NONLOCAL DOUBLE PHASE IMPLICIT OBSTACLE PROBLEMS WITH MULTIVALUED BOUNDARY CONDITIONS

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Abstract. In this paper, we introduce and investigate a new kind of nonlinear double phase implicit obstacle problems involving a nonlinear convection term (a reaction term depending on the gradient), three highly nonlinear and nonlocal functions, and multivalued boundary conditions. Under very general assumptions on the data, we develop a generalized framework to explore the existence of weak solutions as well as the compactness of the solution set to the nonlocal double phase implicit obstacle problem. The results established in this paper improve, generalize and extend some results of the existing literature. Our method is based on the theory of multivalued analysis, Tychonoff's fixed point principle and variational methods.

1. Introduction

This paper is concerned with the investigation of an elliptic inclusion problem with a nonlinear and nonhomogeneous partial differential operator (called double phase differential operator), a nonlinear convection term (a reaction term depending on the gradient), an implicit obstacle constraint, three multivalued terms where two of them are appearing on the boundary and the other one is formulated in the domain, and three nonlocal operators in which two of them are described in the domain and the other one is appearing on the boundary. More precisely, we consider the following nonlocal double phase implicit obstacle problem:

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in U_1(x,u)+N(u)(x)+f(x,u,\nabla u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c\phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
L(u) &\le J(u),
\end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary Γ such that Γ is divided into four disjoint measurable parts Γ_1, Γ_2, Γ_3 and Γ_4 with Γ_1 having positive measure, $\mu\colon\Omega\to[0,+\infty)$ and $1<p<q$. Here the nonlinear and nonlocal partial differential operator D_M is given by

$$D_M u := \operatorname{div}\left(M(u)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right) \text{ for all } u\in W^{1,\mathcal{H}}(\Omega),$$

and

$$\frac{\partial u}{\partial \nu_a} := \left(M(u)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu,$$

with ν being the unit normal vector on Γ, $U_1\colon\Omega\times\mathbb{R}\to 2^{\mathbb{R}}$ and $U_2\colon\Gamma_2\times\mathbb{R}\to 2^{\mathbb{R}}$ are two multivalued mappings, $M\colon L^{p^*}(\Omega)\to(0,+\infty)$, $N\colon L^{\zeta_1}(\Omega)\to L^{\zeta_1'}(\Omega)$ and $G\colon L^{\zeta_2}(\Gamma_4)\to L^{\zeta_2'}(\Gamma_4)$ are three continuous functions, $\partial_c\phi(x,u)$ is the convex subdifferential of $s\mapsto\phi(x,s)$, and

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$L, J\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ are given functions defined on the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, see Section 2 for its precise definition.

Such class of problems include different interesting special cases which have not been studied largely in the literature. Initially, the treatment of obstacle problems goes back to the groundbreaking work by Stefan [46] in which the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water, was studied. We also mention the pioneering work of Lions [27] who studied the equilibrium position of an elastic membrane which lies above a given obstacle and which turns out as the unique minimizer of the Dirichlet energy functional.

It should be mentioned that if $M(u) = 1$, $N(u) = 0$ for all $u \in W^{1,\mathcal{H}}(\Omega)$ and $\Gamma_4 = \emptyset$, then problem (1.1) reduces to the following double phase implicit obstacle inclusion problem:

$$\begin{aligned} -D_\mu u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_1,\\ \frac{\partial u}{\partial \nu_\mu} &\in U_2(x,u) && \text{on } \Gamma_2,\\ -\frac{\partial u}{\partial \nu_\mu} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\ L(u) &\leq J(u), \end{aligned} \tag{1.2}$$

where D_μ is the well-known double phase differential operator

$$D_\mu u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \text{ for all } u \in W^{1,\mathcal{H}}(\Omega), \tag{1.3}$$

and

$$\frac{\partial u}{\partial \nu_\mu} := \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nu,$$

In fact, this problem has been considered and studied by Zeng-Rădulescu-Winkert [50] and they used the Kakutani-Ky Fan fixed point theorem in a multivalued version for examining the existence of a solution to problem (1.2) under the condition

$$(f(x,s,\xi) - f(x,t,\xi))(s-t) \leq e_f|s-t|^p \text{ for a. a. } x \in \Omega, \text{ for all } s,t \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N. \tag{1.4}$$

Moreover, when $p = 2$, it is not hard to see that the function $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(x,s,\xi) = \sum_{i=1}^N \zeta_i \xi_i + \kappa_1 s^{\frac{1}{2}} + \omega(x)$$

for all $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, does not satisfy inequality (1.4), where $\omega \in L^2(\Omega)$, $\kappa_1 > 0$ and $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{R}^N$ is a given vector. However, in the present paper, on the one hand, we remove the assumption (1.4) in order to extend the scope of applications to the theoretical results concerning the existence of weak solutions to double phase implicit obstacle problems; on the other hand, we develop a generalized framework to explore the existence of weak solutions as well as the compactness of the solution set to the nonlocal double phase implicit obstacle problem (1.1).

Note that the double phase operator defined in (1.3) is related to the energy functional

$$\omega \mapsto \int_\Omega \left(|\nabla \omega|^p + \mu(x)|\nabla \omega|^q\right)\mathrm{d}x. \tag{1.5}$$

Functionals of type (1.5) have first been studied by Zhikov [53] in order to provide models for strongly anisotropic materials. The main characteristic of the functional defined in (1.5) is the change of ellipticity on the set where the weight function is zero, that is, on the set $\{x \in \Omega : \mu(x) = 0\}$. To be more precise, the energy density of (1.5) exhibits ellipticity in the gradient of order q on the points x where $\mu(x)$ is positive and of order p on the points x where $\mu(x)$ vanishes. Further results on regularity of minimizers of (1.5) can be found in the papers of Baroni-Colombo-Mingione [3, 4], Colombo-Mingione [9, 10], De Filippis-Mingione [12, 13, 14, 15], Marcellini [31, 32] and Ragusa-Tachikawa [44]. We also mention the overview articles of Rădulescu [43] about isotropic and anisotropic problems and of Mingione-Rădulescu [33] about recent developments for problems with nonstandard growth and nonuniform ellipticity.

The main objective of the paper is the development of a general framework for determining the existence of a weak solution to the nonlinear nonlocal implicit obstacle problems (1.1) via Tychonoff's fixed point theorem for multivalued operators, the theory of nonsmooth analysis and variational methods for pseudomonotone operators. As far as we know this is the first work for nonlocal implicit obstacle problems in the double phase setting with mixed boundary conditions.

It should be mentioned that the combination of an implicit obstacle effect with mixed boundary conditions along with multivalued mappings occur in several engineering and economic models, such as Nash equilibrium problems with shared constraints and transport route optimization with feedback control. For more models related to nonsmooth mechanical problems we refer to books of Panagiotopoulos [41, 40] and Naniewicz-Panagiotopoulos [39].

In the content of (implicit) obstacle effects involving Clarke's generalized gradient or general multivalued mappings but without nonlocal term there are several papers using different methods. We refer to the works of Alleche-Rădulescu [1], Aussel-Sultana-Vetrivel [2], Bonanno-Motreanu-Winkert [5], Carl-Le-Winkert [8], Iannizzotto-Papageorgiou [24], Migórski-Khan-Zeng [35, 36], Zeng-Bai-Gasiński-Winkert [48, 49], Zeng-Rădulescu-Winkert [51, 52], see also the recent monograph of Carl-Le [7] about multivalued variational inequalities and inclusions. In the single-valued case with gradient dependent right-hand sides (so-called convection term) we mention the papers of Faraci-Motreanu-Puglisi [16], Faraci-Puglisi [17], Figueiredo-Madeira [18], Gasiński-Papageorgiou [19], Gasiński-Winkert [20], Liu-Motreanu-Zeng [29], Marano-Winkert [30], Papageorgiou-Rădulescu-Repovš [42], see also the references therein.

The paper is organized as follows. Section 2 presents a detailed overview about Musielak-Orlicz Lebesgue and Musielak-Orlicz Sobolev spaces, the p-Laplacian eigenvalue problem with Steklov boundary condition and we state some results from nonsmooth analysis, the properties of Clarke's generalized gradient and Tychonoff's fixed point theorem for multivalued operators which will be used in next sections to establish the existence theorems to various nonlocal double phase obstacle problems. In Section 3, in order to establish the solvability of the nonlocal double phase implicit obstacle problem (1.1), we first introduce an auxiliary problem defined in (3.1), a variational mapping $\mathcal{S}$ driven by problem (3.1), and two multivalued mappings $\mathcal{U}_1$ and $\mathcal{U}_2$ which are exactly the Nemitskij operators of U_1 and U_2, respectively. After that, we prove the complete continuity of $\mathcal{S}$ and upper semicontinuity of $\mathcal{U}_1$ and $\mathcal{U}_2$. Finally, via employing Tychonoff's fixed point theorem for multivalued operators along with the theory of nonsmooth analysis, we establish the nonemptiness and compactness of the solution set of problem (1.1). However, in Section 4, we move our attention to study several special and interesting cases of our problem (1.1), and we deliver the corresponding existence results to these special cases. Also, we make further discussion to some particular problems of (1.1), and obtain several generalized existence theorems for various nonlocal double phase obstacle problems.

2. Mathematical background

In this section we give some necessary notations and preliminary materials which will be used in the next sections from at several places.

Throughout this paper, we suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ such that Γ is separated by four disjoint measurable parts Γ_1, Γ_2, Γ_3 and Γ_4 with Γ_1 having positive Lebesgue measure. Let $1 \leq r < +\infty$ and $D \subset \overline{\Omega}$ be a nonempty set. In what follows, we denote by $L^r(D) := L^r(D;\mathbb{R})$ the usual Lebesgue space equipped with the norm

$\|\cdot\|_{r,D}$ defined by

$$\|u\|_{r,D} := \left(\int_D |u|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \quad \text{for all } u \in L^r(D).$$

Also, we introduce the set $L^r(D)_+ := \{u \in L^r(D) \,:\, u(x) \geq 0 \text{ for a.a. } x \in D\}$. By $W^{1,r}(\Omega)$ we define the corresponding Sobolev space endowed with the norm $\|\cdot\|_{1,r,\Omega}$ defined by

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

The conjugate of $r > 1$ is denoted by $r' > 1$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. Additionally, the critical exponents of $r > 1$ in the domain and on the boundary, denoted by r^* and r_*, are defined by

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N, \end{cases} \quad \text{and} \quad r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N, \end{cases} \tag{2.1}$$

respectively. For the sake of convenience, in the entire paper, the symbols "$\xrightarrow{w}$" and "$\to$" stand for the weak and the strong convergences, respectively, to various function spaces. Recalling that the measure of Γ_1 is positive, so it follows from Korn's inequality that there exists a constant $\hat{\lambda} > 0$ such that

$$\|u\|_{p,\Omega}^p \leq \hat{\lambda} \|\nabla u\|_{p,\Omega}^p \tag{2.2}$$

for all $u \in W$, where W is the subspace of $W^{1,p}(\Omega)$ given by

$$W := \left\{u \in W^{1,p}(\Omega) \,:\, u = 0 \text{ for a.a. } x \in \Gamma_1\right\}.$$

For any $r \geq 2$ fixed, from Simon [45, formula (2.2)], we are able to find a constant $k(r) > 0$ such that the inequality holds

$$\left(|x|^{r-2}x - |y|^{r-2}y\right) \cdot (x - y) \geq k(r)|x - y|^r \tag{2.3}$$

for all $x, y \in \mathbb{R}^N$. Furthermore, we consider the eigenvalue problem of the r-Laplacian ($r > 1$) with Steklov boundary condition formulated by

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u && \text{in } \Omega, \\ |u|^{r-2}u \cdot \nu &= \lambda |u|^{r-2}u && \text{on } \Gamma. \end{aligned} \tag{2.4}$$

From Lê [26], we know that the eigenvalue problem (2.4) has a smallest eigenvalue $\lambda_{1,r}^S > 0$ which is isolated and simple. Also, it is easy to prove that the following variational identity holds

$$\lambda_{1,r}^S = \inf_{u \in W^{1,r}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{r,\Omega}^r + \|u\|_{r,\Omega}^r}{\|u\|_{r,\Gamma}^r}. \tag{2.5}$$

In the whole paper, we suppose that the following hypothesis holds.

H(1): $1 < p < N$, $p < q < p^*$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

Under the above assumption, let us introduce the nonlinear function $\mathcal{H}\colon \Omega \times [0,\infty) \to [0,\infty)$ described by the exponents p, q and weight-function μ defined by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q \quad \text{for all } (x,t) \in \Omega \times [0,\infty).$$

We are now in a position to recall the well-known Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ given by

$$L^{\mathcal{H}}(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ is measurable } \mid \rho_{\mathcal{H}}(u) < +\infty\},$$

where the modular function $\rho_{\mathcal{H}}\colon L^{\mathcal{H}}(\Omega) \to [0,+\infty)$ is formulated by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x \quad \text{for all } u \in L^{\mathcal{H}}(\Omega).$$

It follows from Liu-Dai [28] that Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 \mid \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1\right\} \text{ for all } u \in L^{\mathcal{H}}(\Omega)$$

becomes a reflexive Banach space, because it is uniformly convex. Moreover, we consider the seminormed space $L^q_\mu(\Omega)$

$$L^q_\mu(\Omega) = \left\{u\colon \Omega \to \mathbb{R} \text{ measurable } \mid \int_\Omega \mu(x)|u|^q \,\mathrm{d}x < +\infty\right\}$$

endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \quad \text{for all } u \in L^q_\mu(\Omega).$$

Because problem (1.1) has mixed boundary conditions, the basic function space in the present paper is considered by

$$V := \left\{u \in W^{1,\mathcal{H}}(\Omega) \mid u = 0 \text{ on } \Gamma_1\right\},$$

where $W^{1,\mathcal{H}}(\Omega)$ is the well-known Musielak-Orlicz Sobolev space defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \mid |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}.$$

It is not difficult to prove that V endowed with the norm $\|\cdot\|_V$

$$\|u\|_V := \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}} \quad \text{for all } u \in V$$

is a reflexive Banach space, where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$.

Let us recall some embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, see Gasiński-Winkert [21] or Liu-Dai [28].

Proposition 2.1. *Let* H(1) *be satisfied and denote by p^*, p_* the critical exponents to p as given in* (2.1) *for $s = p$. Then, we have*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ *are continuous for all* $r \in [1,p]$*;*
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1,p^*]$ *and compact for all* $r \in [1,p^*)$*;*
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is continuous for all* $r \in [1,p_*]$ *and compact for all* $r \in [1,p_*)$*;*
(iv) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_\mu(\Omega)$ *is continuous;*
(v) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

It should be mentioned that when the space $W^{1,\mathcal{H}}(\Omega)$ is replaced by V in Proposition 2.1, then the embeddings (ii) and (iii) remain valid.

The following proposition is due to Liu-Dai [28, Proposition 2.1].

Proposition 2.2. *Let* H(1) *be satisfied and let $y \in L^{\mathcal{H}}(\Omega)$. Then the following hold:*

(i) *if $y \neq 0$, then $\|y\|_{\mathcal{H}} = \lambda$ if and only if $\rho_{\mathcal{H}}\left(\frac{y}{\lambda}\right) = 1$;*
(ii) *$\|y\|_{\mathcal{H}} < 1$ (resp. > 1 and $= 1$) if and only if $\rho_{\mathcal{H}}(y) < 1$ (resp. > 1 and $= 1$);*
(iii) *if $\|y\|_{\mathcal{H}} < 1$, then $\|y\|^q_{\mathcal{H}} \le \rho_{\mathcal{H}}(y) \le \|y\|^p_{\mathcal{H}}$;*
(iv) *if $\|y\|_{\mathcal{H}} > 1$, then $\|y\|^p_{\mathcal{H}} \le \rho_{\mathcal{H}}(y) \le \|y\|^q_{\mathcal{H}}$;*
(v) *$\|y\|_{\mathcal{H}} \to 0$ if and only if $\rho_{\mathcal{H}}(y) \to 0$;*
(vi) *$\|y\|_{\mathcal{H}} \to +\infty$ if and only if $\rho_{\mathcal{H}}(y) \to +\infty$.*

Let $w \in V$ be fixed and $M\colon V \to (0,+\infty)$. Next, we introduce the nonlinear operator $\mathcal{H}_w\colon V \to V^*$ given by

$$\begin{aligned}\langle \mathcal{H}_w(u), v\rangle := &\int_\Omega \left(M(w)|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x\\ &+ \int_\Omega \left(|u|^{p-2}u + \mu(x)|u|^{q-2}u\right) v \,\mathrm{d}x,\end{aligned} \tag{2.6}$$

for $u, v \in V$ with $\langle \cdot, \cdot \rangle$ being the duality pairing between V and its dual space V^*. The following proposition states the main properties of $\mathcal{H}_w \colon V \to V^*$. We refer to Crespo-Blanco-Gasiński-Harjulehto-Winkert [11].

Proposition 2.3. *Let the hypotheses* H(1) *be satisfied. Then, for each $w \in V$ the operator $\mathcal{H}_w$ defined by* (2.6) *is bounded, continuous, monotone (hence maximal monotone) and of type* (S_+)*, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{and} \quad \limsup_{n\to\infty} \langle \mathcal{H}_w u_n, u_n - u\rangle \leq 0,$$

imply $u_n \to u$ in V.

In the last part of this section we are going to recall some results from nonsmooth analysis and multivalued analysis. In the following, let E be real Banach space with norm $\|\cdot\|_E$. A function $\varphi \colon E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be proper, convex and lower semicontinuous, if the following conditions are fulfilled:

- $D(\varphi) := \{u \in E : \varphi(u) < +\infty\} \neq \emptyset$;
- for any $u, v \in E$ and $t \in (0,1)$, it holds $\varphi(tu + (1-t)v) \leq t\varphi(u) + (1-t)\varphi(v)$;
- $\liminf_{n\to\infty} \varphi(u_n) \geq \varphi(u)$ where the sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ is such that $u_n \to u$ in E as $n \to \infty$ for some $u \in E$.

Let φ be a convex mapping. An element $x^* \in E^*$ is said to be a subgradient of φ at $u \in E$ if

$$\langle x^*, v - u\rangle \leq \varphi(v) - \varphi(u) \tag{2.7}$$

holds for all $v \in E$. The set of all elements $x^* \in E^*$ which satisfies (2.7) is called the convex subdifferential of φ at u and is denoted by $\partial_c \varphi(u)$.

Moreover, a function $j \colon E \to \mathbb{R}$ is said to be locally Lipschitz at $x \in E$ if there is a neighborhood $O(x)$ of x and a constant $L_x > 0$ such that

$$|j(y) - j(z)| \leq L_x \|y - z\|_E \quad \text{for all } y, z \in O(x).$$

We denote by

$$j^\circ(x; y) := \limsup_{z\to x,\, \lambda\downarrow 0} \frac{j(z + \lambda y) - j(z)}{\lambda},$$

the generalized directional derivative of j at the point x in the direction y and $\partial j \colon E \to 2^{E^*}$ given by

$$\partial j(x) := \{\, \xi \in E^* : j^\circ(x; y) \geq \langle \xi, y\rangle_{E^*\times E} \ \text{ for all } y \in E\} \quad \text{for all } x \in E$$

is the generalized gradient of j at x in the sense of Clarke.

The next proposition summarizes the properties of generalized gradients and generalized directional derivatives of a locally Lipschitz function. We refer to Migórski-Ochal-Sofonea [37, Proposition 3.23] for its proof.

Proposition 2.4. *Let $j \colon E \to \mathbb{R}$ be locally Lipschitz with Lipschitz constant $L_x > 0$ at $x \in E$. Then we have the following:*

(i) *The function $y \mapsto j^\circ(x; y)$ is positively homogeneous, subadditive, and satisfies*
$$|j^\circ(x; y)| \leq L_x \|y\|_E \quad \text{for all } y \in E.$$
(ii) *The function $(x, y) \mapsto j^\circ(x; y)$ is upper semicontinuous.*
(iii) *For each $x \in E$, $\partial j(x)$ is a nonempty, convex, and weak* compact subset of E^* with $\|\xi\|_{E^*} \leq L_x$ for all $\xi \in \partial j(x)$.*
(iv) *$j^\circ(x; y) = \max\{\langle \xi, y\rangle_{E^*\times E} \mid \xi \in \partial j(x)\}$ for all $y \in E$.*
(v) *The multivalued function $E \ni x \mapsto \partial j(x) \subset E^*$ is upper semicontinuous from E into the subsets of E^* with weak* topology.*

We end this section to recall the Tychonoff's fixed point theorem for multivalued operators, its proof can be found in Granas-Dugundji [22, Theorem 8.6].

Theorem 2.5. *Let D be a bounded, closed and convex subset of a reflexive Banach space E, and $\Lambda\colon D \to 2^D$ be a multivalued map such that*

(i) *Λ has bounded, closed and convex values,*
(ii) *Λ is weakly-weakly u.s.c.*

Then Λ has a fixed point in D.

3. Existence and compactness

This section is devoted to explore the nonemptiness and compactness of the solution set to problem (1.1). As mentioned before, our method is based on the theory of multivalued analysis, Tychonoff's fixed point principle and variational methods.

In order to state the existence and compactness results for problem (1.1), we first impose the following assumptions on the data of problem (1.1).

We assume that the nonlocal functions $M\colon L^{p^*}(\Omega) \to (0,+\infty)$, $N\colon L^{\zeta_1}(\Omega) \to L^{\zeta_1'}(\Omega)$ and $G\colon L^{\zeta_2}(\Gamma_4) \to L^{\zeta_2'}(\Gamma_4)$ satisfy the following conditions:

H(M): $M\colon L^{p^*}(\Omega) \to (0,+\infty)$ is such that M is weakly continuous in V, namely, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset V \subset L^{p^*}(\Omega)$ and $u \in V$ such that $u_n \xrightarrow{w} u$ in V as $n \to \infty$, we have

$$M(u) = \lim_{n\to\infty} M(u_n),$$

and there exists a constant $c_M > 0$ such that

$$M(u) \geq c_M \quad \text{for all } u \in V,$$

where p^* is the critical exponent p^* of p in the domain Ω given in (2.1) with $r = p$.

H(N): The function $N\colon L^{\zeta_1}(\Omega) \to L^{\zeta_1'}(\Omega)$ is continuous such that there exist constants $a_N, b_N \geq 0$ and $0 < \kappa_1 < p-1$ satisfying

$$\|N(w)\|_{\zeta_1',\Omega} \leq a_N + b_N\|w\|_{\zeta_1,\Omega}^{\kappa_1} \quad \text{for all } w \in L^{\zeta_1}(\Omega),$$

where $1 < \zeta_1 < p^*$.

H(G): The function $G\colon L^{\zeta_2}(\Gamma_4) \to L^{\zeta_2'}(\Gamma_4)$ is continuous such that there exist constants $a_G, b_G \geq 0$ and $0 < \kappa_2 < p-1$ satisfying

$$\|G(w)\|_{\zeta_2',\Gamma_4} \leq a_G + b_G\|w\|_{\zeta_2,\Gamma_4}^{\kappa_2} \quad \text{for all } w \in L^{\zeta_2}(\Gamma_4),$$

where $1 < \zeta_2 < p_*$ and p_* is the critical exponent of p on the boundary Γ given in (2.1) with $r = p$.

For the convection term f, we suppose the following conditions.

H(f): $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) there exist two constants $a_f, b_f \geq 0$ and a function $\alpha_f \in L^{p'}(\Omega)_+$ satisfying

$$|f(x,s,\xi)| \leq a_f|\xi|^{p-1} + b_f|s|^{p-1} + \alpha_f(x)$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$;

(ii) there exists a constant $e_f \geq 0$ such that

$$|f(x,s,\xi_1) - f(x,s,\xi_2)| \leq e_f|\xi_1 - \xi_2|^{p-1}$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

The multivalued mappings $U_1\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ and $U_2\colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ are assumed to satisfy the following conditions:

H(U_1): The multivalued function $U_1\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that

(i) $U_1(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. a. $x \in \Omega$ and all $s \in \mathbb{R}$;
(ii) $x \mapsto U_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$;
(iii) $s \mapsto U_1(x,s)$ is u.s.c. for a. a. $x \in \Omega$;

(iv) there exist a function $\alpha_{U_1} \in L^{p'}(\Omega)_+$ and a constant $a_{U_1} \geq 0$ such that

$$|\eta| \leq \alpha_{U_1}(x) + a_{U_1}|s|^{p-1}$$

for all $\eta \in U_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

H(U_2): The multivalued function $U_2\colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that

(i) $U_2(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. a. $x \in \Gamma_2$ and all $s \in \mathbb{R}$;

(ii) $x \mapsto U_2(x,s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$;

(iii) $s \mapsto U_2(x,s)$ is u.s.c. for a. a. $x \in \Gamma_2$;

(iv) there exist a function $\alpha_{U_2} \in L^{p'}(\Gamma_2)_+$ and a constant $a_{U_2} > 0$ such that

$$|\xi| \leq \alpha_{U_2}(x) + a_{U_2}|s|^{p-1}$$

for all $\xi \in U_2(x,s)$, for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$.

On the boundary Γ_3, the function $\phi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ fulfills the following assumptions:

H(ϕ): The function $\phi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $x \mapsto \phi(x,r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$;

(ii) $r \mapsto \phi(x,r)$ is convex and l.s.c. for a. a. $x \in \Gamma_3$;

(iii) for each function $u \in L^{p_*}(\Gamma_3)$ the function $x \mapsto \phi(x,u(x))$ belongs to $L^1(\Gamma_3)$.

With respect to the nonlocal functions $L\colon V \to \mathbb{R}$ and $J\colon V \to (0,+\infty)$, we suppose the following:

H(L): $L\colon V \to \mathbb{R}$ is positively homogeneous and subadditive such that

$$L(u) \leq \limsup_{n\to\infty} L(u_n),$$

whenever $\{u_n\}_{n\in\mathbb{N}} \subset V$ is such that $u_n \xrightarrow{w} u$ in V for some $u \in V$.

H(J): $J\colon V \to (0,+\infty)$ is weakly continuous, that is, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$ such that $u_n \xrightarrow{w} u$ for some $u \in V$, we have

$$J(u_n) \to J(u).$$

Moreover, we state the following compatibility conditions.

H(2): The inequalities

$$\begin{aligned}
&0 < k(p)c_M - e_f\hat{\lambda}^{\frac{1}{p}},\\
&0 < \min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}}, 1\} - (a_{U_1} + b_f)\, c_p(\Omega)^p - a_{U_2}c_p(\Gamma_2)^p
\end{aligned}$$

hold, where $k(p)$ and $\hat{\lambda} > 0$ are given in (2.3) and (2.2), and $c_p(\Omega) > 0$ and $c_p(\Gamma_2) > 0$ are the smallest constants satisfying the following inequalities (because of the continuity of the embeddings of V to $L^p(\Omega)$ and of V to $L^p(\Gamma_2)$)

$$\|u\|_{p,\Omega} \leq c_p(\Omega)\|u\|_V \quad \text{and} \quad \|u\|_{p,\Gamma_2} \leq c_p(\Gamma_2)\|u\|_V \quad \text{for all } u \in V.$$

Remark 3.1. *The compatibility inequalities in H(2) are usually called to be smallness conditions, which have been applied in many literatures, for example,* [23, 34] *(nonsmooth contact mechanics problems) and* [50, 35] *(nonlinear partial differential equations). Essentially speaking, the compatibility inequalities in H(2) will play a critical role to guarantee that the variational selection $\mathcal{S}$ is self-map on a bounded closed set (see (3.19), below), and to reveal that the problem (1.1) has coercive framework. The following functions fulfill assumptions H(M):*

- $M(u) = c_M + r_1(\|u\|_{\pi_1,\Omega})$ *for all* $u \in V$*, where* $r_1\colon [0,+\infty) \to [0,+\infty)$ *is a continuous function,* $c_M > 0$ *and* $1 < \pi_1 < p^*$*;*
- $M(u) = a_M + r_2(\|u\|_{\pi_2,\Gamma})$ *for all* $u \in V$*, where* $r_2\colon [0,+\infty) \to [0,+\infty)$ *is a continuous function,* $c_M > 0$ *and* $1 < \pi_2 < p_*$*.*

It is not difficult to see that the following functions $N\colon L^{\zeta_1}(\Omega)\to L^{\zeta_1'}(\Omega)$ *and* $G\colon L^{\zeta_2}(\Gamma_4)\to L^{\zeta_2'}(\Gamma_4)$ *satisfy the conditions H(N) and H(G), respectively:*

$$N(u)(x) := \left(\int_\Omega \varpi_1(x)|u(x)|\,\mathrm{d}x\right)^{\frac{p-1}{2}} + \varpi_2(x) \quad \text{for all } x\in\Omega \text{ and all } u\in L^{\zeta_1}(\Omega),$$

and

$$G(w)(x) := \int_{\Gamma_4} \varpi_3(x)|w(x)|^{\zeta_1-1}\,\mathrm{d}x + \varpi_4(x) \quad \text{for all } x\in\Gamma_4 \text{ and } w\in L^{\zeta_2}(\Gamma_4),$$

where $\varpi_1\in L^{\zeta_1'}(\Omega)_+$, $\varpi_2\in L^p(\Omega)$ *and* $\varpi_3,\varpi_4\in L^{\zeta_2}(\Gamma_4)$.

Let $p=2$ *and* $f\colon \Omega\times\mathbb{R}^N\to\mathbb{R}$ *be defined by*

$$f(x,s,\xi)=\sum_{i=1}^N \zeta_i\xi_i-\kappa_1 s+\omega(x)$$

for all $x\in\Omega$, *for all* $s\in\mathbb{R}$ *and for all* $\xi\in\mathbb{R}^N$, *where* $\omega\in L^2(\Omega)$, $\kappa_1>0$ *and* $\zeta=(\zeta_1,\ldots,\zeta_N)\in\mathbb{R}^N$ *is a given vector. Then,* f *satisfies hypotheses H(f).*

Let $\alpha_1\in L^{p'}(\Omega)$ *and* $\alpha_2\in L^{p'}(\Gamma_2)$. *Then, the multivalued functions defined by*

$$\begin{aligned} U_1(x,s)&=[-1,1]\alpha_1(x)+s^{p-1} \text{ for all } x\in\Omega \text{ and all } s\in\mathbb{R},\\ U_2(x,s)&=[0,2]s^{p-1}+\alpha_2(x) \text{ for all } x\in\Gamma_2 \text{ and all } s\in\mathbb{R}, \end{aligned}$$

satisfy hypotheses H(U_1*) and H(*U_2*), respectively.*

Let $\varpi_5\in L^{p'}(\Gamma_3)_+$. *Then, the function defined by*

$$\varphi(x,s):=\varpi_5(x)|s| \text{ for all } x\in\Gamma_3 \text{ and } s\in\mathbb{R},$$

satisfies hypotheses H(ϕ*).*

It is obvious that the functions $L(u)=\|u\|_V$ *and* $J(u)=e^{1+\|u\|_{p,\Omega}}$ *for all* $u\in V$ *fulfill hypotheses H(L) and H(J), respectively.*

Let us consider the multivalued mapping $K\colon V\to 2^V$ defined by

$$K(u)=\{v\in V \mid L(v)\le J(u)\} \quad \text{for all } u\in V.$$

Under the hypotheses H(L) and H(J), we have the following important auxiliary result which delivers several significant properties for the multivalued mapping $K\colon V\to 2^V$. More precisely, this lemma reveals an essential characteristic that K is Mosco continuous (see Mosco [38], i.e., K is sequentially weakly-weakly closed and sequentially weakly-strongly l.s.c.). The detailed proof of this lemma can be found in Lemma 3.3 of Zeng-Rădulescu-Winkert [50].

Lemma 3.2. *Let* $J\colon V\to(0,+\infty)$ *and* $L\colon V\to\mathbb{R}$ *be two functions such that* H(L) *and* H(J) *are satisfied. Then, the following statements hold:*

(i) *for each* $u\in V$, $K(u)$ *is closed and convex in* V *such that* $0\in K(u)$;
(ii) *the graph* $\operatorname{Gr}(K)$ *of* K *is sequentially closed in* $V_w\times V_w$, *that is,* K *is sequentially closed from* V *with the weak topology into the subsets of* V *with the weak topology;*
(iii) *if* $\{u_n\}_{n\in\mathbb{N}}\subset V$ *is a sequence such that*
$$u_n \xrightarrow{w} u \quad \text{in } V$$
for some $u\in V$, *then for each* $v\in K(u)$ *there exists a sequence* $\{v_n\}_{n\in\mathbb{N}}\subset V$ *such that*
$$v_n\in K(u_n) \quad \text{and} \quad v_n\to v \quad \text{in } V.$$

We are now in a position to give the definition of weak solutions to problem (1.1) as follows.

Definition 3.3. *We say that a function $u \in V$ is a weak solution of problem (1.1) if $u \in K(u)$ and there exist functions $\eta \in L^{p'}(\Omega)$, $\xi \in L^{p'}(\Gamma_2)$ such that $\eta(x) \in U_1(x,u(x))$ for a. a. $x \in \Omega$, $\xi(x) \in U_2(x,u(x))$ for a. a. $x \in \Gamma_2$ and the inequality*

$$\begin{aligned}
&M(u)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x+\int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x\\
&+\int_\Omega(|u|^{p-2}u+\mu(x)|u|^{q-2}u)(v-u)\,\mathrm{d}x+\int_\Omega N(u)(x)(v-u)\,\mathrm{d}x\\
&+\int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma+\int_{\Gamma_4}G(u)(x)(v-u)\,\mathrm{d}\Gamma\\
&\geq\int_\Omega\eta(x)(v-u)\,\mathrm{d}x+\int_{\Gamma_2}\xi(x)(v-u)\,\mathrm{d}\Gamma+\int_\Omega f(x,u,\nabla u)(v-u)\,\mathrm{d}x
\end{aligned}$$

holds for all $v \in K(u)$.

For the convenience of the reader, in the sequel, we use the following notion

$$X=L^p(\Omega),\quad Y=L^p(\Gamma_2),\quad X^*=L^{p'}(\Omega)\quad\text{and}\quad Y^*=L^{p'}(\Gamma_2).$$

For any $(w,\eta,\xi)\in V\times X^*\times Y^*$ fixed, let us consider the following auxiliary problem:

$$\begin{aligned}
-D_{M(w)}u+|u|^{p-2}u+\mu(x)|u|^{q-2}u&=\eta(x)+N(w)(x)+f(x,w,\nabla u) &&\text{in } \Omega,\\
u&=0 &&\text{on } \Gamma_1,\\
\frac{\partial u}{\partial\nu_w}&=\xi(x) &&\text{on } \Gamma_2,\\
-\frac{\partial u}{\partial\nu_w}&\in\partial_c\phi(x,u) &&\text{on } \Gamma_3,\\
-\frac{\partial u}{\partial\nu_w}&=G(w)(x) &&\text{on } \Gamma_4,\\
L(u)&\leq J(w),
\end{aligned}\tag{3.1}$$

where the differential operator $D_{M(w)}$ is defined by

$$D_{M(w)}u:=\operatorname{div}\left(M(w)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\text{ for all } u\in W^{1,\mathcal{H}}(\Omega),$$

and $\frac{\partial u(x)}{\partial\nu_w}$ stands for

$$\frac{\partial u}{\partial\nu_w}:=\left(M(w)|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu.$$

From Definition 3.3 we can see that $u\in V$ is a weak solution of problem (3.1) if $u\in K(w)$ and the following inequality is satisfied

$$\begin{aligned}
&M(w)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x+\int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x\\
&+\int_\Omega(|u|^{p-2}u+\mu(x)|u|^{q-2}u)(v-u)\,\mathrm{d}x+\int_\Omega N(w)(x)(v-u)\,\mathrm{d}x\\
&+\int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma+\int_{\Gamma_4}G(w)(x)(v-u)\,\mathrm{d}\Gamma\\
&\geq\int_\Omega\eta(x)(v-u)\,\mathrm{d}x+\int_{\Gamma_2}\xi(x)(v-u)\,\mathrm{d}\Gamma+\int_\Omega f(x,w,\nabla u)(v-u)\,\mathrm{d}x
\end{aligned}\tag{3.2}$$

for all $v\in K(w)$.

The following lemma shows that problem (3.1) is uniquely solvable.

Proposition 3.4. *Let $p \geq 2$. Assume that* H(1), H(ϕ), H(f), H(L) *and* H(J) *hold. If $M(w) \geq c_M$ for each $w \in V$, $N(w) \in L^{\zeta_1'}(\Omega)$ with $1 < \zeta_1 < p^*$, $G(w) \in L^{\zeta_2'}(\Gamma_4)$ with $1 < \zeta_2 < p_*$, and the inequality $0 < k(p)c_M - e_f\hat{\lambda}^{\frac{1}{p}}$ holds, then problem (3.1) admits a unique solution.*

Proof. First we introduce the following nonlinear mappings $\mathcal{G}_w \colon V \to V^*$, $\varphi \colon V \to \overline{\mathbb{R}}$ and $\mathcal{F}_w \colon V \subset L^p(\Omega) \to L^{p'}(\Omega) \subset V^*$ defined by

$$\begin{aligned}\langle \mathcal{G}_w(u), v\rangle := M(w)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla v\,\mathrm{d}x\\ + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v\,\mathrm{d}x + \int_\Omega N(w)(x)v\,\mathrm{d}x\\ + \int_{\Gamma_4} G(w)(x)v\,\mathrm{d}\Gamma - \int_\Omega \eta(x)v\,\mathrm{d}x - \int_{\Gamma_2}\xi(x)v\,\mathrm{d}\Gamma\end{aligned}$$

for all $u, v \in V$,

$$\varphi(u) := \int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma$$

for all $u \in V$, and

$$\langle \mathcal{F}_w u, v\rangle_{L^{p'}(\Omega)\times L^p(\Omega)} := \int_\Omega f(x,w,\nabla u)v\,\mathrm{d}x$$

for all $u \in V$ and $v \in L^p(\Omega)$. Using the notations above, it is not difficult to prove that inequality (3.2) can be equivalently rewritten by the following nonlinear variational inequality with constraint

$$\langle \mathcal{G}_w u, v-u\rangle + \varphi(v) - \varphi(u) \geq \langle i^*\mathcal{F}_w u, v-u\rangle$$

for all $v \in K(w)$, where $i \colon V \to L^p(\Omega)$ is the embedding operator of V into $L^p(\Omega)$ and $i^* \colon L^{p'}(\Omega) \to V^*$ is the dual operator of i. Arguing as in the proof of Theorem 3.4 of Zeng-Bai-Gasiński [47], we can show that problem (3.1) has at least one solution.

Next, we are going to prove the uniqueness of problem (3.1). Let $u_1, u_2 \in V$ be two weak solutions of problem (3.1). So, for every $i = 1, 2$, it holds $u_i \in K(w)$ and

$$\begin{aligned}M(w)\int_\Omega |\nabla u_i|^{p-2}\nabla u_i\cdot\nabla(v-u_i)\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_i|^{q-2}\nabla u_i\cdot\nabla(v-u_i)\,\mathrm{d}x\\ + \int_\Omega (|u_i|^{p-2}u_i + \mu(x)|u_i|^{q-2}u_i)(v-u_i)\,\mathrm{d}x + \int_\Omega N(w)(x)(v-u_i)\,\mathrm{d}x\\ + \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_i)\,\mathrm{d}\Gamma + \int_{\Gamma_4} G(w)(x)(v-u_i)\,\mathrm{d}\Gamma\\ \geq \int_\Omega \eta(x)(v-u_i)\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)(v-u_i)\,\mathrm{d}\Gamma + \int_\Omega f(x,w,\nabla u_i)(v-u_i)\,\mathrm{d}x\end{aligned}$$

for all $v \in K(w)$. Putting $v = u_2$ and $v = u_1$ into the above inequalities with $i = 1$ and $i = 2$, respectively, we use the resulting inequalities to get

$$\begin{aligned}M(w)\int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\ + \int_\Omega \mu(x)\left(|\nabla u_1|^{q-2}\nabla u_1 - |\nabla u_2|^{q-2}\nabla u_2\right)\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\ + \int_\Omega \left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right)(u_1-u_2)\,\mathrm{d}x\\ + \int_\Omega \mu(x)\left(|u_1|^{q-2}u_1 - |u_2|^{q-2}u_2\right)(u_1-u_2)\,\mathrm{d}x\end{aligned}$$

$$\leq \int_\Omega (f(x,w,\nabla u_1) - f(x,w,\nabla u_2))(u_1-u_2)\,\mathrm{d}x.$$

The latter combined with (2.3), hypothesis H(f)(ii) and Hölder's inequality implies that

$$\begin{aligned}
&k(p)\left(c_M\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p + \|u_1-u_2\|_{p,\Omega}^p\right)\\
&\leq \int_\Omega e_f|\nabla u_1-\nabla u_2|^{p-1}|u_1-u_2|\,\mathrm{d}x\\
&\leq e_f\|\nabla u_1-\nabla u_2\|_{p,\Omega}^{p-1}\|u_1-u_2\|_{p,\Omega}\\
&\leq e_f\hat{\lambda}^{\frac{1}{p}}\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p.
\end{aligned}$$

This means that

$$\left(k(p)c_M - e_f\hat{\lambda}^{\frac{1}{p}}\right)\|\nabla u_1-\nabla u_2\|_{p,\Omega}^p + k(p)\|u_1-u_2\|_{p,\Omega}^p \leq 0.$$

Employing the inequality $e_f\hat{\lambda}^{\frac{1}{p}} < c_M k(p)$, we infer that $u_1 = u_2$.

Consequently, for every $(w,\eta,\xi)\in V\times X^*\times Y^*$ fixed, problem (3.1) is uniquely solvable. ☐

Lemma 3.4 allows us to introduce the solution mapping $\mathcal{S}\colon V\times X^*\times Y^*\to V$ of problem (3.1) formulated by

$$\mathcal{S}(w,\eta,\xi) = u_{w,\eta,\xi} \quad \text{for all } (w,\eta,\xi)\in V\times X^*\times Y^*,$$

where $u_{w,\eta,\xi}$ is the unique solution of problem (3.1) associated with $(w,\eta,\xi)\in V\times X^*\times Y^*$.

The following lemma says that $\mathcal{S}\colon V\times X^*\times Y^*\to V$ is a completely continuous operator.

Lemma 3.5. *Let $p\geq 2$. Assume that* H(1), H(2), H(M), H(N), H(G), H(ϕ), H(f), H(L) *and* H(J) *are fulfilled. Then, the solution map $\mathcal{S}\colon V\times X^*\times Y^*\to V$ of problem* (3.1) *is completely continuous.*

Proof. Assume that $\{(w_n,\eta_n,\xi_n)\}_{n\in\mathbb{N}}\subset V\times X^*\times Y^*$ and $(w,\eta,\xi)\in V\times X^*\times Y^*$ satisfy

$$(w_n,\eta_n,\xi_n) \xrightarrow{w} (w,\eta,\xi) \quad \text{in } V\times X^*\times Y^*.$$

Let $u_n := \mathcal{S}(w_n,\eta_n,\xi_n)$ for each $n\in\mathbb{N}$. So, for each $n\in\mathbb{N}$, we have $u_n\in K(w_n)$ and

$$\begin{aligned}
&M(w_n)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(v-u_n)\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(v-u_n)\,\mathrm{d}x\\
&\quad + \int_\Omega (|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n)(v-u_n)\,\mathrm{d}x + \int_\Omega N(w_n)(x)(v-u_n)\,\mathrm{d}x\\
&\quad + \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_4} G(w_n)(x)(v-u_n)\,\mathrm{d}\Gamma\\
&\geq \int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x + \int_{\Gamma_2}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma + \int_\Omega f(x,w_n,\nabla u_n)(v-u_n)\,\mathrm{d}x
\end{aligned} \tag{3.3}$$

for all $v\in K(w_n)$. Using hypothesis H(f)(i) gives

$$\begin{aligned}
&\int_\Omega f(x,w_n,\nabla u_n)u_n(x)\,\mathrm{d}x\\
&\leq \int_\Omega \left(a_f|\nabla u_n|^{p-1} + b_f|w_n(x)|^{p-1} + \alpha_f(x)\right)|u_n(x)|\,\mathrm{d}x\\
&\leq a_f\|\nabla u_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} + b_f\|w_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} + \|\alpha_f\|_{p',\Omega}\|u_n\|_{p,\Omega}\\
&\leq a_f\hat{\lambda}^{\frac{1}{p}}\|\nabla u_n\|_{p,\Omega}^p + b_f\|w_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} + \|\alpha_f\|_{p',\Omega}\|u_n\|_{p,\Omega}.
\end{aligned} \tag{3.4}$$

From Brézis [6, Proposition 1.10] and Hölder's inequality, we can find two constants $\alpha_\varphi,\beta_\varphi\geq 0$ such that

$$\varphi(v) \geq -\alpha_\varphi\|v\|_V - \beta_\varphi \text{ for all } v\in V, \tag{3.5}$$

and

$$\begin{cases} \displaystyle\int_\Omega \eta_n(x)u_n\,\mathrm{d}x \le \|\eta_n\|_{p',\Omega}\|u_n\|_{p,\Omega}, \\ \displaystyle\int_{\Gamma_2} \xi_n(x)u_n\,\mathrm{d}\Gamma \le \|\xi_n\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2}, \\ \displaystyle\left|\int_\Omega N(w_n)(x)u_n\,\mathrm{d}x\right| \le \|N(w_n)\|_{\zeta_1',\Omega}\|u_n\|_{\zeta_1,\Omega} \le \left(a_N + b_N\|w_n\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u_n\|_{\zeta_1,\Omega}, \\ \displaystyle\left|\int_{\Gamma_4} G(w_n)(x)u_n\,\mathrm{d}x\right| \le \|G(w_n)\|_{\zeta_2',\Gamma_4}\|u_n\|_{\zeta_2,\Gamma_4} \le \left(a_G + b_G\|w_n\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u_n\|_{\zeta_2,\Gamma_4}. \end{cases} \tag{3.6}$$

Letting $v=0$ in (3.3) and using the estimates (3.4), (3.5) and (3.6), it yields

$$\begin{aligned}
&a_f\hat{\lambda}^{\frac{1}{p}}\|\nabla u_n\|_{p,\Omega}^p + b_f\|w_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} + \|\alpha_f\|_{p',\Omega}\|u_n\|_{p,\Omega} + \|\eta_n\|_{p',\Omega}\|u_n\|_{p,\Omega} + \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma \\
&\quad + \|\xi_n\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2} + \left(a_N + b_N\|w_n\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u_n\|_{\zeta_1,\Omega} + \left(a_G + b_G\|w_n\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u_n\|_{\zeta_2,\Gamma_4} \\
&\ge -\int_\Omega N(w_n)(x)u_n\,\mathrm{d}x + \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma - \int_{\Gamma_4} G(w_n)(x)u_n\,\mathrm{d}\Gamma + \int_\Omega \eta_n(x)u_n\,\mathrm{d}x \\
&\quad + \int_{\Gamma_2}\xi_n(x)u_n\,\mathrm{d}\Gamma + \int_\Omega f(x,w_n,\nabla u_n)u_n\,\mathrm{d}x \\
&\ge \int_\Omega M(w_n)|\nabla u_n|^p + \mu(x)|\nabla u_n|^q + |u_n|^p + \mu(x)|u_n|^q\,\mathrm{d}x + \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma \\
&\ge c_M\|\nabla u_n\|_{p,\Omega}^p + \|\nabla u_n\|_{q,\mu}^q + \|u_n\|_{p,\Omega}^p + \|u_n\|_{q,\mu}^q - \alpha_\varphi\|u_n\|_V - \beta_\varphi.
\end{aligned}$$

Then, from Proposition 2.2, we have

$$\begin{aligned}
0 &\ge (c_M - a_f\hat{\lambda}^{\frac{1}{p}})\|\nabla u_n\|_{p,\Omega}^p + \|\nabla u_n\|_{q,\mu}^q + \|u_n\|_{p,\Omega}^p + \|u_n\|_{q,\mu}^q - b_f\|w_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} \\
&\quad - \|\alpha_f\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|\eta_n\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|\xi\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2} - \left(a_N + b_N\|w_n\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u_n\|_{\zeta_1,\Omega} \\
&\quad - \left(a_G + b_G\|w_n\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u_n\|_{\zeta_2,\Gamma_4} - \alpha_\varphi\|u_n\|_V - \beta_\varphi - \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma \\
&\ge \min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}}, 1\}\min\{\|u_n\|_V^p, \|u_n\|_V^q\} - b_f\|w_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} - \|\alpha_f\|_{p',\Omega}\|u_n\|_{p,\Omega} \\
&\quad - \|\eta_n\|_{p',\Omega}\|u_n\|_{p,\Omega} - \|\xi_n\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2} - \left(a_N + b_N\|w_n\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u_n\|_{\zeta_1,\Omega} \\
&\quad - \left(a_G + b_G\|w_n\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u_n\|_{\zeta_2,\Gamma_4} - \alpha_\varphi\|u_n\|_V - \beta_\varphi - \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma.
\end{aligned}$$

The latter combined with the boundedness of $\{w_n\}_{n\in\mathbb{N}} \subset V$, $\{\eta_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset Y^*$ implies that solution sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in V.

Passing to a subsequence if necessary, we may find a function $u \in V$ satisfying

$$u_n \xrightarrow{w} u \quad \text{in } V \text{ as } n \to \infty.$$

We assert that $u = \mathcal{S}(w,\eta,\xi)$, i.e., u is the unique solution of problem (3.1) corresponding to $(w,\eta,\xi) \in V \times X^* \times Y^*$.

Recall that $w_n \xrightarrow{w} w$ in V and $u_n \xrightarrow{w} u$ in V, we are now in a position to invoke Lemma 3.2(ii) to get that $u \in K(w)$. However, it follows from Lemma 3.2(iii) that there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$ satisfying

$$v_n \in K(w_n) \quad \text{for every } n \in \mathbb{N} \quad \text{and} \quad v_n \to u \quad \text{in } V.$$

Letting $v=v_n$ in (3.3), one has

$$\begin{aligned}
&M(w_n)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(v_n-u_n)\,\mathrm{d}x+\int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(v_n-u_n)\,\mathrm{d}x\\
&\quad+\int_\Omega(|u_n|^{p-2}u_n+\mu(x)|u_n|^{q-2}u_n)(v_n-u_n)\,\mathrm{d}x+\int_\Omega N(w_n)(x)(v_n-u_n)\,\mathrm{d}x\\
&\quad+\int_{\Gamma_3}\phi(x,v_n)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma+\int_{\Gamma_4}G(w_n)(x)(v_n-u_n)\,\mathrm{d}\Gamma\\
&\geq\int_\Omega \eta_n(x)(v_n-u_n)\,\mathrm{d}x+\int_{\Gamma_2}\xi_n(x)(v_n-u_n)\,\mathrm{d}\Gamma+\int_\Omega f(x,w_n,\nabla u_n)(v_n-u_n)\,\mathrm{d}x.
\end{aligned}\tag{3.7}$$

From the boundedness of $\{N(w_n)\}_{n\in\mathbb{N}}$, $\{G(w_n)\}_{n\in\mathbb{N}}$, $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$, it can directly be obtained that

$$\begin{cases}
\displaystyle\lim_{n\to\infty}\int_\Omega N(w_n)(x)(v_n-u_n)\,\mathrm{d}x=0,\\
\displaystyle\lim_{n\to\infty}\int_{\Gamma_4} G(w_n)(x)(v_n-u_n)\,\mathrm{d}\Gamma=0,\\
\displaystyle\lim_{n\to\infty}\int_\Omega \eta_n(x)(v_n-u_n)\,\mathrm{d}x=0,\\
\displaystyle\lim_{n\to\infty}\int_{\Gamma_2} \xi_n(x)(v_n-u_n)\,\mathrm{d}\Gamma=0,
\end{cases}\tag{3.8}$$

where we have used the compactness of the embeddings of V into $L^{\zeta_1}(\Omega)$, of V into $L^{\zeta_2}(\Gamma_4)$, of V into $L^p(\Gamma_2)$, and of V into $L^p(\Omega)$. By hypothesis H(f)(i), we can see that sequence $\{f(\cdot,w_n,\nabla u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\Omega)$. Hence, it holds

$$\lim_{n\to\infty}\int_\Omega f(x,w_n,\nabla u_n)(v_n-u_n)\,\mathrm{d}x=0.\tag{3.9}$$

From hypotheses H(ϕ), it admits that $V\ni u\mapsto\varphi(u):=\displaystyle\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma$ is continuous and convex, so, it is weakly l.s.c., because of $V\subset \operatorname{int}D(\varphi)$. Therefore, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left[\int_{\Gamma_3}\phi(x,v_n)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma\right]\\
&\leq\lim_{n\to\infty}\int_{\Gamma_3}\phi(x,v_n)\,\mathrm{d}\Gamma-\liminf_{n\to\infty}\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma=0.
\end{aligned}\tag{3.10}$$

Recall that M is weakly continuous in V (see hypothesis H(M)), it yields

$$\begin{aligned}
&\limsup_{n\to\infty}\Big[\int_\Omega \big(M(w_n)|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\big)\cdot\nabla(u_n-v_n)\,\mathrm{d}x\\
&\quad+\int_\Omega(|u_n|^{p-2}u_n+\mu(x)|u_n|^{q-2}u_n)(u_n-v_n)\,\mathrm{d}x\Big]\\
&\geq\limsup_{n\to\infty}\Big[\int_\Omega \big(M(w)|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega(|u_n|^{p-2}u_n+\mu(x)|u_n|^{q-2}u_n)(u_n-u)\,\mathrm{d}x\Big]\\
&\quad-\limsup_{n\to\infty}|M(w_n)-M(w)|\left|\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n-v_n)\,\mathrm{d}x\right|\\
&\quad-\limsup_{n\to\infty}\left|\int_\Omega\mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(u-v_n)\,\mathrm{d}x\right|\\
&\geq\limsup_{n\to\infty}\langle\mathcal{H}_w(u),u_n-u\rangle-\limsup_{n\to\infty}|M(w_n)-M(w)|\,\|u_n\|_{p,\Omega}^{p-1}\|u_n-v_n\|_{p,\Omega}\\
&\quad-\limsup_{n\to\infty}\|u_n\|_{q,\mu}^{q-1}\|u-v_n\|_{q,\mu}\\
&=\limsup_{n\to\infty}\langle\mathcal{H}_w(u),u_n-u\rangle.
\end{aligned}\tag{3.11}$$

Passing to the upper limit as $n\to\infty$ to inequality (3.7) and using (3.8), (3.9), (3.10), (3.11) and (3.15), one has

$$\limsup_{n\to\infty}\langle\mathcal{H}_w(u),u_n-u\rangle\leq 0.$$

The latter combined with Proposition 2.3 (i.e., $\mathcal{H}_w$ is of type (S_+)) implies that $u_n\to u$ in V.

Let $z\in K(w)$ be arbitrary. By Lemma 3.2(iii), we are able to choose a sequence $\{z_n\}_{n\in\mathbb{N}}\subset V$ such that $z_n\in K(w_n)$ for any $n\in\mathbb{N}$ and $z_n\to z$ in V. Inserting $v=z_n$ into (3.3) and passing to the upper limit as $n\to\infty$ for the resulting inequality, we obtain

$$\begin{aligned}
&M(w)\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla(z-u)\,\mathrm{d}x+\int_\Omega\mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla(z-u)\,\mathrm{d}x\\
&\quad+\int_\Omega(|u|^{p-2}u+\mu(x)|u|^{q-2}u)(z-u)\,\mathrm{d}x+\int_\Omega N(w)(x)(z-u)\,\mathrm{d}x\\
&\quad+\int_{\Gamma_3}\phi(x,z)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma+\int_{\Gamma_4}G(w)(x)(z-u)\,\mathrm{d}\Gamma\\
&\geq\limsup_{n\to\infty}\Big[M(w_n)\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(z_n-u_n)\,\mathrm{d}x+\int_\Omega\mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(z_n-u_n)\,\mathrm{d}x\\
&\quad+\int_\Omega(|u_n|^{p-2}u_n+\mu(x)|u_n|^{q-2}u_n)(z_n-u_n)\,\mathrm{d}x+\int_\Omega N(w_n)(x)(z_n-u_n)\,\mathrm{d}x\\
&\quad+\int_{\Gamma_3}\phi(x,z_n)\,\mathrm{d}\Gamma-\int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma+\int_{\Gamma_4}G(w_n)(x)(z_n-u_n)\,\mathrm{d}\Gamma\Big]\\
&\geq\limsup_{n\to\infty}\Big[\int_\Omega\eta_n(x)(z_n-u_n)\,\mathrm{d}x+\int_{\Gamma_2}\xi_n(x)(z_n-u_n)\,\mathrm{d}\Gamma+\int_\Omega f(x,w_n,\nabla u_n)(z_n-u_n)\,\mathrm{d}x\Big]\\
&=\int_\Omega\eta(x)(z-u)\,\mathrm{d}x+\int_{\Gamma_2}\xi(x)(z-u)\,\mathrm{d}\Gamma+\int_\Omega f(x,w,\nabla u)(z-u)\,\mathrm{d}x,
\end{aligned}$$

where we have applied the continuity of M, N and G. Because $z\in K(w)$ is arbitrary, we conclude that $u\in K(w)$ is the unique solution of problem (3.1) corresponding to $(w,\eta,\xi)\in V\times X^*\times Y^*$, namely, $u=\mathcal{S}(w,\eta,\xi)$.

Since every convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to the same limit $u = \mathcal{S}(w,\eta,\xi)$, this implies that the whole sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u. Thus,

$$\mathcal{S}(w_n,\eta_n,\xi_n) = u_n \to u = \mathcal{S}(w,\eta,\xi).$$

Therefore, we have proved that the solution map $\mathcal{S}\colon V\times X^*\times Y^*\to V$ of problem (3.1) is completely continuous. □

With view to hypotheses H(U_1) and H(U_2), it is now natural to introduce the following multivalued mappings $\mathcal{U}_1\colon X\to 2^{X^*}$ and $\mathcal{U}_2\colon Y\to 2^{Y^*}$ given by

$$\begin{aligned}
\mathcal{U}_1(u) &:= \left\{\eta\in X^* \,:\, \eta(x)\in U_1(x,u(x)) \text{ a. a. in } \Omega\right\},\\
\mathcal{U}_2(v) &:= \left\{\xi\in Y^* \,:\, \xi(x)\in U_2(x,v(x)) \text{ a. a. on } \Gamma_2\right\},
\end{aligned}$$

for all $(u,v)\in X\times Y$, respectively. As before, by $i\colon V\to X$ and $\gamma\colon V\to Y$, we denote the embedding operator of V to X and the trace operator from V to Y, respectively. It follows from Proposition 2.1 that the operators $i\colon V\to X$ and $\gamma\colon V\to Y$ are linear, bounded and compact. Therefore, we can see that their dual operators $i^*\colon X^*\to V^*$ and $\gamma^*\colon Y^*\to V^*$ are linear, bounded and compact as well. The following lemma is a direct consequence of Lemma 3.6 of Zeng-Rădulescu-Winkert [50].

Lemma 3.6. *Let* H(U_1) *and* H(U_2) *be satisfied. Then, the following statements hold:*

(i) *$\mathcal{U}_1$ and $\mathcal{U}_2$ are well-defined and for each $u\in X$ and $v\in Y$, the sets $\mathcal{U}_1(u)$ and $\mathcal{U}_2(v)$ are bounded, closed and convex in X^* and Y^*, respectively;*
(ii) *$\mathcal{U}_1$ and $\mathcal{U}_2$ are strongly-weakly u.s.c., i.e., $\mathcal{U}_1$ is u.s.c. from X with the strong topology to the subsets of X^* with the weak topology, and $\mathcal{U}_2$ is u.s.c. from Y with the strong topology to the subsets of Y^* with the weak topology.*

The following theorem states the main results of this section which indicates that the set of weak solutions to problem (1.1) is nonempty and compact in V.

Theorem 3.7. *Let $2\le p$. Assume that* H(1), H(2), H(M), H(f), H(N), H(G), H(U_1), H(U_2), H(ϕ), H(L) *and* H(J) *are satisfied. Then, the solution set of problem* (1.1)*, denoted by $\coprod$, is nonempty and compact in V.*

Proof. First we prove the following claims.

Claim 1: The solution set $\coprod$ of problem (1.1) is bounded, when $\coprod$ is nonempty.

Assume that $\coprod$ is nonempty and let $u\in\coprod$ be arbitrary. Then we can find functions $(\eta,\xi)\in X^*\times Y^*$ satisfying $\eta(x)\in U_1(x,u(x))$ for a. a. $x\in\Omega$ and $\xi(x)\in U_2(x,u(x))$ for a. a. $x\in\Gamma_2$ and the inequality holds

$$\begin{aligned}
&M(u)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x\\
&\quad + \int_\Omega (|u|^{p-2}u+\mu(x)|u|^{q-2}u)(v-u)\,\mathrm{d}x + \int_\Omega N(u)(x)(v-u)\,\mathrm{d}x\\
&\quad + \int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma + \int_{\Gamma_4} G(u)(x)(v-u)\,\mathrm{d}\Gamma\\
&\ge \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)(v-u)\,\mathrm{d}\Gamma + \int_\Omega f(x,u,\nabla u)(v-u)\,\mathrm{d}x
\end{aligned}$$

for all $v \in K(u)$. Recall that $0 \in K(u)$. So we can put $v = 0$ into the above inequality in order to get that

$$\begin{aligned} &M(u)\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu} + \int_\Omega N(u)u\,\mathrm{d}x + \int_{\Gamma_4} G(u)u\,\mathrm{d}\Gamma \\ &\quad + \int_{\Gamma_3} \phi(x,u)\,\mathrm{d}\Gamma - \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x \\ &\le \int_{\Gamma_3} \phi(x,0)\,\mathrm{d}\Gamma + \int_\Omega \eta(x)u\,\mathrm{d}x + \int_{\Gamma_2} \xi(x)u\,\mathrm{d}\Gamma. \end{aligned} \tag{3.12}$$

From hypotheses H(U_1)(iv) and H(U_2)(iv) it follows that

$$\begin{aligned} \int_\Omega \eta(x)u(x)\,\mathrm{d}x &\le \int_\Omega |\eta(x)||u(x)|\,\mathrm{d}x \\ &\le \int_\Omega \left(\alpha_{U_1}(x) + a_{U_1}|u(x)|^{p-1}\right)|u(x)|\,\mathrm{d}x \\ &\le a_{U_1}\|u\|_{p,\Omega}^p + \|\alpha_{U_1}\|_{p',\Omega}\|u\|_{p,\Omega} \\ &\le a_{U_1}c_p(\Omega)^p\|u\|_V^p + \|\alpha_{U_1}\|_{p',\Omega}c_p(\Omega)\|u\|_V, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \int_{\Gamma_2} \xi(x)u(x)\,\mathrm{d}\Gamma &\le \int_{\Gamma_2} |\xi(x)||u(x)|\,\mathrm{d}\Gamma \\ &\le \int_{\Gamma_2} \left(\alpha_{U_2}(x) + a_{U_2}|u(x)|^{p-1}\right)|u(x)|\,\mathrm{d}\Gamma \\ &\le a_{U_2}\|u\|_{p,\Gamma_2}^p + \|\alpha_{U_2}\|_{p',\Gamma_2}\|u\|_{p,\Gamma_2} \\ &\le a_{U_2}c_p(\Gamma_2)^p\|u\|_V^p + \|\alpha_{U_2}\|_{p',\Gamma_2}c_p(\Gamma_2)\|u\|_V. \end{aligned} \tag{3.14}$$

By hypotheses H(f)(i), H(N) and H(G), we have

$$\begin{aligned} \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x &\le \int_\Omega \left(a_f|\nabla u|^{p-1} + b_f|u|^{p-1} + \alpha_f(x)\right)|u|\,\mathrm{d}x \\ &\le a_f\|\nabla u\|_{p,\Omega}^{p-1}\|u\|_{p,\Omega} + b_f\|u\|_{p,\Omega}^p + \|\alpha_f\|_{p',\Omega}\|u\|_{p,\Omega}, \\ &\le a_f\hat{\lambda}^{\frac{1}{p}}\|\nabla u\|_{p,\Omega}^p + b_fc_p(\Omega)^p\|u\|_V^p + \|\alpha_f\|_{p',\Omega}c_p(\Omega)\|u\|_V, \end{aligned} \tag{3.15}$$

and

$$\int_\Omega N(u)(x)u\,\mathrm{d}x \ge -\|N(u)\|_{\zeta_1',\Omega}\|u\|_{\zeta_1,\Omega} \ge -(a_N + b_N\|u\|_{\zeta_1,\Omega}^{\kappa_1})\|u\|_{\zeta_1,\Omega}, \tag{3.16}$$

and

$$\int_{\Gamma_4} G(u)(x)u\,\mathrm{d}\Gamma \ge -\|G(u)\|_{\zeta_2',\Gamma_4}\|u\|_{\zeta_2,\Gamma_4} \ge -(a_G + b_G\|u\|_{\zeta_2,\Gamma_4}^{\kappa_2})\|u\|_{\zeta_2,\Gamma_4}. \tag{3.17}$$

Taking into account (3.12), (3.13), (3.14), (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned} &\left(c_M - a_f\hat{\lambda}^{\frac{1}{p}}\right)\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu} - a_{U_1}c_p(\Omega)^p\|u\|_V^p \\ &\quad - a_{U_2}c_p(\Gamma_2)^p\|u\|_V^p - b_fc_p(\Omega)^p\|u\|_V^p \\ &\le (a_N + b_N\|u\|_{\zeta_1,\Omega}^{\kappa_1})\|u\|_{\zeta_1,\Omega} + (a_G + b_G\|u\|_{\zeta_2,\Gamma_4}^{\kappa_2})\|u\|_{\zeta_2,\Gamma_4} + \|\alpha_{U_1}\|_{p',\Omega}c_p(\Omega)\|u\|_V \\ &\quad + \|\alpha_{U_2}\|_{p',\Gamma_2}c_p(\Gamma_2)\|u\|_V + \|\alpha_f\|_{p',\Omega}c_p(\Omega)\|u\|_V + \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma + \alpha_\varphi\|u\|_V + \beta_\varphi. \end{aligned}$$

Therefore, if $\|u\|_V > 1$, then we have

$$\begin{aligned} &\left(\min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}}, 1\} - (a_{U_1} + b_f)c_p(\Omega)^p - a_{U_2}c_p(\Gamma_2)^p\right)\|u\|_V^p \\ &\le m_0\left(1 + \|u\|_V + \|u\|_V^{\kappa_1+1} + \|u\|_V^{\kappa_2+1}\right), \end{aligned} \tag{3.18}$$

with some $m_0 > 0$ which is independent of u, where we have used the continuity of embeddings of V to $L^{\zeta_1}(\Omega)$, of V to $L^p(\Omega)$, of V to $L^{\zeta_2}(\Gamma_4)$ and of V to $L^p(\Gamma_2)$. Using the inequalities

$$1 < \kappa_1 < p-1, \quad 1 < \kappa_2 < p-1,$$
$$\min\{c_M - a_f \hat{\lambda}^{\frac{1}{p}}, 1\} - (a_{U_1} + b_f) c_p(\Omega)^p - a_{U_2} c_p(\Gamma_2)^p > 0,$$

and (3.18), we conclude that the solution set $\coprod$ of problem (1.1) is bounded, when $\coprod$ is nonempty.

Claim 2: Let $C > 0$ and $\overline{B_V(0,C)} := \{u \in V : \|u\|_V \leq C\}$. Then we can find a positive constant $\mathcal{C}^* > 0$ satisfying

$$\mathcal{S}(\overline{B_V(0,\mathcal{C}^*)}, \ \mathcal{U}_1(i\overline{B_V(0,\mathcal{C}^*)}), \ \mathcal{U}_2(\gamma\overline{B_V(0,\mathcal{C}^*)})) \subset \overline{B_V(0,\mathcal{C}^*)}. \tag{3.19}$$

We prove it by contradiction. Suppose there is no such constant $\mathcal{C}^*$ to satisfy the inclusion (3.19). Therefore, for every $n > 1$, we are able to find elements $w_n, z_n, y_n \in \overline{B_V(0,n)}$ and $(\eta_n, \xi_n) \in X^* \times Y^*$ such that $\eta_n \in \mathcal{U}_1(iz_n)$, $\xi_n \in \mathcal{U}_2(\gamma y_n)$ and

$$u_n = \mathcal{S}(w_n, \eta_n, \xi_n) \quad \text{and} \quad \|u_n\|_V > n.$$

By the definition of u_n, we have

$$\begin{aligned}
&M(w_n)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(v-u_n)\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(v-u_n)\,\mathrm{d}x\\
&+\int_\Omega (|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n)(v-u_n)\,\mathrm{d}x + \int_\Omega N(w_n)(x)(v-u_n)\,\mathrm{d}x\\
&+\int_{\Gamma_3}\phi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_4} G(w_n)(x)(v-u_n)\,\mathrm{d}\Gamma\\
&\geq \int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x + \int_{\Gamma_2}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma + \int_\Omega f(x,w_n,\nabla u_n)(v-u_n)\,\mathrm{d}x
\end{aligned}$$

for all $v \in K(w_n)$. In the inequality above we take $v = 0$ to obtain

$$\begin{aligned}
&M(w_n)\|\nabla u_n\|_{p,\Omega}^p + \|\nabla u_n\|_{q,\mu}^q + \|u_n\|_{p,\Omega}^p + \|u_n\|_{q,\mu} + \int_\Omega N(w_n)u_n\,\mathrm{d}x\\
&+\int_{\Gamma_4} G(u_n)u_n\,\mathrm{d}\Gamma + \int_{\Gamma_3}\phi(x,u_n)\,\mathrm{d}\Gamma - \int_\Omega f(x,w_n,\nabla u_n)u_n\,\mathrm{d}x\\
&\leq \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma + \int_\Omega \eta_n(x)u_n\,\mathrm{d}x + \int_{\Gamma_2}\xi_n(x)u_n\,\mathrm{d}\Gamma.
\end{aligned} \tag{3.20}$$

It follows from hypotheses H(U_1)(iv) and H(U_2)(iv) that

$$\begin{aligned}
\int_\Omega \eta_n(x)u_n(x)\,\mathrm{d}x &\leq \int_\Omega |\eta_n(x)||u_n(x)|\,\mathrm{d}x\\
&\leq \int_\Omega \left(\alpha_{U_1}(x) + a_{U_1}|z_n(x)|^{p-1}\right)|u_n(x)|\,\mathrm{d}x\\
&\leq \|\alpha_{U_1}\|_{p',\Omega}\|u_n\|_{p,\Omega} + a_{U_1}\|z_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega}\\
&\leq c_p(\Omega)\|\alpha_{U_1}\|_{p',\Omega}\|u_n\|_V + a_{U_1}c_p(\Omega)^p\|z_n\|_V^{p-1}\|u_n\|_V,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\int_{\Gamma_2}\xi_n(x)u_n(x)\,\mathrm{d}x &\leq \int_{\Gamma_2}|\xi_n(x)||u_n(x)|\,\mathrm{d}x\\
&\leq \int_{\Gamma_2}\left(\alpha_{U_2}(x) + a_{U_2}|y_n(x)|^{p-1}\right)|u_n(x)|\,\mathrm{d}x\\
&\leq \|\alpha_{U_2}\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2} + a_{U_2}\|y_n\|_{p,\Gamma_2}^{p-1}\|u_n\|_{p,\Gamma_2}\\
&\leq c_p(\Gamma_2)\|\alpha_{U_2}\|_{p',\Gamma_2}\|u_n\|_V + a_{U_2}c_p(\Gamma_2)^p\|y_n\|_V^{p-1}\|u_n\|_V.
\end{aligned} \tag{3.22}$$

Moreover, hypotheses H(N) and H(G) imply that

$$\int_\Omega N(w_n)u_n\,\mathrm{d}x \le \|N(w_n)\|_{\zeta_1',\Omega}\|u_n\|_{\zeta_1,\Omega} \le (a_N + b_N\|w_n\|_{\zeta_1,\Omega}^{\kappa_1})\|u_n\|_{\zeta_1,\Omega}, \tag{3.23}$$

and

$$\int_{\Gamma_4} G(w_n)u_n\,\mathrm{d}\Gamma \le \|G(w_n)\|_{\zeta_2',\Gamma_4}\|u_n\|_{\zeta_2,\Gamma_4} \le (a_G + b_G\|w_n\|_{\zeta_2,\Gamma_4}^{\kappa_2})\|u_n\|_{\zeta_2,\Gamma_4}. \tag{3.24}$$

Finally, by hypothesis H(f)(i), we have

$$\begin{aligned}
&\int_\Omega f(x,w_n,\nabla u_n)u_n\,\mathrm{d}x\\
&\le \int_\Omega \left(a_f|\nabla u_n|^{p-1} + b_f|w_n|^{p-1} + \alpha_f(x)\right)|u_n|\,\mathrm{d}x\\
&\le a_f\|\nabla u_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} + b_f\|w_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} + \|\alpha_f\|_{p',\Omega}\|u_n\|_{p,\Omega}\\
&\le a_f\hat{\lambda}^{\frac{1}{p}}\|\nabla u_n\|_{p,\Omega}^{p} + b_f c_p(\Omega)^p\|w_n\|_V^{p-1}\|u_n\|_V + \|\alpha_f\|_{p',\Omega}c_p(\Omega)\|u_n\|_V.
\end{aligned} \tag{3.25}$$

Since $n>1$ and $\|y_n\|_V \le n < \|u_n\|_V$, we insert (3.21), (3.22), (3.23), (3.24), (3.25) into (3.20) to obtain

$$\begin{aligned}
&\left(\min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}},1\} - (a_{U_1}+b_f)c_p(\Omega)^p - a_{U_2}c_p(\Gamma_2)^p\right)\|u_n\|_V^p\\
&\le (a_N + b_N\|w_n\|_{\zeta_1,\Omega}^{\kappa_1})\|u_n\|_{\zeta_1,\Omega} + (a_G + b_G\|w_n\|_{\zeta_2,\Gamma_4}^{\kappa_2})\|u_n\|_{\zeta_2,\Gamma_4} + \|\alpha_{U_1}\|_{p',\Omega}c_p(\Omega)\|u_n\|_V\\
&\quad + \|\alpha_{U_2}\|_{p',\Omega}c_p(\Gamma_2)\|u_n\|_V + \|\alpha_f\|_{p',\Omega}c_p(\Omega)\|u_n\|_V + \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma + \alpha_\varphi\|u_n\|_V + \beta_\varphi,
\end{aligned}$$

where we have used inequality (3.5). Passing to the limit as $n\to\infty$ to the inequality above, one has

$$+\infty = \lim_{n\to\infty}\left(\min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}},1\} - (a_{U_1}+b_f)c_p(\Omega)^p - a_{U_2}c_p(\Gamma_2)^p\right)\|u_n\|_V^{p-\max\{\kappa_1,\kappa_2\}-1} \le 0,$$

a contradiction. Therefore, we conclude that there exists a positive constant $\mathcal{C}^*>0$ such that (3.19) holds. This proves Claim 2.

As mentioned before, the main tool in the proof of the existence of a solution to problem (1.1) is Tychonoff's fixed point theorem for multivalued operators, see Theorem 2.5. For this purpose, let us consider the multivalued mapping $\Lambda\colon V\times X^*\times Y^*\to 2^{V\times X^*\times Y^*}$ defined by

$$\Lambda(u,\eta,\xi) := (\mathcal{S}(u,\eta,\xi),\mathcal{U}_1(iu),\mathcal{U}_2(\gamma u)).$$

Observe that if (u,η,ξ) is a fixed point of Λ, then we have $u=\mathcal{S}(u,\eta,\xi)$ and $(\eta,\xi)\in\mathcal{U}_1(iu)\times\mathcal{U}_2(\gamma u)$. It is obvious from the definitions of $\mathcal{S}$, $\mathcal{U}_1$ and $\mathcal{U}_2$ that u is also a weak solution of problem (1.1). Therefore, we are going to examine the validity of the conditions of Theorem 2.5.

Invoking Lemmas 3.4 and 3.5, we can see that for each $(w,\eta,\xi)\in V\times X^*\times Y^*$, the set $\Lambda(w,\eta,\xi)$ is a nonempty, bounded, closed and convex subset of $V\times X^*\times Y^*$.

Employing hypotheses H(U_1)(iv) and H(U_2)(iv), it is not difficult to prove that $\mathcal{U}_1\colon X\to 2^{X^*}$ and $\mathcal{U}_2\colon Y\to 2^{Y^*}$ are two bounded operators, and there exist two constants $M_1>0$ and $M_2>0$ satisfying

$$\|\mathcal{U}_1(i\overline{B_V(0,\mathcal{C}^*)})\|_{X^*}\le M_1 \quad\text{and}\quad \|\mathcal{U}_2(\gamma\overline{B_V(0,\mathcal{C}^*)})\|_{Y^*}\le M_2.$$

Additionally, we introduce a bounded, closed and convex subset D of $V\times X^*\times Y^*$ defined by

$$D = \{(u,\eta,\xi)\in V\times X^*\times Y^* \,:\, \|u\|_V\le\mathcal{C}^*,\ \|\eta\|_{X^*}\le M_1 \text{ and } \|\xi\|_{Y^*}\le M_2\}.$$

From this and (3.19) we know that Λ maps D into itself.

Next, we are going to prove that the multivalued mapping Λ is weakly-weakly u.s.c. For any weakly closed set E in $V\times X^*\times Y^*$ such that $\Lambda^-(E)\neq\emptyset$, let $\{(w_n,\eta_n,\xi_n)\}_{n\in\mathbb{N}}\subset\Lambda^-(E)$ be such that $(w_n,\eta_n,\xi_n)\xrightarrow{w}(w,\eta,\xi)$ in $V\times X^*\times Y^*$ for some $(w,\eta,\xi)\in V\times X^*\times Y^*$. Our goal is to show that $(w,\eta,\xi) \in \Lambda^{-}(E)$, namely, there exists $(u,\delta,\sigma) \in \Lambda(w,\eta,\xi) \cap E$. Indeed, for each $n \in \mathbb{N}$, we are able to find $(u_n,\delta_n,\sigma_n) \in \Lambda(w_n,\eta_n,\xi_n) \cap E$, so, $u_n = \mathcal{S}(w_n,\eta_n,\xi_n)$, $\delta_n \in \mathcal{U}_1(iw_n)$ and $\sigma_n \in \mathcal{U}_2(\gamma w_n)$. From the boundedness of $\mathcal{U}_1$ and $\mathcal{U}_2$, one has that the sequences $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ are bounded in X^* and Y^*, respectively. Passing to a subsequence if necessary, we may assume that

$$\delta_n \xrightarrow{w} \delta \quad \text{in } X^* \quad \text{and} \quad \sigma_n \xrightarrow{w} \sigma \quad \text{in } Y^*$$

for some $(\delta,\sigma) \in X^* \times Y^*$. Recall that $\mathcal{S}$ is completely continuous. So, it holds $u_n = \mathcal{S}(w_n,\eta_n,\xi_n) \to \mathcal{S}(w,\eta,\xi) := u$ in V. Note that i and γ are both compact. Hence $iw_n \to iw$ in X and $\gamma w_n \to \gamma w$ in Y. Since $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is strongly-weakly u.s.c. and has nonempty, bounded, closed and convex values, it follows from Theorem 1.1.4 of Kamenskii-Obukhovskii-Zecca [25] that $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is strongly-weakly closed. The latter combined with the convergences above implies that $\delta \in \mathcal{U}_1(iw)$ and $\sigma \in \mathcal{U}_2(\gamma w)$, namely, $(u,\delta,\sigma) \in \Lambda(w,\eta,\xi) \cap E$, because of the weak closedness of E. Therefore, we conclude that Λ is weakly-weakly u.s.c.

Therefore, all conditions of Theorem 2.5 are satisfied. Using this theorem, we conclude that Λ has at least a fixed point, say $(u^*,\eta^*,\xi^*) \in V \times X^* \times Y^*$. Hence, $u^* \in V$ is a weak solution of problem (1.1).

Next, let us prove the compactness of the solution set $\coprod$. From Claim 1, we can see that the solution set $\coprod$ of problem (1.1) is bounded in V. By the definitions of a weak solution (see Definition 3.3) and of Λ, there exist $(\eta,\xi) \in X^* \times Y^*$ such that $u = \mathcal{S}(u,\eta,\xi)$, $\eta \in \mathcal{U}_1(iu)$ and $\xi \in \mathcal{U}_2(\gamma u)$, that is, $(u,\eta,\xi) \in \Lambda(u,\eta,\xi)$. Let $\{u_n\}_{n\in\mathbb{N}}$ be any sequence of solutions to problem (1.1). Then, there are two sequences $\{\eta_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset Y^*$ such that $\eta_n \in \mathcal{U}_1(iu_n)$, $\xi_n \in \mathcal{U}_2(\gamma u_n)$ such that $u_n = \mathcal{S}(u_n,\eta_n,\xi_n)$ for all $n \in \mathbb{N}$. From the boundedness of $\coprod$ we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } V$$

for some $u \in V$. This together with the boundedness of $\mathcal{U}_1$ and $\mathcal{U}_2$ deduces that $\{\eta_n\}_{n\in\mathbb{N}} \subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}} \subset Y^*$ are both bounded. So, passing to a subsequence if necessary, we suppose that

$$\eta_n \xrightarrow{w} \eta \quad \text{in } X^* \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } Y^*$$

for some $\eta \in \mathcal{U}_1(iu)$ and $\xi \in \mathcal{U}_2(\gamma u)$, owing to the compactness of i and γ as well as the strongly-weakly closedness of $\mathcal{U}_1$ and $\mathcal{U}_2$. Using the complete continuity of $\mathcal{S}$, we conclude that

$$u_n = \mathcal{S}(u_n,\eta_n,\xi_n) \to \mathcal{S}(u,\eta,\xi) = u.$$

This means that u is a solution to problem (1.1). Consequently, the solution set $\coprod$ of problem (1.1) is compact. $\square$

4. Special cases of the original problem

In this section, we are going to study several special cases of problem (1.1) and discuss some particular situations.

First, we move our attention to consider the special case of problem (1.1) formed as follows:

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u \in r_1(u)\partial j_1(x,u) + N(u)(x) + f(x,u,\nabla u) \quad & \text{in } \Omega,\\
u = 0 \quad & \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} \in r_2(u)\partial j_2(x,u) \quad & \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} \in \partial_c \phi(x,u) \quad & \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} = G(u)(x) \quad & \text{on } \Gamma_4,\\
L(u) \le J(u),
\end{aligned} \tag{4.1}$$

where the terms ∂j_1 and ∂j_2 stand for the Clarke's generalized gradients of locally Lipschitz functions $s \mapsto j_1(x,s)$ and $s \mapsto j_2(x,s)$, respectively. Here, the functions $j_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 \colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ are supposed to satisfy the following properties:

H(j_1): The functions $j_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $r_1 \colon \mathbb{R} \to \mathbb{R}$ are such that
- (i) $x \mapsto j_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$ with $x \mapsto j_1(x,0)$ belonging to $L^1(\Omega)$;
- (ii) $s \mapsto j_1(x,s)$ is locally Lipschitz continuous for a. a. $x \in \Omega$ and the function $r_1 \colon \mathbb{R} \to \mathbb{R}$ is continuous;
- (iii) there exist a function $\alpha_{j_1} \in L^{p'}(\Omega)_+$ and a constant $a_{j_1} \geq 0$ such that
$$|r_1(s)\eta| \leq \alpha_{j_1}(x) + a_{j_1}|s|^{p-1}$$
for all $\eta \in \partial j_1(x,s)$, for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

H(j_2): The functions $j_2 \colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ and $r_2 \colon \mathbb{R} \to \mathbb{R}$ are such that
- (i) $x \mapsto j_2(x,s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$ with $x \mapsto j_2(x,0)$ belonging to $L^1(\Gamma_2)$;
- (ii) $s \mapsto j_2(x,s)$ is locally Lipschitz continuous for a. a. $x \in \Gamma_2$ and the function $r_2 \colon \mathbb{R} \to \mathbb{R}$ is continuous;
- (iii) there exist a function $\alpha_{j_2} \in L^{p'}(\Gamma_2)_+$ and a constant $a_{j_2} \geq 0$ such that
$$|r_2(s)\xi| \leq \alpha_{j_2}(x) + a_{j_2}|s|^{p-1}$$
for all $\xi \in \partial j_2(x,s)$, for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$.

Using the same arguments as in the proof of Theorem 3.11 of Zeng-Rădulescu-Winkert [50], we have the following lemma.

Lemma 4.1. *Assume that* H(j_1) *and* H(j_2) *are fulfilled. Then, the multivalued mappings* $U_1 \colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ *and* $U_2 \colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ *defined by*
$$U_1(x,s) := r_1(s)\partial j_1(x,s) \quad \textit{and} \quad U_2(y,s) := r_2(s)\partial j_2(y,s)$$
for all $s \in \mathbb{R}$, *for a. a.* $x \in \Omega$ *and for a. a.* $y \in \Gamma_2$, *satisfy* H(U_1) *and* H(U_2), *respectively.*

By Theorem 3.7 and Lemma 4.1, we have the following existence theorem to problem (4.1).

Theorem 4.2. *Let* $p \geq 2$. *Assume that* H(1), H(M), H(f), H(N), H(G), H(j_1), H(j_2), H(ϕ), H(L), H(J) *and the inequalities*
$$\begin{aligned}
&0 < k(p)c_M - e_f\hat{\lambda}^{\frac{1}{p}},\\
&0 < \min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}}, 1\} - (a_{j_1} + b_f)\, c_p(\Omega)^p - a_{j_2}c_p(\Gamma_2)^p
\end{aligned}$$
are satisfied. Then, the solution set of problem (4.1) *is nonempty and compact in* V.

When f is independent of the third variable (i.e., f is formulated by $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$), then problem (1.1) becomes to the following problem:
$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + N(u)(x) + f(x,u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c\phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
L(u) &\leq J(u).
\end{aligned} \tag{4.2}$$

A careful reading of the proofs in Section 3 gives the following results to problem (4.2).

Theorem 4.3. *Let* $p \geq 2$. *Assume that* H(1), H(M), H(N), H(G), H(U_1), H(U_2), H(ϕ), H(L) *and* H(J) *are satisfied. If, in addition,* f *satisfies the following conditions*

H(f'): *$f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that there exist a constant $b_f\geq 0$ and a function $\alpha_f\in L^{\frac{p}{p-1}}(\Omega)_+$ satisfying*

$$|f(x,s)|\leq b_f|s|^{p-1}+\alpha_f(x)$$

for a. a. $x\in\Omega$, for all $s\in\mathbb{R}$

and the following inequality is satisfied

$$0<\min\{c_M,1\}-(a_{U_1}+b_f)\,c_p(\Omega)^p-a_{U_2}c_p(\Gamma_2)^p,$$

then the solution set of problem (4.2) is nonempty and compact in V.

Therefore, from Theorems 4.2 and 4.3, we can directly obtain the existence of a weak solution to the following implicit obstacle inclusion problem:

$$\begin{aligned}
-D_Mu+|u|^{p-2}u+\mu(x)|u|^{q-2}u&\in r_1(u)\partial j_1(x,u)+N(u)(x)+f(x,u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial\nu_a}&\in r_2(u)\partial j_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial\nu_a}&\in\partial_c\phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial\nu_a}&=G(u)(x) && \text{on } \Gamma_4,\\
L(u)&\leq J(u).
\end{aligned}\tag{4.3}$$

Theorem 4.4. *Let $p\geq 2$. Assume that* H(1), H(M), H(N), H(G), H(j_1), H(j_2), H(ϕ), H(L) *and* H(J) *are satisfied. If, in addition,* H(f') *and the following inequality are satisfied*

$$0<\min\{c_M,1\}-(a_{j_1}+b_f)\,c_p(\Omega)^p-a_{j_2}c_p(\Gamma_2)^p,$$

then the solution set of problem (4.3) is nonempty and compact in V.

Particularly, if $\Gamma_2=\Gamma_3=\Gamma_4=\emptyset$ (namely, $\Gamma_1=\Gamma$), then problem (1.1) reduces to the following nonlocal implicit obstacle problem with Dirichlet boundary condition:

$$\begin{aligned}
-D_Mu+|u|^{p-2}u+\mu(x)|u|^{q-2}u&\in U_1(x,u)+N(u)(x)+f(x,u,\nabla u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma,\\
L(u)&\leq J(u).
\end{aligned}\tag{4.4}$$

Obviously, the function space considered in problem (4.4) is the closed subspace

$$W_0^{1,\mathcal{H}}(\Omega):=\{u\in W^{1,\mathcal{H}}(\Omega)\,:\,u=0\text{ on }\Gamma\}$$

of $W^{1,\mathcal{H}}(\Omega)$. It is well-known that $V_0:=W_0^{1,\mathcal{H}}(\Omega)$ endowed the norm $\|u\|_{V_0}:=\||\nabla u|\|_{\mathcal{H}}$ for all $u\in V_0$ becomes a reflexive Banach space. Therefore, we have the following existence theorem to problem (4.4).

Theorem 4.5. *Let $p\geq 2$. Assume that* H(1), H(M), H(f), H(N), H(U_1), H(L), H(J) *and the following inequalities*

$$\begin{aligned}
&0<k(p)c_M-e_f\hat{\lambda}^{\frac{1}{p}},\\
&0<\min\{c_M-a_f\hat{\lambda}^{\frac{1}{p}},1\}-(a_{U_1}+b_f)\,c_p(\Omega)^p
\end{aligned}$$

are satisfied. Then, the solution set of problem (4.4), denoted by $\coprod$, is nonempty and compact in V_0.

More particularly, if f is independent of the third variable and U_1 is specialized by the formulation $U_1(x,s) = r_1(s)\partial j_1(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$, then problem (4.4) reduces to the following implicit obstacle problems, respectively:

$$\begin{aligned} -D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + N(u)(x) + f(x,u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma,\\ L(u) &\le J(u), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} -D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in r_1(u)\partial j_1(x,u) + N(u)(x) + f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma,\\ L(u) &\le J(u). \end{aligned} \tag{4.6}$$

Therefore, we have the following existence theorems to problems (4.5) and (4.6), respectively.

Theorem 4.6. *Let $p \ge 2$. Assume that* H(1), H(M), H(f'), H(N), H(U_1), H(L), H(J) *and the following inequality*

$$0 < \min\{c_M, 1\} - (a_{U_1} + b_f)\, c_p(\Omega)^p$$

is satisfied. Then, the solution set of problem (4.5), denoted by $\coprod$, is nonempty and compact in V_0.

Theorem 4.7. *Let $p \ge 2$. Assume that* H(1), H(M), H(f), H(N), H(j_1), H(L), H(J) *and the following inequalities*

$$\begin{aligned} &0 < k(p)c_M - e_f\hat{\lambda}^{\frac{1}{p}},\\ &0 < \min\{c_M - a_f\hat{\lambda}^{\frac{1}{p}}, 1\} - (a_{j_1} + b_f)\, c_p(\Omega)^p \end{aligned}$$

are satisfied. Then, the solution set of problem (4.6), denoted by $\coprod$, is nonempty and compact in V_0.

Let $c_J \ge 0$ be a given constant. When $J(u) = c_J$ for all $u \in V$, then problem (1.1) can be rewritten as the following nonlocal elliptic system:

$$\begin{aligned} -D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + N(u)(x) + f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_1,\\ \frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\ -\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\ -\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\ L(u) &\le c_J, \end{aligned} \tag{4.7}$$

With respect to problem (4.7), the constraint set is denoted by the following one

$$K := \{u \in V \,:\, L(u) \le c_J\}.$$

Observe that the following condition

H(L'): $L\colon V \to \mathbb{R}$ is a l.s.c. and convex function,

is weaker than hypothesis H(L). Without loss of generality, in the sequel, we suppose that $L(0) \le c_J$. Therefore, it is not difficult to prove that if H(L') holds, then the constraint set K is a nonempty, closed and convex subset of V with $0 \in K$.

In Theorem 3.7, the inequalities given in H(2) play critical role to prove the existence of weak solutions to problem (1.1). But, in some sense, such inequalities restrict the scope of applications to our theoretical results. A natural question arises whether we can drop hypotheses H(2).

However, this is still an open problem for the equations with the implicit obstacle effect (for example, problem (1.1)). But, fortunately, if the obstacle constraint is formulated by the form $L(u) \leq c_J$ and M is a coercive in V, i.e., $M(u) \to +\infty$ as $\|u\|_V \to \infty$, then hypotheses H(2) can be removed. More precisely, if the obstacle constraint is formulated by $L(u) \leq c_J$, then hypothesis H(M) can be relaxed to the following condition:

H(M'): $M\colon L^{p^*}(\Omega) \to (0,+\infty)$ is bounded and continuous in V such that $\inf_{u\in V} M(u) > 0$.

Theorem 4.8. *Assume that* H(1), H(f)(i), H(N), H(G), H(U_1), H(U_2), H(ϕ), H(M') *and* H(L') *are satisfied. If, moreover, $M\colon L^{p^*}(\Omega) \to (0,+\infty)$ is coercive in V, then the solution set of problem* (4.7)*, denoted by $\coprod$, is nonempty and compact in V.*

Proof. Let $\mathcal{A}\colon V\times V \to V^*$, $\mathcal{F}\colon V \to L^{p'}(\Omega) \subset V^*$ and $\mathcal{G}\colon V \to V^*$ be the functions defined by

$$\begin{aligned}
\langle \mathcal{A}(u,u), v\rangle &:= M(u)\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla v\,\mathrm{d}x\\
&\qquad + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v\,\mathrm{d}x\\
\langle \mathcal{F}u, v\rangle &:= \int_\Omega f(x,u,\nabla u)v\,\mathrm{d}x,\\
\langle \mathcal{G}(u), v\rangle &:= \int_\Omega N(u)(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_4} G(u)(x)v\,\mathrm{d}\Gamma,
\end{aligned}$$

for all $u,v\in V$. Applying a standard procedure, it is easily to show that $u\in V$ is a weak solution to problem (4.7) if and only if it solves the following inclusion problem:

$$\mathcal{A}(u,u) + \mathcal{G}(u) + \mathcal{F}(u) + i^*\mathcal{U}_1(iu) + \gamma^*\mathcal{U}_2(\gamma u) + \partial_c\varphi_K(u) \ni 0 \quad \text{in } V^*,$$

where $\partial_c\varphi_K$ is the convex differential operator of $\varphi_K := \varphi + I_K$ and I_K is the indicator function of K.

We assert that the multivalued mapping $V \ni u \mapsto \mathcal{A}(u,u)+\mathcal{G}(u)+\mathcal{F}(u)+i^*\mathcal{U}_1(u)+\gamma^*\mathcal{U}_2(u)+\partial_c\varphi_K(u) \subset V^*$ is coercive. Let $u\in K$, $\eta\in\mathcal{U}_1(iu)$ and $\xi\in\mathcal{U}_2(\gamma u)$ be arbitrary. A simple calculating gives

$$\begin{aligned}
&\int_\Omega M(u)|\nabla u|^p + \mu(x)|\nabla u|^q + |u|^p + \mu(x)|u|^q\,\mathrm{d}x + \int_\Omega N(u)(x)u\,\mathrm{d}x + \int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma\\
&\quad - \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma + \int_{\Gamma_4} G(u)(x)u\,\mathrm{d}\Gamma + \int_\Omega \eta(x)u\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)u\,\mathrm{d}\Gamma + \int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x\\
&\geq M(u)\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu}^q - a_f\hat\lambda^{\frac1p}\|\nabla u\|_{p,\Omega}^p - b_f\|u\|_{p,\Omega}^p\\
&\quad - \|\alpha_f\|_{p',\Omega}\|u\|_{p,\Omega} - \left(a_N + b_N\|u\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u\|_{\zeta_1,\Omega} - \left(a_G + b_G\|u\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u\|_{\zeta_2,\Gamma_4} - \alpha_\varphi\|v\|_V\\
&\quad - \beta_\varphi - \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma - \|\alpha_{U_1}\|_{p',\Omega}\|u\|_{p,\Omega} - a_{U_1}\|u\|_{p,\Omega}^p - \|\alpha_{U_2}\|_{p',\Gamma_2}\|u\|_{p,\Omega} - a_{U_2}\|u\|_{p,\Gamma_2}^p\\
&\geq \left(M(u) - a_f\hat\lambda^{\frac1p} - b_f\hat\lambda - a_{U_1}\hat\lambda - a_{U_2}\lambda_{1,p}^S(1+\hat\lambda)\right)\|\nabla u\|_{p,\Omega}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p,\Omega}^p + \|u\|_{q,\mu}^q\\
&\quad - \|\alpha_f\|_{p',\Omega}\|u\|_{p,\Omega} - \left(a_N + b_N\|u\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u\|_{\zeta_1,\Omega} - \left(a_G + b_G\|u\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u\|_{\zeta_2,\Gamma_4} - \alpha_\varphi\|v\|_V\\
&\quad - \beta_\varphi - \int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma,
\end{aligned}$$

where we have used the variational identity (2.5). Hence, if $\|u\|_V > 1$ is such that

$$M(u) - a_f\hat\lambda^{\frac1p} - b_f\hat\lambda - a_{U_1}\hat\lambda - a_{U_2}\lambda_{1,p}^S(1+\hat\lambda) > 1,$$

then we have

$$\int_\Omega M(u)|\nabla u|^p + \mu(x)|\nabla u|^q + |u|^p + \mu(x)|u|^q\,\mathrm{d}x + \int_\Omega N(u)(x)u\,\mathrm{d}x + \int_{\Gamma_3}\phi(x,u)\,\mathrm{d}\Gamma$$

$$
\begin{aligned}
&-\int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma+\int_{\Gamma_4}G(u)(x)u\,\mathrm{d}\Gamma+\int_\Omega \eta(x)u\,\mathrm{d}x+\int_{\Gamma_2}\xi(x)u\,\mathrm{d}\Gamma+\int_\Omega f(x,u,\nabla u)u\,\mathrm{d}x\\
\geq{}& \|u\|_V^p-\|\alpha_f\|_{p',\Omega}\|u\|_{p,\Omega}-\left(a_N+b_N\|u\|_{\zeta_1,\Omega}^{\kappa_1}\right)\|u\|_{\zeta_1,\Omega}\\
&-\left(a_G+b_G\|u\|_{\zeta_2,\Gamma_4}^{\kappa_2}\right)\|u\|_{\zeta_2,\Gamma_4}-\alpha_\varphi\|v\|_V-\beta_\varphi-\int_{\Gamma_3}\phi(x,0)\,\mathrm{d}\Gamma.
\end{aligned}
$$

Recall that $\kappa_1+1<p$ and $\kappa_2+1<p$. Therefore, we have

$$
\frac{\langle \mathcal{A}(u,u)+\mathcal{G}(u)+\mathcal{F}(u)+i^*\mathcal{U}_1(u)+\gamma^*\mathcal{U}_2(u)+\partial_c\varphi_K(u),u\rangle}{\|u\|_V}\to\infty \quad \text{as } \|u\|_V\to\infty.
$$

This means that the multivalued mapping $V\ni u\mapsto \mathcal{A}(u,u)+\mathcal{G}(u)+\mathcal{F}(u)+i^*\mathcal{U}_1(u)+\gamma^*\mathcal{U}_2(u)+\partial_c\varphi_K(u)\subset V^*$ is coercive.

From the proof of Theorem 3.4 of Zeng-Bai-Gasiński [47] and Theorem 3.7, we can see that the weak continuity of M plays an important role to prove the pseudomonotonicity of $V\ni u\mapsto \mathcal{A}(u,u)+\mathcal{G}(u)+\mathcal{F}(u)+i^*\mathcal{U}_1(iu)+\gamma^*\mathcal{U}_2(\gamma u)\subset V^*$. More exactly, it directly effects the validity of the condition that

- if $\{u_n\}_{n\in\mathbb{N}}\subset V$ with $u_n \xrightarrow{w} u$ in V and $u_n^*\in \mathcal{A}(u_n,u_n)+\mathcal{G}(u_n)+\mathcal{F}(u_n)+i^*\mathcal{U}_1(iu_n)+\gamma^*\mathcal{U}_2(\gamma u_n)$ are such that
$$
\limsup_{n\to\infty}\langle u_n^*,u_n-u\rangle\leq 0, \tag{4.8}
$$
then to each element $v\in V$, there exists $u^*(v)\in \mathcal{A}(u,u)+\mathcal{G}(u)+\mathcal{F}(u)+i^*\mathcal{U}_1(iu)+\gamma^*\mathcal{U}_2(\gamma u)$ with
$$
\langle u^*(v),u-v\rangle\leq \liminf_{n\to\infty}\langle u_n^*,u_n-v\rangle. \tag{4.9}
$$

Let $\{u_n\}_{n\in\mathbb{N}}\subset V$ and $\{u_n^*\}_{n\in\mathbb{N}}\subset V^*$ be sequences such that $u_n^*\in\mathcal{A}(u_n,u_n)+\mathcal{G}(u_n)+\mathcal{F}(u_n)+i^*\mathcal{U}_1(iu_n)+\gamma^*\mathcal{U}_2(\gamma u_n)$ and suppose inequality (4.8) holds. Then, there exist sequences $\{\eta_n\}_{n\in\mathbb{N}}\subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}}\subset Y^*$ satisfying $\eta_n\in\mathcal{U}_1(iu_n)$, $\xi_n\in\mathcal{U}_2(\gamma u_n)$ and

$$
u_n^*=\mathcal{A}(u_n,u_n)+\mathcal{G}(u_n)+\mathcal{F}(u_n)+i^*\eta_n+\gamma^*\xi_n \quad \text{for all } n\in\mathbb{N}.
$$

Using hypotheses H(U_1) and H(U_2), we can observe that the sequences $\{\eta_n\}_{n\in\mathbb{N}}\subset X^*$ and $\{\xi_n\}_{n\in\mathbb{N}}\subset Y^*$ are both bounded. Passing to a subsequence if necessary, we may assume that

$$
\eta_n \xrightarrow{w} \eta \quad \text{in } X^* \quad \text{and} \quad \xi_n \xrightarrow{w} \xi \quad \text{in } Y^* \tag{4.10}
$$

for some $(\eta,\xi)\in X^*\times Y^*$. Besides, hypothesis H($f$)(i) reveals that the sequence $\{\mathcal{F}(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\Omega)$. Then, we use the compactness of i and γ as well as of the embedding from V into $L^p(\Omega)$ to obtain

$$
\begin{aligned}
0&\geq \limsup_{n\to\infty}\langle u_n^*,u_n-u\rangle\\
&\geq \limsup_{n\to\infty}\langle \mathcal{A}(u_n,u_n),u_n-u\rangle+\liminf_{n\to\infty}\langle\mathcal{G}(u_n),u_n-u\rangle+\liminf_{n\to\infty}\langle\mathcal{F}(u_n),u_n-u\rangle\\
&\quad -\limsup_{n\to\infty}\langle\eta_n,u_n-u\rangle_{L^{p'}(\Omega)\times L^p(\Omega)}-\limsup_{n\to\infty}\langle\xi_n,u_n-u\rangle_{L^{p'}(\Gamma_2)\times L^p(\Gamma_2)}\\
&\geq \limsup_{n\to\infty}\langle\mathcal{A}(u_n,u_n),u_n-u\rangle.
\end{aligned}
$$

Let $c_M:=\inf_{u\in V}M(u)>0$, and $0<\varepsilon<c_M$ arbitrary. Recall that $u_n \xrightarrow{w} u$ in V and M is bounded in V, we have

$$
\begin{aligned}
0&\geq \limsup_{n\to\infty}\langle\mathcal{A}(u_n,u_n),u_n-u\rangle\\
&=\limsup_{n\to\infty}\int_\Omega \left(M(u_n)|\nabla u_n|^{p-2}\nabla u_n+\mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\\
&\quad +\left(|u_n|^{p-2}u_n+\mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\geq \liminf_{n\to\infty} \left(M(u_n)-\varepsilon\right) \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\left(u_n-u\right)\,\mathrm{d}x\\
&\quad + \limsup_{n\to\infty} \int_\Omega \left(\varepsilon|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\\
&\quad + \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x\\
&\geq \liminf_{n\to\infty} \left(M(u_n)-\varepsilon\right) \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(u_n-u\right)\,\mathrm{d}x\\
&\quad + \limsup_{n\to\infty} \int_\Omega \left(\varepsilon|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\\
&\quad + \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x\\
&\geq \limsup_{n\to\infty} \int_\Omega \left(\varepsilon|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right)\cdot\nabla(u_n-u)\\
&\quad + \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(u_n-u)\,\mathrm{d}x.
\end{aligned}
$$

Let us define the function $\mathscr{A}\colon V\to V^*$

$$
\begin{aligned}
\langle \mathscr{A}w, v\rangle := \int_\Omega &\left(\varepsilon|\nabla w|^{p-2}\nabla w + \mu(x)|\nabla w|^{q-2}\nabla w\right)\cdot\nabla v\\
&+ \left(|w|^{p-2}w + \mu(x)|w|^{q-2}w\right)v\,\mathrm{d}x,
\end{aligned}
$$

it is of type (S_+) (see Proposition 2.3). This implies that $u_n \to u$ in V.

Recall that $\mathcal{U}_1$ and $\mathcal{U}_2$ are strongly-weakly closed. Therefore, from (4.10) it follows that $\eta \in \mathcal{U}_1(iu)$ and $\xi \in \mathcal{U}_2(\gamma u)$. For any $v\in V$, we have

$$
\lim_{n\to\infty} \langle u_n^*, u_n - v\rangle = \langle \mathcal{A}(u,u) + \mathcal{G}(u) + \mathcal{F}(u) - i^*\eta - \gamma^*\xi, u-v\rangle.
$$

The latter combined with the fact that $\eta \in \mathcal{U}_1(iu)$ and $\xi\in\mathcal{U}_2(\gamma u)$ implies that $u^* \in \mathcal{A}(u,u) + \mathcal{G}(u) + \mathcal{F}(u) + i^*\mathcal{U}_1(iu) + \gamma^*\mathcal{U}_2(\gamma u)$. Therefore, we conclude that (4.9) holds.

Using the same arguments as in the proof of Theorem 3.7 and Theorem 3.4 of Zeng-Bai-Gasiński [47], it is not difficult to prove that the solution set of problem (4.7) is nonempty and compact in V. □

Remark 4.9. *In fact, there are a several of functions which satisfy the hypotheses H(M′) such that M is coercive in V. For example, the following functions are coercive in V and fulfill hypothesis H(M′)*

$$
M(u) = c_a + \|u\|_V, \quad M(u) = c_a + \ln(1+\|u\|_V), \quad M(u) = c_a + \|u\|_V^{\|u\|_V}, \quad \text{and} \quad M(u) = e^{\|u\|_V}
$$

for all $u\in V$ with $c_a>0$.

Let $\mathcal{D}\subset\overline{\Omega}$ be a nonempty set with positive measure and $\Psi\colon\mathcal{D}\to\mathbb{R}$ be a given obstacle function. Furthermore, when $J(u)\equiv 0$ (i.e., $c_J = 0$) and L is formulated by

$$
L(u) = \int_{\mathcal{D}} (u(x) - \Psi(x))^+\,\mathrm{d}x \quad \text{for all } u\in V,
$$

then problem (4.7) can be written by the following obstacle problem:

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + N(u)(x) + f(x,u,\nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
u(x) &\le \Psi(x) && \text{in } \mathcal{D}.
\end{aligned} \tag{4.11}$$

Therefore, we have the following corollary.

Corollary 4.10. *Assume that* H(1), H(f)(i), H(N), H(G), H(U_1), H(U_2), H(M') *and* H(ϕ) *are satisfied. If, moreover, M is coercive in V, and $\Phi\colon \Omega \to \mathbb{R}$ is a measurable function, then the solution set of problem (4.11), denoted by $\coprod$, is nonempty and compact in V.*

Under the analysis above, we have the following theorems and corollaries.

Theorem 4.11. *Assume that* H(1), H(f)(i), H(N), H(G), H(j_1), H(j_2), H(ϕ), H(M') *and* H(L') *are satisfied. If, moreover, M is coercive in V, then the solution set of the following nonlocal obstacle problem*

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in r_1(u)\partial j_1(x,u) + N(u)(x) + f(x,u,\nabla u) && \textit{in } \Omega,\\
u &= 0 && \textit{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in r_2(u)\partial j_2(x,u) && \textit{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \textit{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \textit{on } \Gamma_4,\\
L(u) &\le c_J,
\end{aligned}$$

is nonempty and compact in V.

Corollary 4.12. *Assume that* H(1), H(f)(i), H(N), H(G), H(j_1), H(j_2), H(M') *and* H(ϕ) *are satisfied. If, moreover, M is coercive in V, and $\Phi\colon \Omega \to \mathbb{R}$ is a measurable function, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in r_1(u)\partial j_1(x,u) + N(u)(x) + f(x,u,\nabla u) && \textit{in } \Omega,\\
u &= 0 && \textit{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in r_2(u)\partial j_2(x,u) && \textit{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \textit{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \textit{on } \Gamma_4,\\
u(x) &\le \Psi(x) && \textit{in } \mathcal{D},
\end{aligned}$$

is nonempty and compact in V.

Theorem 4.13. *Assume that* H(1), H(f'), H(N), H(G), H(U_1), H(U_2), H(ϕ), H(M') *and* H(L') *are satisfied. If, moreover, M is coercive in V, then the solution set of the following* *obstacle problem*

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + N(u)(x) + f(x,u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
L(u) &\le c_J,
\end{aligned}$$

is nonempty and compact in V*.*

Corollary 4.14. *Assume that* H(1), H(f'), H(N), H(G), H(U_1), H(U_2), H(M') *and* H(ϕ) *are satisfied. If, moreover,* M *is coercive in* V*, and* $\Phi\colon \Omega \to \mathbb{R}$ *is a measurable function, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in U_1(x,u) + N(u)(x) + f(x,u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
u(x) &\le \Psi(x) && \text{in } \mathcal{D},
\end{aligned}$$

is nonempty and compact in V*.*

Theorem 4.15. *Assume that* H(1), H(f'), H(N), H(G), H(j_1), H(j_2), H(ϕ), H(M') *and* H(L') *are satisfied. If, moreover,* M *is coercive in* V*, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u &\in r_1(u)\partial j_1(x,u) + N(u)(x) + f(x,u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in r_2(u)\partial j_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
L(u) &\le c_J,
\end{aligned}$$

is nonempty and compact in V*.*

Corollary 4.16. *Assume that* H(1), H(f'), H(N), H(G), H(j_1), H(j_2), H(M') *and* H(ϕ) *are satisfied. If, moreover,* M *is coercive in* V*, and* $\Phi\colon \Omega \to \mathbb{R}$ *is a measurable function, then the* *solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in r_1(u)\partial j_1(x,u)+N(u)(x)+f(x,u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in r_2(u)\partial j_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c\phi(x,u) && \text{on } \Gamma_3,\\
-\frac{\partial u}{\partial \nu_a} &= G(u)(x) && \text{on } \Gamma_4,\\
u(x) &\le \Psi(x) && \text{in } \mathcal{D},
\end{aligned}$$

is nonempty and compact in V.

Theorem 4.17. *Assume that* H(1), H(f)(i), H(N), H(U_1), H(M') *and* H(L') *are satisfied. If, moreover, M is coercive in V_0, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in U_1(x,u)+N(u)(x)+f(x,u,\nabla u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma,\\
L(u) &\le c_J,
\end{aligned}$$

is nonempty and compact in V_0.

Corollary 4.18. *Let $\mathcal{D}$ be a nonempty and measurable subset of Ω. Assume that* H(1), H(f)(i), H(N), H(M') *and* H(U_1) *are satisfied. If, moreover, M is coercive in V_0, and $\Phi\colon \Omega\to\mathbb{R}$ is a measurable function, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in U_1(x,u)+N(u)(x)+f(x,u,\nabla u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma,\\
u(x) &\le \Psi(x) && \text{in } \mathcal{D},
\end{aligned}$$

is nonempty and compact in V_0.

Theorem 4.19. *Assume that* H(1), H(f'), H(N), H(U_1), H(M') *and* H(L') *are satisfied. If, moreover, M is coercive in V_0, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in U_1(x,u)+N(u)(x)+f(x,u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma,\\
L(u) &\le c_J,
\end{aligned}$$

is nonempty and compact in V_0.

Corollary 4.20. *Assume that* H(1), H(f'), H(N), H(M') *and* H(U_1) *are satisfied. If, moreover, M is coercive in V_0, and $\Phi\colon \Omega\to\mathbb{R}$ is a measurable function, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in U_1(x,u)+N(u)(x)+f(x,u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma,\\
u(x) &\le \Psi(x) && \text{in } \mathcal{D},
\end{aligned}$$

is nonempty and compact in V_0.

Theorem 4.21. *Assume that* H(1), H(f)(i), H(N), H(j_1), H(M') *and* H(L') *are satisfied. If, moreover, M is coercive in V_0, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u+|u|^{p-2}u+\mu(x)|u|^{q-2}u &\in r_1(u)\partial j_1(x,u)+N(u)(x)+f(x,u,\nabla u) && \text{in } \Omega,\\
u&=0 && \text{on } \Gamma,\\
L(u) &\le c_J.
\end{aligned}$$

is nonempty and compact in V_0.

Corollary 4.22. *Assume that* H(1)*,* H(f)*,* H(N)*,* H(M') *and* H(j_1) *are satisfied. If, moreover,* M *is coercive in* V_0 *and* $\Phi\colon \Omega \to \mathbb{R}$ *is a measurable function, then the solution set of the following obstacle problem*

$$\begin{aligned}
-D_M u + |u|^{p-2}u + \mu(x)|u|^{q-2}u \in r_1(u)\partial j_1(x,u) + N(u)(x) + f(x,u,\nabla u) \qquad & \text{in } \Omega,\\
u = 0 \qquad & \text{on } \Gamma,\\
u(x) \leq \Psi(x) \qquad & \text{in } \mathcal{D}.
\end{aligned}$$

is nonempty and compact in V_0.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interests

There is no conflict of interests.

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