theta} \phi^\theta_{\mathcal{H}}(\nabla u_n) - \frac{1}{q\theta} m(\phi_{\mathcal{H}}(\nabla u_n)) \left\langle \mathcal{L}^a_{p,q}(u_n), u_n \right\rangle \\ &\quad - \lambda \left(\frac{1}{1-\gamma} - \frac{1}{q\theta} \right) \int_\Omega u_n^{1-\gamma} \,\mathrm{d}x + \left(\frac{1}{q\theta} - \frac{1}{p^*} \right) \int_\Omega u_n^{p^*} \,\mathrm{d}x \\ &\geq \left(\frac{1}{q\theta} - \frac{1}{p^*} \right) \|u_n\|_{p^*}^{p^*} - \lambda \left(\frac{1}{1-\gamma} - \frac{1}{q\theta} \right) \int_\Omega u_n^{1-\gamma} \,\mathrm{d}x. \end{aligned}$$

From this, as $n \to \infty$, by (3.47), (3.40), Hölder's and Young's inequality, we derive

$$\begin{aligned} c &= \lim_{n\to\infty} \left(J_\lambda(u_n) - \frac{1}{q\theta} \langle J'_\lambda(u_n), u_n \rangle \right) \\ &\geq \alpha_0 \left(\ell^{p^*} + \|u_\lambda\|_{p^*}^{p^*} \right) - \lambda \left(\frac{1}{1-\gamma} - \frac{1}{q\theta} \right) |\Omega|^{\frac{p^*-1+\gamma}{p^*}} \|u_\lambda\|_{p^*}^{1-\gamma} \\ &\geq \alpha_0 \ell^{p^*} - \alpha_1 \lambda^{\frac{p^*}{p^*-1+\gamma}} \\ &\geq \alpha_0 \left(\frac{S b_0}{p^{\theta-1}} \right)^{\frac{p^*}{p^*-p}} \left[S^{p^*} \left(\frac{b_0}{p^{\theta-1}} \right)^p \right]^{\frac{(\theta-1)p^*}{(p^*-p\theta)(p^*-p)}} - \alpha_1 \lambda^{\frac{p^*}{p^*-1+\gamma}} = c_\lambda, \end{aligned}$$

where α_0, α_1 are defined in (3.32). The above estimates gives a contradiction to (3.34). Hence $\ell = 0$ and using (3.41) and Proposition 2.1(v), we conclude the proof. ☐

Remark 3.11. *By taking $\lambda \in (0, \Lambda_*)$ with $\Lambda_* := \left(\alpha_2 \alpha_1^{-1}\right)^{\frac{p^*-1+\gamma}{p^*}}$ and α_1, α_2 are defined in (3.32) and (3.33) respectively, we have $c_\lambda > 0$.*

Proof of Theorem 1.1. Fix $\lambda < \lambda^* := \min\{\Lambda^*, \Lambda_*\}$. From Lemma 3.1(ii) and Ekeland's variational principle there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{N}^+_\lambda \setminus \{0\}$ verifying (3.8), (3.9), (3.10) and (3.34) with $c = \Theta^+_\lambda$. Hence, by combining Propositions 3.4 and 3.10, we obtain $u_n \to u_\lambda$ strongly in $W^{1,\mathcal{H}}_0(\Omega)$ (up to a subsequence). This further implies that $u_\lambda \in \mathcal{N}_\lambda$ and by Lemma 3.7, we get $u_\lambda \in \mathcal{N}^+_\lambda$ with u_λ achieving Θ^+_λ since J_λ is continuous on $W^{1,\mathcal{H}}_0(\Omega)$. Since $0 \notin \mathcal{N}^+_\lambda$ and $u_n \geq 0$ we have $u_\lambda \not\equiv 0$ and $u_\lambda \geq 0$. Letting $n \to \infty$ in (3.26), we obtain that u satisfies $u_\lambda^{-\gamma}\varphi \in L^1(\Omega)$ and

$$m(\phi_{\mathcal{H}}(\nabla u_\lambda)) \left\langle \mathcal{L}^a_{p,q}(u_\lambda), \varphi \right\rangle = \lambda \int_\Omega u_\lambda^{-\gamma} \varphi \,\mathrm{d}x + \int_\Omega u_\lambda^{r-1} \varphi \,\mathrm{d}x$$

for all $\varphi \in W^{1,\mathcal{H}}_0(\Omega)$. Finally, by using Proposition 3.4, Lemma 3.5 and by repeating the proof of [2, Proposition 4.3 and Proposition 4.4, Step 1], we obtain $u_\lambda > 0$ a. e. in Ω. ☐

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