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An explicit parameter range for a singular double phase problem

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ABSTRACT

We investigate a class of singular quasilinear elliptic equations driven by double phase operators with variable coefficients. Under suitable assumptions on the parameter λ, the coefficients, and the exponents, we employ the fibering method to analyze the variational structure of the associated energy functional on the Nehari manifold. More precisely, we derive an explicit sufficient threshold value Λ and prove that the two components of the Nehari manifold are quantitatively separated whenever $\lambda < \Lambda$. This separation enables us to establish the existence of two distinct positive solutions, obtained as minimizers of the energy functional on the two components of the Nehari manifold. Furthermore, we derive explicit upper and lower bounds for both the L^p-norms of the gradients and the $L^{\alpha+1}$-norms of these solutions. Our estimates quantify the influence of the parameter λ, the geometry of the domain, and the exponent α on the size of the corresponding solutions.

1. Introduction

In order to investigate the behaviour of strongly anisotropic materials, Zhikov [51] was the first to introduce and study the prototype functional

$$u \mapsto \int_U \left(\frac{|\nabla u|^p}{p} + \mu(x) \frac{|\nabla u|^q}{q} \right) \mathrm{d}x, \tag{1.1}$$

where $U \subset \mathbb{R}^N$ is a domain, $1 < p < q$, and $\mu \in L^\infty(U)$ is a nonnegative weight function. The functional (1.1) belongs to the class of integral functionals with nonstandard growth, more specifically, with unbalanced growth, introduced by Marcellini [34,35]. Its Euler-Lagrange equation is governed by the so-called double phase operator

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$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u). \tag{1.2}$$

A distinctive feature of this operator is the spatial phase transition induced by the coefficient μ. More precisely, the coefficient partitions the domain into two qualitatively different regions. On the set $\{\mu = 0\}$, the operator reduces to the classical p-Laplacian, whereas on $\{\mu > 0\}$ it exhibits (p, q)-growth, with the q-Laplacian determining the asymptotic behaviour. Consequently, the governing equation is nonuniformly elliptic, and its ellipticity degenerates precisely on the zero set of the modulating coefficient. Nonlinear problems involving the operator have attracted considerable attention in recent years. We refer the reader to the works of Bai–Gasiński–Papageorgiou [4], Baroni–Colombo–Mingione [6–8], Baroni–Kuusi–Mingione [9], Byun–Oh [12], Cai–Papageorgiou–Rădulescu [13], Colombo–Mingione [14,15], Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], De Filippis–Palatucci [17], Kumar–Rădulescu–Sreenadh [25], Liu–Dai–Winkert [28], Liu–Dai–Winkert–Zeng [29], Liu–Winkert [33], Marcellini [34,35], Mingione–Rădulescu [36], Ok [37], Papageorgiou–Rădulescu–Yuan [39], Ragusa–Tachikawa [45,46], and the references therein.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. In [27], Liu–Dai–Papageorgiou–Winkert studied the singular double phase problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u) = h(x)u^{-\gamma} + \lambda g(x)u^{\alpha} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

They assumed that $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$, and $0 \leq \mu(\cdot) \in C^{0,1}(\overline{\Omega})$, $0 < \gamma < 1$ and $q - 1 < \alpha < p^* - 1$. Moreover, $h \in L^\infty(\Omega)$ is assumed to be nonnegative a.e. in Ω and not identically zero, whereas $g \equiv 1$ in Ω. The Nehari manifold method was employed to prove the existence of a positive constant λ_0 such that, for all $0 < \lambda < \lambda_0$, problem (1.3) admits two positive solutions, one with negative energy and the other with positive energy. Nevertheless, no explicit value of λ_0, and hence no explicit admissible range for the parameter λ, was provided in [27]. The main objective of the present paper is to fill this gap.

More precisely, the aim of the present paper is to determine an explicit range of values of the parameter λ for which problem (1.3) admits two positive solutions. To this end, we impose the following assumptions:

(H) (i) $1 < p < N$ and $p < q < p^* = \frac{Np}{N-p}$;
(ii) $0 \leq \mu(\cdot) \in L^\infty(\Omega)$;
(iii) $0 < \gamma < 1$;
(iv) $q - 1 < \alpha < p^* - 1$;
(v) $h \in L^\infty(\Omega)$ and $h(x) > 0$ for a.a. $x \in \Omega$;
(vi) $g \in L^\infty(\Omega)$ and $m(\{g > 0\}) > 0$, where m denotes the Lebesgue measure.

We say that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a solution of problem (1.3) if $u(x) > 0$ for a.a. $x \in \Omega$, $hu^{-\gamma}\varphi \in L^1(\Omega)$, and

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega h(x)u^{-\gamma}\varphi\,\mathrm{d}x + \lambda\int_\Omega g(x)u^{\alpha}\varphi\,\mathrm{d}x$$

for any $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$.

For each $\lambda > 0$, we define the energy functional $J_\lambda \colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ associated with problem (1.3) by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\,\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}\,\mathrm{d}x - \frac{\lambda}{\alpha+1}\int_\Omega g(x)|u|^{\alpha+1}\,\mathrm{d}x.$$

For $u \in W_0^{1,\mathcal{H}}(\Omega)\setminus\{0\}$, let $\varphi_u \colon \mathbb{R}_+ \to \mathbb{R}$ denote the associated fibering function defined by $\varphi_u(t) := J_\lambda(tu)$. The corresponding Nehari manifold is defined by

$$\mathcal{N}_\lambda := \left\{ u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\} \colon \varphi_u'(1) = 0 \right\}.$$

According to the sign of the second derivative of the fibering function, we decompose $\mathcal{N}_\lambda$ into the three mutually disjoint sets

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda \colon \varphi_u''(1) > 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda \colon \varphi_u''(1) = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda \colon \varphi_u''(1) < 0\}. \end{aligned} \tag{1.4}$$

To derive an explicit admissible range for the parameter λ, we introduce the constant

$$\Lambda := \left[\frac{\alpha - p + 1}{\gamma + \alpha}\right]^{\frac{\alpha-p+1}{p-1+\gamma}} \left(\frac{p-1+\gamma}{\gamma+\alpha}\right) \frac{1}{\|g\|_\infty} \left(\frac{1}{\|h\|_\infty}\right)^{\frac{\alpha-p+1}{p-1+\gamma}} \left(\frac{S}{|\Omega|^{\frac{p}{N}}}\right)^{\frac{\gamma+\alpha}{p-1+\gamma}}, \tag{1.5}$$

together with the auxiliary constants

$$\begin{aligned} C_1(p,\alpha,\gamma,h,N) &= \left(\frac{\gamma+\alpha}{\alpha-p+1}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \frac{|\Omega|^{\frac{p^*-(1-\gamma)}{p^*(p-1+\gamma)}}}{S^{\frac{1-\gamma}{p(p-1+\gamma)}}}, \\ C_2(p,\alpha,\gamma,h,N) &= \left(\frac{\gamma+\alpha}{\alpha-p+1}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \left[\frac{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}+\frac{\gamma+\alpha}{\alpha+1}}}{S}\right]^{\frac{1}{p-1+\gamma}}, \\ C_3(p,\alpha,\gamma,g,N,\lambda) &= \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{\alpha-p+1}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{\alpha-p+1}} \frac{S^{\frac{\alpha+1}{p(\alpha-p+1)}}}{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha-p+1)}}}, \\ C_4(p,\alpha,\gamma,g,N,\lambda) &= \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{\alpha-p+1}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{\alpha-p+1}} \left[\frac{S}{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}\right]^{\frac{1}{\alpha-p+1}}, \end{aligned}$$

where S denotes the best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Our main result is stated as follows.

Theorem 1.1. *Under assumption (H), for every $\lambda \in (0, \Lambda)$, problem (1.3) admits at least two distinct positive solutions:*

(i) *A minimizer u_λ of J_λ on $\mathcal{N}_\lambda^+$ satisfying the a priori upper bounds*

$$\|\nabla u_\lambda\|_p < C_1(p,\alpha,\gamma,h,N) \quad \text{and} \quad \|u_\lambda\|_{\alpha+1} < C_2(p,\alpha,\gamma,h,N).$$

(ii) *A minimizer v_λ of J_λ on $\mathcal{N}_\lambda^-$ satisfying the lower bounds*

$$\|\nabla v_\lambda\|_p > C_3(p,\alpha,\gamma,g,N,\lambda) \quad \text{and} \quad \|v_\lambda\|_{\alpha+1} > C_4(p,\alpha,\gamma,g,N,\lambda).$$

Remark 1.2. In Theorem 1.1, the solution $v_\lambda \in \mathcal{N}_\lambda^-$ becomes unbounded as $\lambda \to 0^+$. More precisely,

$$\|\nabla v_\lambda\|_p \to \infty \quad \text{and} \quad \|v_\lambda\|_{\alpha+1} \to \infty \quad \text{as } \lambda \to 0^+.$$

Furthermore, let $\varepsilon \in (0, p^* - q)$, set $\alpha = q - 1 + \varepsilon$, and denote by Λ_ε the constant Λ corresponding to this choice of α. For every $\lambda \in (0, \Lambda_\varepsilon)$, let $v_{\lambda,\varepsilon} \in \mathcal{N}_\lambda^-$ be the solution given by Theorem 1.1. Then we have

$$\|\nabla v_{\lambda,\varepsilon}\|_p > C_\varepsilon \lambda^{-\frac{1}{q-p+\varepsilon}},$$

where

$$C_\varepsilon = \left[\frac{p-1+\gamma}{(q-1+\gamma+\varepsilon)\|g\|_\infty}\right]^{\frac{1}{q-p+\varepsilon}} \frac{S^{\frac{q+\varepsilon}{p(q-p+\varepsilon)}}}{|\Omega|^{\frac{p^*-(q+\varepsilon)}{p^*(q-p+\varepsilon)}}}.$$

In particular, for every fixed $\varepsilon > 0$, the lower bound exhibits a polynomial blow-up of order $\lambda^{-\frac{1}{q-p+\varepsilon}}$ as $\lambda \to 0^+$. Moreover,

$$C_\varepsilon \to \left[\frac{p-1+\gamma}{(q-1+\gamma)\|g\|_\infty}\right]^{\frac{1}{q-p}} \frac{S^{\frac{q}{p(q-p)}}}{|\Omega|^{\frac{p^*-q}{p^*(q-p)}}} \quad \text{as } \varepsilon \to 0^+,$$

and the corresponding blow-up exponent converges to $\frac{1}{q-p}$.

Theorem 1.1 provides an explicit sufficient range for the parameter λ. In particular, if $h, g \equiv 1$, then problem (1.3) admits at least two distinct positive solutions whenever

$$0 < \lambda < \left[\frac{-p+\alpha+1}{\alpha+\gamma}\right]^{\frac{-p+\alpha+1}{p+\gamma-1}} \left(\frac{p+\gamma-1}{\alpha+\gamma}\right) \left(\frac{S}{|\Omega|^{\frac{p}{N}}}\right)^{\frac{\alpha+\gamma}{p+\gamma-1}}.$$

The proof of Theorem 1.1 is again based on the Nehari manifold method. In [27], the Nehari manifold was decomposed into the three sets $\mathcal{N}^+$, $\mathcal{N}^-$, and $\mathcal{N}^0$, and two solutions were obtained by minimizing the energy functional over $\mathcal{N}^+$ and $\mathcal{N}^-$, respectively. The distinction between these solutions relied on their energy levels: the minimizer on $\mathcal{N}^+$ has negative energy, whereas the minimizer on $\mathcal{N}^-$ has positive energy. The present paper adopts a different viewpoint. Rather than distinguishing the two solutions by their energy levels, we exploit the geometric structure of the Nehari manifold. More precisely, we show that the relative position of $\mathcal{N}^+$ and $\mathcal{N}^-$ depends crucially on the parameter λ. As λ decreases, the two sets move farther apart, and for sufficiently small values of λ they become quantitatively separated. This immediately implies that the corresponding minimizers are distinct. The explicit sufficient threshold obtained by our quantitative estimates is the constant Λ. This geometric perspective was first introduced by Sun–Li [49] in the context of nonsingular semilinear elliptic equations involving the Laplacian.

In recent years, the qualitative analysis of Dirichlet problems driven by double phase operators has attracted considerable attention, particularly in the presence of singular reaction terms. A prototype problem is given by

$$\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &= \lambda_1 \xi(x) u^{-\alpha} + \lambda_2 f(x,u) && \text{in } \Omega,\\
u &> 0 && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{1.6}$$

The main analytical difficulty of problem (1.6) stems from the interaction between the nonuniform ellipticity of the double phase operator, the singular term $u^{-\alpha}$, and the nonlinear reaction $f(x,u)$. Assuming $1 < p < N$ and $0 < \alpha < 1$, the existing literature may be broadly classified according to the growth conditions imposed on the reaction term f. For $(q-1)$-superlinear reactions, Liu–Dai–Papageorgiou–Winkert [27] considered the case of a fixed singular coefficient $\lambda_1 = 1$ together with the power-type nonlinearity $f(x,u) = u^{r-1}$ with $r > q$. Using the Nehari manifold method, they proved the existence of a constant $\lambda_2^* > 0$ such that problem (1.6) admits at least two positive solutions for every $\lambda_2 \in (0, \lambda_2^*)$. This result was subsequently extended to the whole space $\mathbb{R}^N$ by Liu–Winkert [32]. Bai–Gasiński–Papageorgiou [4] generalized this result by replacing the power-type nonlinearity with a general class of $(q-1)$-superlinear nonlinearities, while Liu–Papageorgiou [31] obtained a similar multiplicity result in the particular case $\xi(\cdot) = \lambda_2 = 1$ with the weighted reaction term $\eta(x)u^{r-1}$, where $r > q$. For the case $p > q$, Papageorgiou–Repovš–Vetro [42] established the existence of multiple positive solutions for superlinear reaction terms satisfying $r > p$. The dependence of the solution set on the parameters also gives rise to bifurcation phenomena. Assuming $\xi(\cdot) = 1$ and a general $(q-1)$-superlinear reaction term, Bai–Papageorgiou–Zeng [5] established a bifurcation-type result with respect to the parameter λ_2. More precisely, they proved the existence of a critical value $\lambda_2^* > 0$ such that problem (1.6) admits at least two positive solutions for $\lambda_2 \in (0, \lambda_2^*)$, at least one positive solution for $\lambda_2 = \lambda_2^*$, and no positive solutions whenever $\lambda_2 > \lambda_2^*$. Moving beyond constant exponents, Papageorgiou–Rădulescu–Zhang [40] extended this bifurcation result to a nonhomogeneous setting involving Lipschitz continuous variable exponents $p(\cdot)$, $q(\cdot)$, and $\alpha(\cdot) \in (0, 1)$. Failla–Gasiński–Papageorgiou–Skupien [19] considered the case $\lambda_1 = \lambda_2 = 1$ together with a $(p-1)$-sublinear perturbation $f(x, u)$. They established the existence of a positive solution and further proved that this solution is unique under an additional monotonicity assumption on $f(x, \cdot)$. Papageorgiou–Rădulescu–Yuan [39] proved the existence of positive solutions in the case $\alpha > 1$, $\lambda_1 = \lambda_2 = 1$, and $f(x, u) = \eta(x)u^{r-1}$ with $r < p$. Finally, Papageorgiou–Rădulescu–Zhang [41] considered $(q-1)$-superlinear reaction terms with $\lambda_2 = 1$ and proved the existence of a positive solution for every $\lambda_1 > 0$. For further developments, we refer the reader to the works of Arora–Dwivedi [1], Arora–Fiscella–Mukherjee–Winkert [2], Bahrouni–Rădulescu–Repovš [3], Biagi–Esposito–Vecchi [11], Farkas–Winkert [21], Garain–Mukherjee [22], Guarnotta–Winkert [23], Han–Liu–Papageorgiou [24], Papageorgiou–Peng [38], and Pimenta–Winkert [44] for further developments on singular double phase problems.

The aforementioned existence and multiplicity results establish the existence of a critical parameter value λ_2^*. However, they do not provide explicit estimates for this critical threshold. Liu–Feng–Sun–Winkert [30] investigated problem (1.6) under the assumptions $\lambda_1 = \lambda_2$, $0 < \alpha < 1$, a $(p-1)$-sublinear perturbation, and $N < p$. By means of a refined quantitative analysis of the associated energy functional, they established the existence of three bounded positive solutions for sufficiently small values of λ_1. Moreover, they derived explicit estimates characterizing the range of the parameter for which three positive solutions exist. Farkas–Fiscella–Ho–Winkert [20] considered the case $\lambda_1 = 0$ and $\lambda_2 f(x, u) = \tau_1 w(x)|u|^{s-2}u + \tau_2 g(x, u)$, where $p < s < q$. Overcoming the lack of compactness caused by the critical growth of $g(x, u)$, they proved that, whenever $\tau_1 > \tau_0$ and τ_2 is sufficiently small, problem (1.6) admits a negative energy solution. In addition, they obtained an explicit computable lower bound for the threshold parameter τ_0. Finally, we mention the one-dimensional case. Son–Sim [48] studied problem (1.6) with $\lambda_1 = 0$ and $f(x, u) = g(x)h(u)$. They derived sufficient conditions guaranteeing that the problem admits no solution, exactly one solution, exactly two solutions, or exactly three solutions. Furthermore, they obtained explicit estimates for the range of the parameter λ_2 ensuring the existence of three solutions; see also Sim–Son [47].

The remainder of this paper is organized as follows. In Section 2, we briefly review the theory of Musielak-Orlicz spaces together with the fundamental properties of double phase operators. In Section 3, we investigate the geometric structure of $\mathcal{N}_\lambda^\pm$ and derive an explicit sufficient range for the parameter λ. Finally, Section 4 is devoted to the proof of Theorem 1.1 and concludes with an example.

2. Preliminaries

In this section, we recall some basic properties of Musielak-Orlicz spaces and the double phase operator (1.2). Most of the material presented here can be found in Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], Liu–Dai [26], and Papageorgiou–Winkert [43]. We denote by $L^r(\Omega)$ and $L^r(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ for every $1 \le r < \infty$. For $1 < r < \infty$, we write $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ for the corresponding Sobolev spaces equipped with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0} = \|\nabla \cdot\|_r$, respectively.

Let $M(\Omega)$ denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. By $L^q_\mu(\Omega)$, we denote the weighted Lebesgue space defined by

$$L^q_\mu(\Omega) = \left\{ u \in M(\Omega) \colon \|u\|_{q,\mu} := \left(\int_\Omega \mu(x)|u(x)|^q \,\mathrm{d}x \right)^{\frac{1}{q}} < +\infty \right\}.$$

Next, we introduce the function $\mathcal{H} \colon \Omega \times [0,\infty) \to [0,\infty)$ by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q.$$

Then, the corresponding Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) \colon \rho_{\mathcal{H}}(u) < +\infty\}$$

and is equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{ \tau > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1 \right\},$$

where the modular $\rho_{\mathcal{H}}(\cdot) \colon L^{\mathcal{H}}(\Omega) \to \mathbb{R}$ is defined by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x. \tag{2.1}$$

The associated Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \colon |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\},$$

and is equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. Furthermore, the completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,\mathcal{H}}(\Omega)$ is denoted by $W_0^{1,\mathcal{H}}(\Omega)$. The spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$, and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive and separable Banach spaces. In what follows, we equip $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}}.$$

Moreover, the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ is continuous.

The norm $\|\cdot\|_{\mathcal{H}}$ and the modular $\rho_{\mathcal{H}}$ are related as follows, see Liu–Dai [26, Proposition 2.1].

Proposition 2.1. *Let (H)(i), (ii) be satisfied, let $y \in L^{\mathcal{H}}(\Omega)$, and let the modular $\rho_{\mathcal{H}}$ be defined by (2.1). Then the following assertions hold:*

(i) *If $y \neq 0$, then $\|y\|_{\mathcal{H}} = \lambda$ if and only if $\rho_{\mathcal{H}}(\frac{y}{\lambda}) = 1$;*
(ii) *$\|y\|_{\mathcal{H}} < 1$ (resp. $> 1, = 1$) if and only if $\rho_{\mathcal{H}}(y) < 1$ (resp. $> 1, = 1$);*
(iii) *If $\|y\|_{\mathcal{H}} < 1$, then $\|y\|^q_{\mathcal{H}} \le \rho_{\mathcal{H}}(y) \le \|y\|^p_{\mathcal{H}}$;*
(iv) *If $\|y\|_{\mathcal{H}} > 1$, then $\|y\|^p_{\mathcal{H}} \le \rho_{\mathcal{H}}(y) \le \|y\|^q_{\mathcal{H}}$;*
(v) *$\|y\|_{\mathcal{H}} \to 0$ if and only if $\rho_{\mathcal{H}}(y) \to 0$;*
(vi) *$\|y\|_{\mathcal{H}} \to +\infty$ if and only if $\rho_{\mathcal{H}}(y) \to +\infty$.*

Let $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), v\rangle := \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x \tag{2.2}$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The operator $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ possesses the following properties, see Liu–Dai [26].

Proposition 2.2. *Let (H)(i), (ii) be satisfied. Then the operator A defined by (2.2) is bounded, continuous, strictly monotone, coercive, a homeomorphism, and satisfies the (S_+)-property, that is,*

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0,$$

imply $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

We recall Ekeland's variational principle [18], which will be a crucial tool in what follows.

Theorem 2.3. *Let (X, d) be a complete metric space and let $J\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous functional that is bounded from below. Let $\varepsilon > 0$ and suppose that $u \in X$ satisfies*

$$J(u) \le \inf_{v\in X} J(v) + \varepsilon.$$

Then, for any $\rho > 0$, there exists $v \in X$ such that

(i) $J(v) \le J(u)$;
(ii) $d(u, v) \le \rho$;
(iii) $J(w) > J(v) - \dfrac{\varepsilon}{\rho} d(v, w)$ *for all $w \in X$ with $w \ne v$.*

3. The number Λ and the geometric structure of $\mathcal{N}_\lambda^\pm$

In this section, we investigate the geometric structure of $\mathcal{N}_\lambda^\pm$ and derive an explicit sufficient range for the parameter λ. Throughout this section, hypotheses (H) are assumed to hold and will not be repeated in the statements below.

Lemma 3.1. *For every $\lambda \in (0, \Lambda)$, we have $\mathcal{N}_\lambda^\pm \ne \emptyset$ and $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. In view of (H)(v) and (H)(vi), we may choose u_0 such that

$$\int_\Omega h(x)|u_0|^{1-\gamma}\,\mathrm{d}x > 0 \quad \text{and} \quad \int_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x > 0.$$

For $t > 0$, define

$$\varphi(t) := t^{p-1-\alpha}\|\nabla u_0\|_p^p + t^{q-1-\alpha}\|\nabla u_0\|_{q,\mu}^q - t^{-\gamma-\alpha}\int_\Omega h(x)|u_0|^{1-\gamma}\,\mathrm{d}x.$$

It is straightforward to verify that

$$\varphi(t) \to -\infty \quad \text{as } t \to 0^+ \quad \text{and} \quad \varphi(t) \to 0 \quad \text{as } t \to \infty.$$

Define

$$t^* := \left[\frac{(\alpha + 1 - p)\|\nabla u_0\|_p^p}{(\gamma + \alpha) \int_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x} \right]^{-\frac{1}{p+\gamma-1}}.$$

Then $0 < t^* < \infty$, and a direct computation shows that

$$\varphi(t^*) \geq \left(\frac{p - 1 + \gamma}{\gamma + \alpha} \right) \left(\frac{\alpha + 1 - p}{\gamma + \alpha} \right)^{\frac{\alpha+1-p}{p-1+\gamma}} \frac{\|\nabla u_0\|_p^{\frac{p(\gamma+\alpha)}{p-1+\gamma}}}{\left(\int_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x \right)^{\frac{\alpha+1-p}{p-1+\gamma}}} > 0.$$

Hence, there exists $\overline{t} \in (0, \infty)$ such that

$$\varphi(\overline{t}) = \max_{t>0} \varphi(t) > 0,$$

which implies that $\varphi'(\overline{t}) = 0$, or equivalently,

$$\begin{aligned} &(\alpha + 1 - p)\overline{t}^{\,p-2-\alpha}\|\nabla u_0\|_p^p + (\alpha + 1 - q)\overline{t}^{\,q-2-\alpha}\|\nabla u_0\|_{q,\mu}^q \\ &= (\gamma + \alpha)\overline{t}^{\,-\gamma-\alpha-1} \int\limits_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x. \end{aligned} \tag{3.1}$$

Multiplying both sides of (3.1) by $\overline{t}^{\,\gamma+\alpha+1}$ and rearranging terms, we obtain

$$(\alpha + 1 - p)\overline{t}^{\,p+\gamma-1}\|\nabla u_0\|_p^p + (\alpha + 1 - q)\overline{t}^{\,q+\gamma-1}\|\nabla u_0\|_{q,\mu}^q = (\gamma + \alpha) \int\limits_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x. \tag{3.2}$$

Since $q - 1 < \alpha$, we have $\alpha + 1 - q > 0$. Therefore, the second term on the left-hand side of (3.2) is nonnegative, and omitting it yields

$$(\alpha + 1 - p)\overline{t}^{\,p+\gamma-1}\|\nabla u_0\|_p^p \leq (\gamma + \alpha) \int\limits_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x.$$

Hence, $\overline{t} \leq t^*$.

By Hölder's inequality and the Sobolev embedding theorem, we obtain

$$\int\limits_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x \leq \|h\|_\infty \|u_0\|_{p^*}^{1-\gamma} |\Omega|^{\frac{p^*-(1-\gamma)}{p^*}} \leq \|h\|_\infty \left(\frac{1}{\sqrt[p]{S}} \|\nabla u_0\|_p \right)^{1-\gamma} |\Omega|^{\frac{p^*-(1-\gamma)}{p^*}}.$$

Consequently,

$$\frac{\|\nabla u_0\|_p}{\left(\int_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x \right)^{\frac{1}{1-\gamma}}} \geq \left(\frac{1}{\|h\|_\infty} \right)^{\frac{1}{1-\gamma}} \frac{\sqrt[p]{S}}{|\Omega|^{\frac{p^*-(1-\gamma)}{p^*(1-\gamma)}}}.$$

Since $\overline{t}$ is a global maximizer of φ, we have $\varphi(\overline{t}) \geq \varphi(t^*)$. Therefore,

$$
\begin{aligned}
&\varphi(\bar{t}) - \lambda \int\limits_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x \\
&\geq \varphi(t^*) - \lambda \int\limits_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x \\
&\geq \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{\alpha+1-p}{\gamma+\alpha}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \frac{\|\nabla u_0\|_p^{\frac{p(\gamma+\alpha)}{p-1+\gamma}}}{\left(\int_\Omega h(x)|u_0|^{1-\gamma}\,\mathrm{d}x\right)^{\frac{\alpha+1-p}{p-1+\gamma}}} \\
&\quad - \lambda\|g\|_\infty \left[\frac{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}{\sqrt[p]{S}}\right]^{\alpha+1} \|\nabla u_0\|_p^{\alpha+1} \\
&= \left[\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{\alpha+1-p}{\gamma+\alpha}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \frac{\|\nabla u_0\|_p^{\frac{(\alpha+1-p)(1-\gamma)}{p-1+\gamma}}}{\left(\int_\Omega h(x)|u_0|^{1-\gamma}\,\mathrm{d}x\right)^{\frac{\alpha+1-p}{p-1+\gamma}}}\right. \\
&\quad \left. - \lambda\|g\|_\infty \left(\frac{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}{\sqrt[p]{S}}\right)^{\alpha+1}\right] \|\nabla u_0\|_p^{\alpha+1} \\
&\geq \left[\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{\alpha+1-p}{\gamma+\alpha}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \left(\frac{1}{\|h\|_\infty}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \left(\frac{\sqrt[p]{S}}{|\Omega|^{\frac{p^*-(1-\gamma)}{p^*(1-\gamma)}}}\right)^{\frac{(\alpha+1-p)(1-\gamma)}{p-1+\gamma}}\right. \\
&\quad \left. - \lambda\|g\|_\infty \left(\frac{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}{\sqrt[p]{S}}\right)^{\alpha+1}\right] \|\nabla u_0\|_p^{\alpha+1}
\end{aligned}
$$

We now define

$$
E_\lambda = \frac{p-1+\gamma}{\gamma+\alpha}\left(\frac{\alpha+1-p}{(\gamma+\alpha)\|h\|_\infty}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \left(\frac{\sqrt[p]{S}}{|\Omega|^{\frac{p^*-(1-\gamma)}{p^*(1-\gamma)}}}\right)^{\frac{(\alpha+1-p)(1-\gamma)}{p-1+\gamma}} - \lambda\|g\|_\infty \left(\frac{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}{\sqrt[p]{S}}\right)^{\alpha+1}.
$$

Thus,

$$
\varphi(\bar{t}) - \lambda \int\limits_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x \geq E_\lambda \|\nabla u_0\|_p^{\alpha+1}.
$$

Setting $E_\lambda = 0$ and solving for λ, we obtain

$$
\lambda = \left(\frac{\alpha+1-p}{\gamma+\alpha}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \left(\frac{p-1+\gamma}{\gamma+\alpha}\right) \frac{1}{\|g\|_\infty} \left(\frac{1}{\|h\|_\infty}\right)^{\frac{\alpha+1-p}{p-1+\gamma}} \frac{\sqrt[p]{S}^{\frac{(\alpha+1-p)(1-\gamma)}{p-1+\gamma}+(\alpha+1)}}{|\Omega|^{\frac{(p^*-1+\gamma)(\alpha+1-p)}{p^*(p-1+\gamma)}+\frac{p^*-(\alpha+1)}{p^*}}} =: \Lambda.
$$

Observe that

$$
\frac{(\alpha+1-p)(1-\gamma)}{p-1+\gamma} + (\alpha+1) = \frac{p(\gamma+\alpha)}{p-1+\gamma}
$$

and

$$
\frac{(p^*-1+\gamma)(\alpha+1-p)}{p^*(p-1+\gamma)} + \frac{p^*-(\alpha+1)}{p^*} = \frac{p}{N}\frac{\gamma+\alpha}{p-1+\gamma}.
$$

Consequently, Λ can be written in the more compact form

$$\Lambda=\left(\frac{\alpha+1-p}{\gamma+\alpha}\right)^{\frac{\alpha+1-p}{p-1+\gamma}}\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\frac{1}{\|g\|_\infty}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{\alpha+1-p}{p-1+\gamma}}\left(\frac{S}{|\Omega|^{\frac{p}{N}}}\right)^{\frac{\gamma+\alpha}{p-1+\gamma}}. \tag{3.3}$$

By the definition of E_λ, for every $\lambda\in(0,\Lambda)$, we have

$$\varphi(\bar{t})-\lambda\int_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x>0.$$

Moreover, multiplying $\varphi'(t)$ by $t^{\gamma+\alpha+1}$, we obtain

$$t^{\gamma+\alpha+1}\varphi'(t)=(\gamma+\alpha)\int_\Omega h(x)|u_0|^{1-\gamma}\,\mathrm{d}x \\ -(\alpha+1-p)t^{p+\gamma-1}\|\nabla u_0\|_p^p-(\alpha+1-q)t^{q+\gamma-1}\|\nabla u_0\|_{q,\mu}^q.$$

Note that the right-hand side is strictly decreasing on $(0,\infty)$. Hence, $\bar{t}$ is the unique critical point of φ, and φ is strictly increasing on $(0,\bar{t})$ and strictly decreasing on $(\bar{t},\infty)$. Since

$$\varphi(t)\to-\infty \quad \text{as } t\to0^+ \quad \text{and} \quad \varphi(t)\to0 \quad \text{as } t\to\infty,$$

and

$$0<\lambda\int_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x<\varphi(\bar{t}),$$

there exist unique numbers $0<t_{0,\lambda}^-<\bar{t}<t_{0,\lambda}^+$ such that

$$\varphi(t_{0,\lambda}^-)=\lambda\int_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x=\varphi(t_{0,\lambda}^+).$$

Furthermore,

$$\varphi'(t_{0,\lambda}^-)>0 \quad \text{and} \quad \varphi'(t_{0,\lambda}^+)<0.$$

At either of these two points, the identity

$$\varphi(t)=\lambda\int_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x$$

is equivalent to $tu_0\in\mathcal{N}_\lambda$. Moreover, the sign of the second derivative of the fibering map $t\mapsto J_\lambda(tu_0)$ agrees with the sign of $\varphi'(t)$. Therefore,

$$t_{0,\lambda}^-u_0\in\mathcal{N}_\lambda^+ \quad \text{and} \quad t_{0,\lambda}^+u_0\in\mathcal{N}_\lambda^-.$$

Thus, $\mathcal{N}_\lambda^+\neq\emptyset$ and $\mathcal{N}_\lambda^-\neq\emptyset$.

We now show that $\mathcal{N}_\lambda^0=\emptyset$. Assume, for contradiction, that there exists $u_0\neq0$ such that $u_0\in\mathcal{N}_\lambda^0$. Then

$$(p-1+\gamma)\|\nabla u_0\|_p^p+(q-1+\gamma)\|\nabla u_0\|_{q,\mu}^q-\lambda(\gamma+\alpha)\int_\Omega g(x)|u_0|^{\alpha+1}\,\mathrm{d}x=0.$$

Since $u_0 \in \mathcal{N}_\lambda$, we also have

$$\int_\Omega |\nabla u_0|^p \, \mathrm{d}x + \int_\Omega \mu(x)|\nabla u_0|^q \, \mathrm{d}x - \int_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x - \lambda \int_\Omega g(x)|u_0|^{\alpha+1} \, \mathrm{d}x = 0.$$

Combining these two identities yields

$$\frac{-p+1+\alpha}{\gamma+\alpha}\|\nabla u_0\|_p^p + \frac{-q+1+\alpha}{\gamma+\alpha}\|\nabla u_0\|_{q,\mu}^q - \int_\Omega h(x)|u_0|^{1-\gamma} \, \mathrm{d}x = 0.$$

Consequently,

$$\begin{aligned}
&E_\lambda \|\nabla u_0\|_p^{\alpha+1} \\
&\leq \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{\|\nabla u_0\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left(\int_\Omega h(x)|u_0|^{1-\gamma}\,\mathrm{d}x\right)^{\frac{-p+1+\alpha}{p-1+\gamma}}} - \lambda \int_\Omega g(x)|u_0|^{\alpha+1} \, \mathrm{d}x \\
&= \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{\|\nabla u_0\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left(\frac{-p+1+\alpha}{\gamma+\alpha}\|\nabla u_0\|_p^p + \frac{-q+1+\alpha}{\gamma+\alpha}\|\nabla u_0\|_{q,\mu}^q\right)^{\frac{-p+1+\alpha}{p-1+\gamma}}} \\
&\quad - \frac{(p-1+\gamma)}{\gamma+\alpha}\|\nabla u_0\|_p^p - \frac{(q-1+\gamma)}{\gamma+\alpha}\|\nabla u_0\|_{q,\mu}^q \\
&\leq \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{\|\nabla u_0\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left(\frac{-p+1+\alpha}{\gamma+\alpha}\|\nabla u_0\|_p^p\right)^{\frac{-p+1+\alpha}{p-1+\gamma}}} \\
&\quad - \frac{(p-1+\gamma)}{\gamma+\alpha}\|\nabla u_0\|_p^p \\
&= 0.
\end{aligned}$$

This contradicts the fact that $E_\lambda > 0$ whenever $0 < \lambda < \Lambda$ and $\|\nabla u_0\|_p > 0$ for $u_0 \neq 0$. Therefore, $\mathcal{N}_\lambda^0 = \emptyset$. The proof is complete. □

Lemma 3.2. *For every $\lambda \in (0, \Lambda)$, we have*

$$\|\nabla U\|_p - \|\nabla u\|_p > A(\lambda) - A_0 > 0 \quad \textit{and} \quad \|U\|_{\alpha+1} - \|u\|_{\alpha+1} > B(\lambda) - B_0 > 0,$$

for every $U \in \mathcal{N}_\lambda^-$ and $u \in \mathcal{N}_\lambda^+$, where $A(\lambda)$, A_0, $B(\lambda)$, and B_0 are given explicitly in the proof below.

Proof. Let $U \in \mathcal{N}_\lambda^-$. We first derive estimates for $\|\nabla U\|_p$ and $\|U\|_{\alpha+1}$. By (1.4), Hölder's inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}
(p-1+\gamma)\|\nabla U\|_p^p + (q-1+\gamma)\|\nabla U\|_{q,\mu}^q &< \lambda(\gamma+\alpha)\int_\Omega g(x)|U|^{\alpha+1} \, \mathrm{d}x \\
&\leq \lambda(\gamma+\alpha)\|g\|_\infty \left[\frac{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}{\sqrt[p]{S}}\right]^{\alpha+1} \|\nabla U\|_p^{\alpha+1},
\end{aligned}$$

which implies

$$(p-1+\gamma)\|\nabla U\|_p^p < \lambda(\gamma+\alpha)\|g\|_\infty \left[\frac{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}{\sqrt[p]{S}}\right]^{\alpha+1} \|\nabla U\|_p^{\alpha+1}.$$

Consequently

$$\|\nabla U\|_p > \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{-p+1+\alpha}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{-p+1+\alpha}} \frac{(\sqrt[p]{S})^{\frac{\alpha+1}{-p+1+\alpha}}}{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(-p+1+\alpha)}}}. \tag{3.4}$$

Note that

$$\frac{p^*-(\alpha+1)}{p^*(-p+1+\alpha)} = \frac{Np-(\alpha+1)(N-p)}{Np(-p+1+\alpha)} = \frac{1}{N}\left(\frac{\alpha+1}{-p+1+\alpha}\right) - \frac{1}{p}$$

and therefore, for convenience, we define the right-hand side of (3.4) by

$$A(\lambda) = \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{-p+1+\alpha}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{-p+1+\alpha}} \frac{(\sqrt[p]{S})^{\frac{\alpha+1}{-p+1+\alpha}}}{|\Omega|^{\frac{1}{N}\left(\frac{\alpha+1}{-p+1+\alpha}\right)-\frac{1}{p}}}.$$

Using a similar argument, we also obtain the following estimate for U, namely

$$\begin{aligned}(p-1+\gamma)\frac{S}{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}\|U\|_{\alpha+1}^p < (p-1+\gamma)\|\nabla U\|_p^p < \lambda(\gamma+\alpha)\int_\Omega g(x)|U|^{\alpha+1}\,\mathrm{d}x \\ \le \lambda(\gamma+\alpha)\|g\|_\infty\|U\|_{\alpha+1}^{\alpha+1},\end{aligned}$$

which implies

$$\|U\|_{\alpha+1} > \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{-p+1+\alpha}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{-p+1+\alpha}} \left[\frac{S}{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}\right]^{\frac{1}{-p+1+\alpha}}. \tag{3.5}$$

Furthermore,

$$\begin{aligned}p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)} &= \frac{Np-(\alpha+1)(N-p)}{N(\alpha+1)} = \frac{p(\alpha+1)-N(-p+1+\alpha)}{N(\alpha+1)} \\ &= \frac{p}{N} - \frac{-p+1+\alpha}{\alpha+1}.\end{aligned}$$

We denote the right-hand side of (3.5) by

$$B(\lambda) = \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{-p+1+\alpha}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{-p+1+\alpha}} \left[\frac{S}{|\Omega|^{\frac{p}{N}-\frac{-p+1+\alpha}{\alpha+1}}}\right]^{\frac{1}{-p+1+\alpha}}.$$

Next, we derive analogous estimates for $\|\nabla u\|_p$ and $\|u\|_{\alpha+1}$ with $u \in \mathcal{N}_\lambda^+$. By the definition of $\mathcal{N}_\lambda^+$, Hölder's inequality, and the Sobolev embedding theorem, we obtain

$$(-p+1+\alpha)\|\nabla u\|_p^p < (\gamma+\alpha)\int_\Omega h(x)|u|^{1-\gamma}\,\mathrm{d}x < (\gamma+\alpha)\left(\frac{1}{\sqrt[p]{S}}\right)^{1-\gamma} |\Omega|^{\frac{p^*-(1-\gamma)}{p^*}} \|h\|_\infty \|\nabla u\|_p^{1-\gamma}$$

which implies

$$\|\nabla u\|_p < \left(\frac{\gamma+\alpha}{-p+1+\alpha}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \frac{|\Omega|^{\frac{p^*-(1-\gamma)}{p^*(p-1+\gamma)}}}{(\sqrt[p]{S})^{\frac{1-\gamma}{p-1+\gamma}}}. \tag{3.6}$$

Furthermore,

$$\frac{p^*-(1-\gamma)}{p^*(p-1+\gamma)} = \frac{Np-(1-\gamma)(N-p)}{Np(p-1+\gamma)} = \frac{1}{p} + \frac{1}{N}\frac{1-\gamma}{p-1+\gamma}$$

For convenience, we denote the right-hand side of (3.6) by

$$A_0 = \left(\frac{\gamma+\alpha}{-p+1+\alpha}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \frac{|\Omega|^{\frac{1}{p}+\frac{1}{N}\frac{1-\gamma}{p-1+\gamma}}}{(\sqrt[p]{S})^{\frac{1-\gamma}{p-1+\gamma}}}.$$

Similarly, we derive the estimate

$$\begin{aligned}(-p+1+\alpha)\frac{S}{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}\|u\|_{\alpha+1}^p < (-p+1+\alpha)\|\nabla u\|_p^p < (\gamma+\alpha)\int\limits_\Omega h(x)|u|^{1-\gamma}\,\mathrm{d}x \\ \le (\gamma+\alpha)|\Omega|^{\frac{\gamma+\alpha}{\alpha+1}}\|h\|_\infty\|u\|_{\alpha+1}^{1-\gamma},\end{aligned}$$

from which it follows that

$$\|u\|_{\alpha+1} < \left(\frac{\gamma+\alpha}{-p+1+\alpha}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \left[\frac{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}+\frac{\gamma+\alpha}{\alpha+1}}}{S}\right]^{\frac{1}{p-1+\gamma}}. \tag{3.7}$$

A straightforward calculation shows that

$$\begin{aligned}p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)} + \frac{\gamma+\alpha}{\alpha+1} &= \frac{Np-(\alpha+1)(N-p)+N(\gamma+\alpha)}{N(\alpha+1)} = \frac{N(p-1+\gamma)+p(\alpha+1)}{N(\alpha+1)} \\ &= \frac{p}{N} + \frac{p-1+\gamma}{\alpha+1}\end{aligned}$$

Accordingly, we introduce the notation for the right-hand side of equation (3.7)

$$B_0 = \left(\frac{\gamma+\alpha}{-p+1+\alpha}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \left[\frac{|\Omega|^{\frac{p}{N}+\frac{p-1+\gamma}{\alpha+1}}}{S}\right]^{\frac{1}{p-1+\gamma}}.$$

We next compare the quantities $A(\lambda)$, $B(\lambda)$, A_0, and B_0. Since both $A(\lambda)$ and $B(\lambda)$ are strictly decreasing functions of λ, they satisfy $A(\lambda), B(\lambda) \to \infty$ as $\lambda \to 0$, and $A(\lambda), B(\lambda) \to 0$ as $\lambda \to \infty$. It therefore follows that each of the equations $A(\lambda) = A_0$ and $B(\lambda) = B_0$ admits a unique solution. Expanding the identity $A(\lambda) = A_0$, we arrive at

$$\left(\frac{\gamma+\alpha}{-p+1+\alpha}\right)^{\frac{1}{p-1+\gamma}} \|h\|_\infty^{\frac{1}{p-1+\gamma}} \frac{|\Omega|^{\frac{p^*-(1-\gamma)}{p^*(p-1+\gamma)}}}{(\sqrt[p]{S})^{\frac{1-\gamma}{p-1+\gamma}}} = \left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{-p+1+\alpha}} \left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{-p+1+\alpha}} \frac{(\sqrt[p]{S})^{\frac{\alpha+1}{-p+1+\alpha}}}{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*(-p+1+\alpha)}}}.$$

Calculation yields

$$\lambda = \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left[\frac{-p+1+\alpha}{\gamma+\alpha}\right]^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{1}{\|g\|_\infty}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{(\sqrt[p]{S})^{(\alpha+1)+\frac{(1-\gamma)(-p+1+\alpha)}{p-1+\gamma}}}{|\Omega|^{\frac{p^*-(\alpha+1)}{p^*}+\frac{p^*-(1-\gamma)}{p^*(p-1+\gamma)}(-p+1+\alpha)}}.$$

Moreover,

$$\frac{p^*-(\alpha+1)}{p^*}+\frac{p^*-(1-\gamma)}{p^*(p-1+\gamma)}(-p+1+\alpha)=\frac{(p^*-p)(\gamma+\alpha)}{p^*(p-1+\gamma)}=\frac{p}{N}\frac{\gamma+\alpha}{p-1+\gamma}$$

and therefore $\lambda=\Lambda$, see (3.3).

Likewise, the identity $B(\lambda)=B_0$ can be written as

$$\begin{aligned}&\left(\frac{\gamma+\alpha}{-p+1+\alpha}\right)^{\frac{1}{p-1+\gamma}}\|h\|_\infty^{\frac{1}{p-1+\gamma}}\left[\frac{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}+\frac{\gamma+\alpha}{\alpha+1}}}{S}\right]^{\frac{1}{p-1+\gamma}}\\&=\left[\frac{p-1+\gamma}{\lambda(\gamma+\alpha)}\right]^{\frac{1}{-p+1+\alpha}}\left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{-p+1+\alpha}}\left[\frac{S}{|\Omega|^{p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}\right]^{\frac{1}{-p+1+\alpha}}.\end{aligned}$$

Solving for λ, we obtain

$$\lambda=\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left[\frac{-p+1+\alpha}{\gamma+\alpha}\right]^{\frac{-p+1+\alpha}{p-1+\gamma}}\frac{1}{\|g\|_\infty}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}}\frac{S^{\frac{\gamma+\alpha}{p-1+\gamma}}}{|\Omega|^{\left(p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}+\frac{\gamma+\alpha}{\alpha+1}\right)\frac{-p+1+\alpha}{p-1+\gamma}+p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}}}.$$

Furthermore,

$$\begin{aligned}&\left(p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}+\frac{\gamma+\alpha}{\alpha+1}\right)\frac{-p+1+\alpha}{p-1+\gamma}+p\frac{p^*-(\alpha+1)}{p^*(\alpha+1)}\\&=\frac{p^*(\alpha+1)(\gamma+\alpha)-p(\alpha+1)(\gamma+\alpha)}{p^*(\alpha+1)(p-1+\gamma)}=\frac{p}{N}\frac{\gamma+\alpha}{p-1+\gamma}.\end{aligned}$$

Hence, we again obtain $\lambda=\Lambda$, see (3.3). Since both $A(\lambda)$ and $B(\lambda)$ are strictly decreasing, it follows that

$$\begin{aligned}\|\nabla U\|_p-\|\nabla u\|_p&>A(\lambda)-A_0>0,\\\|U\|_{\alpha+1}-\|u\|_{\alpha+1}&>B(\lambda)-B_0>0,\end{aligned}$$

for every $\lambda<\Lambda$, $U\in\mathcal{N}_\lambda^-$, and $u\in\mathcal{N}_\lambda^+$. This proves the asserted quantitative separation. □

4. The proof of the main result

In this section, we prove Theorem 1.1. We begin by establishing two auxiliary results.

Lemma 4.1. *For every $\lambda\in(0,\Lambda)$, the set $\mathcal{N}_\lambda^-$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}\subset\mathcal{N}_\lambda^-$ be a sequence such that $U_n\to U_0$ in $W_0^{1,\mathcal{H}}(\Omega)$. By (3.4),

$$\|\nabla U_0\|_p=\lim_{n\to\infty}\|\nabla U_n\|_p\geq A(\lambda)>0,$$

and hence $U_0\neq0$. Passing to the limit in the defining equation of the Nehari manifold, we conclude that $U_0\in\mathcal{N}_\lambda$. Since $\lambda<\Lambda$, we have $A(\lambda)>A_0$. On the other hand, estimate (3.6) implies that every $u\in\mathcal{N}_\lambda^+$ satisfies $\|\nabla u\|_p<A_0$. Combining the above inequalities, we infer that

$$\|\nabla U_0\|_p>A_0>\|\nabla u\|_p\quad\text{for every } u\in\mathcal{N}_\lambda^+,$$

which shows that $U_0 \notin \mathcal{N}_\lambda^+$. Moreover, Lemma 3.1 ensures that $\mathcal{N}_\lambda^0 = \emptyset$. Consequently, $U_0 \notin \mathcal{N}_\lambda^0$. Since $U_0 \in \mathcal{N}_\lambda$, the only remaining possibility is $U_0 \in \mathcal{N}_\lambda^-$. Hence, $\mathcal{N}_\lambda^-$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$. $\square$

Proposition 4.2. *Let $u \in \mathcal{N}_\lambda^+$ (resp. $u \in \mathcal{N}_\lambda^-$). Then there exist $\varepsilon > 0$ and a continuous function $f\colon B_\varepsilon(0) \subset W_0^{1,\mathcal{H}}(\Omega) \to (0,\infty)$ such that*

$$f(0) = 1, \quad f(v)(u+v) \in \mathcal{N}_\lambda^+ \quad (\textit{resp.}\ f(v)(u+v) \in \mathcal{N}_\lambda^-),$$

for every $v \in B_\varepsilon(0)$.

Proof. We consider only the case $u \in \mathcal{N}_\lambda^+$, since the argument for $u \in \mathcal{N}_\lambda^-$ is analogous. Define $F\colon W_0^{1,\mathcal{H}}(\Omega)\times(0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} F(v,t) = t^{p-1+\gamma} \int_\Omega |\nabla(u+v)|^p \,\mathrm{d}x + t^{q-1+\gamma} \int_\Omega \mu(x)|\nabla(u+v)|^q \,\mathrm{d}x \\ - \lambda t^{\gamma+\alpha} \int_\Omega g(x)|u+v|^{\alpha+1} \,\mathrm{d}x - \int_\Omega h(x)|u+v|^{1-\gamma} \,\mathrm{d}x. \end{aligned}$$

Since $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have $F(0,1) = 0$. Moreover,

$$F_t(0,1) = (p-1+\gamma) \int_\Omega |\nabla u|^p \,\mathrm{d}x + (q-1+\gamma) \int_\Omega \mu(x)|\nabla u|^q \,\mathrm{d}x - \lambda(\gamma+\alpha) \int_\Omega g(x)|u|^{\alpha+1} \,\mathrm{d}x > 0.$$

The implicit function theorem (see, for example, Berger [10, p. 115]) therefore provides $\overline{\varepsilon} > 0$ and a continuous function $f\colon B_{\overline{\varepsilon}}(0) \to (0,\infty)$ such that

$$f(0) = 1, \quad F(v, f(v)) = 0, \quad \text{for every } v \in B_{\overline{\varepsilon}}(0).$$

Equivalently,

$$f(0) = 1, \quad f(v)(u+v) \in \mathcal{N}_\lambda, \quad \text{for every } v \in B_{\overline{\varepsilon}}(0).$$

Since $u \in \mathcal{N}_\lambda^+$ and the defining inequality for $\mathcal{N}_\lambda^+$ is stable under sufficiently small perturbations, we may choose $0 < \varepsilon < \overline{\varepsilon}$ such that

$$f(v)(u+v) \in \mathcal{N}_\lambda^+, \quad \text{for every } v \in B_\varepsilon(0).$$

This completes the proof. $\square$

Proof of Theorem 1.1. From the definition of $\mathcal{N}_\lambda$ and Proposition 2.1(iv), we conclude that

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \,\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma} \,\mathrm{d}x \\ &\quad - \frac{\lambda}{\alpha+1}\int_\Omega g(x)|u|^{\alpha+1} \,\mathrm{d}x \\ &\ge \left(\frac{1}{q} - \frac{1}{\alpha+1}\right)\rho_{\mathcal{H}}(\nabla u) - \left(\frac{1}{1-\gamma} - \frac{1}{\alpha+1}\right)\int_\Omega h(x)|u|^{1-\gamma} \,\mathrm{d}x \\ &\ge c_1\|u\|_{1,\mathcal{H},0}^p - c_2\|u\|_{1,\mathcal{H},0}^{1-\gamma}, \quad \text{for all } u \in \mathcal{N}_\lambda \text{ with } \|u\|_{1,\mathcal{H},0} > 1, \end{aligned}$$

where $c_1, c_2 > 0$ are suitable constants. Here we used that $p-1 < q-1 < \alpha$. Consequently, since $0 < 1-\gamma < p$, the functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.

Step 1. Existence of a solution in $\mathcal{N}_\lambda^+$ for $\lambda \in (0, \Lambda)$.

Since $u \in \mathcal{N}_\lambda^+$, we have

$$(p-1+\gamma)\|\nabla u\|_p^p + (q-1+\gamma)\|\nabla u\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int_\Omega g(x)|u|^{\alpha+1}\,\mathrm{d}x > 0.$$

Hence, for any $u \in \mathcal{N}_\lambda^+$,

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\|\nabla u\|_p^p + \left(\frac{1}{q} - \frac{1}{1-\gamma}\right)\|\nabla u\|_{q,\mu}^q + \lambda\left(\frac{1}{1-\gamma} - \frac{1}{\alpha+1}\right)\int_\Omega g(x)|u|^{\alpha+1}\,\mathrm{d}x \\ &< -\frac{p-1+\gamma}{p(1-\gamma)}\|\nabla u\|_p^p - \frac{q-1+\gamma}{q(1-\gamma)}\|\nabla u\|_{q,\mu}^q + \frac{p-1+\gamma}{(1-\gamma)(\alpha+1)}\|\nabla u\|_p^p + \frac{q-1+\gamma}{(1-\gamma)(\alpha+1)}\|\nabla u\|_{q,\mu}^q \\ &= \left(\frac{p-1+\gamma}{1-\gamma}\right)\left(-\frac{1}{p} + \frac{1}{\alpha+1}\right)\|\nabla u\|_p^p + \left(\frac{q-1+\gamma}{1-\gamma}\right)\left(-\frac{1}{q} + \frac{1}{\alpha+1}\right)\|\nabla u\|_{q,\mu}^q \\ &< 0. \end{aligned}$$

Consequently,

$$m_\lambda^+ := \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) < 0.$$

Choose $c \in (m_\lambda^+, 0)$. We define

$$X_c := \left\{u \in \mathcal{N}_\lambda^+ \colon J_\lambda(u) \le c\right\}.$$

Since $c > m_\lambda^+$, the set X_c is nonempty and

$$\inf_{u \in X_c} J_\lambda(u) = m_\lambda^+.$$

Moreover, X_c is closed in $W_0^{1,\mathcal{H}}(\Omega)$. Indeed, let $\{w_n\}_{n\in\mathbb{N}} \subset X_c$ be such that $w_n \to w$ in $W_0^{1,\mathcal{H}}(\Omega)$. By the continuity of J_λ, we have

$$J_\lambda(w) \le c < 0,$$

and hence $w \ne 0$. Passing to the limit in the Nehari identity yields $w \in \mathcal{N}_\lambda$. By Lemma 3.1, $\mathcal{N}_\lambda^0 = \emptyset$. If $w \in \mathcal{N}_\lambda^-$, then Lemma 3.2 gives

$$\|\nabla w\|_p - \|\nabla w_n\|_p > A(\lambda) - A_0 > 0$$

for every $n \in \mathbb{N}$, which contradicts $w_n \to w$ in $W_0^{1,\mathcal{H}}(\Omega)$. Therefore $w \in \mathcal{N}_\lambda^+$ and hence $w \in X_c$. Thus, X_c is a complete metric space. Let

$$X_c^+ := \{u \in X_c \colon u \ge 0 \text{ a.e. in } \Omega\}.$$

Since $u \mapsto |u|$ preserves $\mathcal{N}_\lambda^+$ and $J_\lambda(|u|) = J_\lambda(u)$, we have

$$\inf_{u\in X_c^+} J_\lambda(u) = \inf_{u\in X_c} J_\lambda(u) = m_\lambda^+.$$

Moreover, X_c^+ is closed in X_c and hence is a complete metric space.

By Ekeland's variational principle (see Theorem 2.3), applied to the complete metric space X_c^+ and J_λ, there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_c^+$ satisfying

(1) $J_\lambda(u_n) < m_\lambda^+ + \frac{1}{n}$;
(2) $J_\lambda(u) \geq J_\lambda(u_n) - \frac{1}{n}\|u-u_n\|_{1,\mathcal{H},0}$ for every $u \in X_c^+$.

The coercivity of J_λ on $\mathcal{N}_\lambda$ implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Hence, there exists a constant $M_1 > 0$ such that

$$\|u_n\|_{1,\mathcal{H},0} \leq M_1.$$

Passing to a subsequence if necessary, we obtain

$$u_n \rightharpoonup u_\lambda \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u_\lambda \quad \text{in } L^{\alpha+1}(\Omega) \text{ and in } L^{1-\gamma}(\Omega).$$

By the weak lower semicontinuity of the convex part of the energy functional, it follows that

$$J_\lambda(u_\lambda) \leq \liminf_{n\to\infty} J_\lambda(u_n) = \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0.$$

Therefore, $u_\lambda \not\equiv 0$. Furthermore, since $u_n \in \mathcal{N}_\lambda^+$, we have

$$(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int_\Omega g(x)|u_n|^{\alpha+1}\,\mathrm{d}x > 0.$$

Passing to the limit inferior and using the strong convergence in $L^{\alpha+1}(\Omega)$, we conclude that

$$\liminf_{n\to\infty}\left[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q\right] - \lambda(\gamma+\alpha)\int_\Omega g(x)|u_\lambda|^{\alpha+1}\,\mathrm{d}x \geq 0.$$

Claim:

$$\liminf_{n\to\infty}\left[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q\right] > \lambda(\gamma+\alpha)\int_\Omega g(x)|u_\lambda|^{\alpha+1}\,\mathrm{d}x. \tag{4.1}$$

We argue by contradiction. Suppose that

$$\liminf_{n\to\infty}\left[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q\right] = \lambda(\gamma+\alpha)\int_\Omega g(x)|u_\lambda|^{\alpha+1}\,\mathrm{d}x. \tag{4.2}$$

Passing to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, we may assume that the limit inferior in (4.2) is attained as a limit. Since $u_n \in \mathcal{N}_\lambda^+$, we have

$$\frac{\alpha+1-p}{\gamma+\alpha}\|\nabla u_n\|_p^p + \frac{\alpha+1-q}{\gamma+\alpha}\|\nabla u_n\|_{q,\mu}^q < \int_\Omega h(x)|u_n|^{1-\gamma}\,\mathrm{d}x.$$

Using the same estimate as in the proof of Lemma 3.1, together with the strong convergence of u_n in $L^{1-\gamma}(\Omega)$, we infer that

$$
\begin{aligned}
0 &< E_\lambda \|\nabla u_\lambda\|_p^{\alpha+1} \\
&\leq \liminf_{n\to\infty} \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{\|\nabla u_n\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left(\int_\Omega h(x)|u_n|^{1-\gamma}\,\mathrm{d}x\right)^{\frac{-p+1+\alpha}{p-1+\gamma}}} - \lambda\int_\Omega g(x)|u_\lambda|^{\alpha+1}\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{\|\nabla u_n\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left[\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)\|\nabla u_n\|_p^p + \left(\frac{-q+1+\alpha}{\gamma+\alpha}\right)\|\nabla u_n\|_{q,\mu}^q\right]^{\frac{-p+1+\alpha}{p-1+\gamma}}} \\
&\quad - \liminf_{n\to\infty}\left[\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\|\nabla u_n\|_p^p + \left(\frac{q-1+\gamma}{\gamma+\alpha}\right)\|\nabla u_n\|_{q,\mu}^q\right] \\
&< \liminf_{n\to\infty}\left[\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}} \frac{\|\nabla u_n\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left[\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)\|\nabla u_n\|_p^p\right]^{\frac{-p+1+\alpha}{p-1+\gamma}}} - \left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\|\nabla u_n\|_p^p\right] \\
&= 0.
\end{aligned}
$$

This contradiction proves (4.1).

Combining (4.1) with the strong convergence $u_n \to u_\lambda$ in $L^{\alpha+1}(\Omega)$, we obtain

$$
\liminf_{n\to\infty}\Big[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int_\Omega g(x)|u_n|^{\alpha+1}\,\mathrm{d}x\Big] > 0.
$$

Consequently, there exist $d_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$
(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int_\Omega g(x)|u_n|^{\alpha+1}\,\mathrm{d}x \geq d_1, \tag{4.3}
$$

for every $n \geq n_0$. After discarding finitely many terms and relabeling the sequence, we may assume that (4.3) holds for every $n \in \mathbb{N}$.

Applying Proposition 4.2 with $u = u_n$, where n is sufficiently large that $\frac{(1-\gamma)M_1}{n} < d_1$, and choosing $v = t\varphi$, where $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ is fixed and nonnegative, for sufficiently small $t > 0$, we obtain a continuous function $f_n(t) := f_n(t\varphi)$ such that $f_n(0) = 1$ and $f_n(t)(u_n + t\varphi) \in \mathcal{N}_\lambda^+$. Since $J_\lambda(u_n) \to m_\lambda^+ < c$, we have $J_\lambda(u_n) < c$ for all sufficiently large n. Therefore, by the continuity of $t \mapsto J_\lambda(f_n(t)(u_n + t\varphi))$, after reducing $t > 0$ if necessary, we also have

$$
f_n(t)(u_n + t\varphi) \in X_c^+.
$$

Since only the continuity of f_n is known, we first show that the right-sided extended derivative at 0 exists. Recall that

$$
F(t\varphi, f_n(t)) = 0 = F(0,1)
$$

for all sufficiently small $t > 0$. By the mean value formula with respect to the second variable,

$$
F(t\varphi, f_n(t)) - F(t\varphi, 1) = a_n(t)\big(f_n(t) - 1\big),
$$

where

$$a_n(t) = \int_0^1 F_t\big(t\varphi, 1 + \theta(f_n(t) - 1)\big)\,\mathrm{d}\theta.$$

Since $f_n(t) \to 1$ as $t \to 0^+$, we have $a_n(t) \to F_t(0, 1)$ as $t \to 0^+$. Moreover, by (4.3),

$$F_t(0,1) = (p - 1 + \gamma)\|\nabla u_n\|_p^p + (q - 1 + \gamma)\|\nabla u_n\|_{q,\mu}^q - \lambda(\gamma + \alpha)\int_\Omega g(x)u_n^{\alpha+1}\,\mathrm{d}x \geq d_1 > 0.$$

Consequently,

$$a_n(t)\frac{f_n(t) - 1}{t} = -\frac{F(t\varphi, 1) - F(0, 1)}{t}.$$

The right-hand side admits a limit in $[-\infty, +\infty]$ as $t \to 0^+$. Indeed, the terms involving the p- and q-gradient contributions and the $(\alpha + 1)$-power term have finite limits, whereas

$$\frac{1}{t}\int_\Omega h(x)\big[(u_n + t\varphi)^{1-\gamma} - u_n^{1-\gamma}\big]\,\mathrm{d}x$$

has a limit in $[0, +\infty]$. To see this, observe that, for a.a. $x \in \Omega$, the function

$$t \mapsto \frac{(u_n(x) + t\varphi(x))^{1-\gamma} - u_n(x)^{1-\gamma}}{t}$$

is nonincreasing in $t > 0$, since $s \mapsto s^{1-\gamma}$ is concave. Hence, as $t \to 0^+$, it converges increasingly to an element of $[0, +\infty]$, and the monotone convergence theorem yields the assertion.

Since $a_n(t) \to F_t(0, 1) > 0$, it follows that the extended limit

$$l_n := \lim_{t \to 0^+} \frac{f_n(t) - 1}{t} \in [-\infty, +\infty]$$

exists.

For notational simplicity, throughout the remainder of the proof we write

$$f_n'(0) := l_n,$$

where $f_n'(0)$ is understood as the right-sided extended derivative. For every $r > 0$, define

$$\psi_r(s) = \begin{cases} \dfrac{s^r - 1}{s - 1}, & s \neq 1, \\ r, & s = 1. \end{cases}$$

Since ψ_r is continuous at $s = 1$ and $f_n(t) \to 1$ as $t \to 0^+$, we have

$$\lim_{t \to 0^+} \frac{f_n^r(t) - 1}{t} = \lim_{t \to 0^+} \psi_r(f_n(t))\frac{f_n(t) - 1}{t} = r f_n'(0),$$

where the equality is understood in the extended sense.

We claim that the sequence $\{|f_n'(0)|\}_{n\in\mathbb{N}}$ is bounded. First, we observe that $\{f_n'(0)\}_{n\in\mathbb{N}}$ is bounded from below. Indeed, in the identity

$$a_n(t)\frac{f_n(t)-1}{t} = -\frac{F(t\varphi,1)-F(0,1)}{t},$$

the contribution of the singular term on the right-hand side is nonnegative, whereas all the remaining terms are uniformly bounded with respect to n. Since $a_n(t)\to F_t(0,1)\ge d_1>0$, the assertion follows.

Suppose now, to the contrary, that $\{|f_n'(0)|\}_{n\in\mathbb{N}}$ is unbounded. Passing to a subsequence if necessary, we may assume that $f_n'(0)\to+\infty$. In particular, for every sufficiently large n, we have $f_n(t)>1$ whenever $t>0$ is sufficiently small. Assertion (2) of Ekeland's variational principle then yields, for $t>0$ small,

$$\begin{aligned}
&\frac{1}{n}\left\|(f_n(t)-1)\,u_n + tf_n(t)\varphi\right\|_{1,\mathcal{H},0}\\
&\ge J_\lambda(u_n) - J_\lambda\left(f_n(t)(u_n+t\varphi)\right)\\
&= \frac{p-1+\gamma}{p(1-\gamma)}\left(\|\nabla(u_n+t\varphi)\|_p^p - \|\nabla u_n\|_p^p\right) + \frac{p-1+\gamma}{p(1-\gamma)}\left[f_n^p(t)-1\right]\|\nabla(u_n+t\varphi)\|_p^p\\
&\quad + \frac{q-1+\gamma}{q(1-\gamma)}\left(\|\nabla(u_n+t\varphi)\|_{q,\mu}^q - \|\nabla u_n\|_{q,\mu}^q\right) + \frac{q-1+\gamma}{q(1-\gamma)}\left[f_n^q(t)-1\right]\|\nabla(u_n+t\varphi)\|_{q,\mu}^q\\
&\quad - \lambda\frac{\gamma+\alpha}{(1-\gamma)(\alpha+1)} f_n^{\alpha+1}(t)\int_\Omega g(x)\left[(u_n+t\varphi)^{\alpha+1} - u_n^{\alpha+1}\right]\,\mathrm{d}x\\
&\quad - \lambda\frac{\gamma+\alpha}{(1-\gamma)(\alpha+1)}\left[f_n^{\alpha+1}(t)-1\right]\int_\Omega g(x)u_n^{\alpha+1}\,\mathrm{d}x.
\end{aligned}$$

Dividing the preceding inequality by $t>0$, we pass to the limit as $t\to0^+$. For the terms involving $f_n^r(t)-1$, we use the limit relation established above, while the remaining terms converge by the corresponding directional derivative formulas. Collecting the terms involving $f_n'(0)$ then yields, in the extended sense,

$$\begin{aligned}
\frac{1}{n}\|\varphi\|_{1,\mathcal{H},0} \ge \frac{f_n'(0)}{1-\gamma}\Bigg[&(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q\\
&- \lambda(\alpha+\gamma)\int_\Omega g(x)u_n^{\alpha+1}\,\mathrm{d}x - \frac{1-\gamma}{n}\|u_n\|_{1,\mathcal{H},0}\Bigg]\\
&+ \frac{p-1+\gamma}{1-\gamma}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\\
&+ \frac{q-1+\gamma}{1-\gamma}\int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\\
&- \lambda\frac{\gamma+\alpha}{1-\gamma}\int_\Omega g(x)u_n^\alpha\varphi\,\mathrm{d}x.
\end{aligned}$$

By (4.3), the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, and the choice of n, the expression in square brackets satisfies

$$\begin{aligned}
&(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int_\Omega g(x)u_n^{\alpha+1}\,\mathrm{d}x - \frac{1-\gamma}{n}\|u_n\|_{1,\mathcal{H},0}\\
&\ge d_1 - \frac{(1-\gamma)M_1}{n} > 0.
\end{aligned}$$

The remaining integral terms are uniformly bounded because $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Since $f_n'(0) \to +\infty$, the right-hand side tends to $+\infty$, whereas the left-hand side tends to zero, which is impossible. Thus, $\{f_n'(0)\}_{n\in\mathbb{N}}$ is bounded from above. Together with the lower bound established above, this proves that there exists a constant $C_5 > 0$, independent of n, such that

$$|f_n'(0)| \le C_5. \tag{4.4}$$

We now show that $u_\lambda \in \mathcal{N}_\lambda^+$ is a solution of (1.3). Applying assertion (2) of Ekeland's variational principle once again, for n sufficiently large and $t > 0$ sufficiently small so that $f_n(t)(u_n + t\varphi) \in X_c^+$, we obtain

$$\begin{aligned}
&\frac{1}{n}\left\|(f_n(t)-1)\,u_n + tf_n(t)\varphi\right\|_{1,\mathcal{H},0} \\
&\ge J_\lambda(u_n) - J_\lambda(f_n(t)(u_n+t\varphi)) \\
&= -\left[\frac{f_n^p(t)-1}{p}\right]\|\nabla u_n\|_p^p - \left[\frac{f_n^q(t)-1}{q}\right]\|\nabla u_n\|_{q,\mu}^q + \left[\frac{f_n^{1-\gamma}(t)-1}{1-\gamma}\right]\int_\Omega h(x)(u_n+t\varphi)^{1-\gamma}\,\mathrm{d}x \\
&\quad + \lambda\left[\frac{f_n^{\alpha+1}(t)-1}{\alpha+1}\right]\int_\Omega g(x)|u_n+t\varphi|^{\alpha+1}\,\mathrm{d}x + \frac{f_n^p(t)}{p}\left(\|\nabla u_n\|_p^p - \|\nabla(u_n+t\varphi)\|_p^p\right) \\
&\quad + \frac{f_n^q(t)}{q}\left(\|\nabla u_n\|_{q,\mu}^q - \|\nabla(u_n+t\varphi)\|_{q,\mu}^q\right) + \frac{\lambda}{\alpha+1}\int_\Omega g(x)\left[(u_n+t\varphi)^{\alpha+1} - u_n^{\alpha+1}\right]\mathrm{d}x \\
&\quad + \frac{1}{1-\gamma}\int_\Omega h(x)[(u_n+t\varphi)^{1-\gamma} - u_n^{1-\gamma}]\,\mathrm{d}x.
\end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0^+$, we infer that

$$\begin{aligned}
&\frac{1}{n}|f_n'(0)|\,\|u_n\|_{1,\mathcal{H},0} + \frac{\|\varphi\|_{1,\mathcal{H},0}}{n} \\
&\ge -f_n'(0)\|\nabla u_n\|_p^p - f_n'(0)\|\nabla u_n\|_{q,\mu}^q + f_n'(0)\int_\Omega h(x)u_n^{1-\gamma}\,\mathrm{d}x \\
&\quad + \lambda f_n'(0)\int_\Omega g(x)u_n^{\alpha+1}\,\mathrm{d}x - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x \\
&\quad + \lambda\int_\Omega g(x)u_n^\alpha\varphi\,\mathrm{d}x + \liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega \frac{h(x)[(u_n+t\varphi)^{1-\gamma} - u_n^{1-\gamma}]}{t}\,\mathrm{d}x \\
&= -f_n'(0)\left[\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,\mu}^q - \int_\Omega h(x)u_n^{1-\gamma}\,\mathrm{d}x - \lambda\int_\Omega g(x)u_n^{\alpha+1}\,\mathrm{d}x\right] \\
&\quad - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + \lambda\int_\Omega g(x)u_n^\alpha\varphi\,\mathrm{d}x \\
&\quad + \liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega \frac{h(x)[(u_n+t\varphi)^{1-\gamma} - u_n^{1-\gamma}]}{t}\,\mathrm{d}x \\
&= -\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + \lambda\int_\Omega g(x)u_n^\alpha\varphi\,\mathrm{d}x
\end{aligned}$$

$$+\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega \frac{h(x)[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]}{t}\,\mathrm{d}x. \tag{4.5}$$

Since $h(x)[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]\geq 0$ for a.a. $x\in\Omega$ and for every $t>0$, by Fatou's Lemma, we have

$$\int_\Omega h(x)u_n^{-\gamma}\varphi\,\mathrm{d}x \leq \liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega \frac{h(x)[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]}{t}\,\mathrm{d}x.$$

Substituting this estimate into (4.5) and using (4.4), we obtain

$$\begin{aligned}\int_\Omega h(x)u_n^{-\gamma}\varphi\,\mathrm{d}x &\leq \frac{1}{n}|f_n'(0)|\,\|u_n\|_{1,\mathcal{H},0}+\frac{\|\varphi\|_{1,\mathcal{H},0}}{n}\\ &\quad+\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x+\int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x-\lambda\int_\Omega g(x)u_n^{\alpha}\varphi\,\mathrm{d}x\\ &\leq \frac{\left(C_5\cdot M_1+\|\varphi\|_{1,\mathcal{H},0}\right)}{n}+\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x\\ &\quad+\int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x-\lambda\int_\Omega g(x)u_n^{\alpha}\varphi\,\mathrm{d}x.\end{aligned}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$ and the operator A is bounded, the sequence $\{A(u_n)\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)^*$. Passing to a subsequence if necessary, there exists $\eta_\lambda\in W_0^{1,\mathcal{H}}(\Omega)^*$ such that

$$A(u_n)\rightharpoonup\eta_\lambda\quad\text{in } W_0^{1,\mathcal{H}}(\Omega)^*.$$

Passing to the limit in the preceding estimate and using the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$ in $L^{\alpha+1}(\Omega)$, we obtain

$$\liminf_{n\to\infty}\int_\Omega h(x)u_n^{-\gamma}\varphi\,\mathrm{d}x\leq\langle\eta_\lambda,\varphi\rangle-\lambda\int_\Omega g(x)u_\lambda^{\alpha}\varphi\,\mathrm{d}x$$

for every nonnegative $\varphi\in W_0^{1,\mathcal{H}}(\Omega)$. Applying Fatou's lemma, we infer that

$$\int_\Omega h(x)u_\lambda^{-\gamma}\varphi\,\mathrm{d}x\leq\langle\eta_\lambda,\varphi\rangle-\lambda\int_\Omega g(x)u_\lambda^{\alpha}\varphi\,\mathrm{d}x.$$

In particular,

$$\int_\Omega h(x)u_\lambda^{-\gamma}\varphi\,\mathrm{d}x<\infty$$

for every nonnegative $\varphi\in W_0^{1,\mathcal{H}}(\Omega)$, and hence $u_\lambda>0$ a.e. in Ω.

Taking $\varphi=u_\lambda$ in the preceding variational inequality, we obtain

$$\lambda\int_\Omega g(x)u_\lambda^{\alpha+1}\,\mathrm{d}x+\int_\Omega h(x)u_\lambda^{1-\gamma}\,\mathrm{d}x\leq\langle\eta_\lambda,u_\lambda\rangle. \tag{4.6}$$

On the other hand, since $u_n \in \mathcal{N}_\lambda$, the strong convergence in $L^{\alpha+1}(\Omega)$ and $L^{1-\gamma}(\Omega)$ yields

$$\begin{aligned}\langle A(u_n), u_n\rangle &= \lambda \int_\Omega g(x) u_n^{\alpha+1}\,\mathrm{d}x + \int_\Omega h(x) u_n^{1-\gamma}\,\mathrm{d}x \\ &\to \lambda \int_\Omega g(x) u_\lambda^{\alpha+1}\,\mathrm{d}x + \int_\Omega h(x) u_\lambda^{1-\gamma}\,\mathrm{d}x.\end{aligned}$$

Moreover, by the monotonicity of A,

$$0 \le \langle A(u_n) - A(u_\lambda), u_n - u_\lambda\rangle.$$

Passing to the limit inferior and using $u_n \rightharpoonup u_\lambda$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $A(u_n) \rightharpoonup \eta_\lambda$ in $W_0^{1,\mathcal{H}}(\Omega)^*$, we obtain

$$0 \le \lambda \int_\Omega g(x) u_\lambda^{\alpha+1}\,\mathrm{d}x + \int_\Omega h(x) u_\lambda^{1-\gamma}\,\mathrm{d}x - \langle \eta_\lambda, u_\lambda\rangle.$$

Together with (4.6), this gives

$$\langle \eta_\lambda, u_\lambda\rangle = \lambda \int_\Omega g(x) u_\lambda^{\alpha+1}\,\mathrm{d}x + \int_\Omega h(x) u_\lambda^{1-\gamma}\,\mathrm{d}x.$$

Furthermore,

$$\langle A(u_n), u_\lambda\rangle \to \langle \eta_\lambda, u_\lambda\rangle,$$

and consequently

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_\lambda\rangle = 0.$$

By the (S_+)-property of A (see Proposition 2.2), we conclude that

$$u_n \to u_\lambda \quad \text{strongly in } W_0^{1,\mathcal{H}}(\Omega).$$

Since A is continuous, it follows that

$$\eta_\lambda = A(u_\lambda).$$

Moreover, the strong convergence, the Nehari identity for u_n, and (4.3) imply that

$$u_\lambda \in \mathcal{N}_\lambda^+.$$

Hence,

$$\langle A(u_\lambda), u_\lambda\rangle = \lambda \int_\Omega g(x) u_\lambda^{\alpha+1}\,\mathrm{d}x + \int_\Omega h(x) u_\lambda^{1-\gamma}\,\mathrm{d}x.$$

Therefore, the preceding variational inequality can be written as

$$\langle A(u_\lambda), \varphi \rangle \geq \lambda \int_\Omega g(x) u_\lambda^\alpha \varphi \, \mathrm{d}x + \int_\Omega h(x) u_\lambda^{-\gamma} \varphi \, \mathrm{d}x$$

for every nonnegative $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$. A standard truncation argument, using $u_\lambda > 0$ a.e. in Ω and the Nehari identity above, yields

$$\langle A(u_\lambda), \varphi \rangle = \lambda \int_\Omega g(x) u_\lambda^\alpha \varphi \, \mathrm{d}x + \int_\Omega h(x) u_\lambda^{-\gamma} \varphi \, \mathrm{d}x$$

for every $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$. Thus, u_λ is a weak solution of problem (1.3). Furthermore,

$$J_\lambda(u_\lambda) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u).$$

Step 2. Existence of a solution in $\mathcal{N}_\lambda^-$ for $\lambda \in (0, \Lambda)$.
Set

$$Y^- := \{u \in \mathcal{N}_\lambda^- : u \geq 0 \text{ a.e. in } \Omega\}.$$

Since $\mathcal{N}_\lambda^-$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$ by Lemma 4.1 and the positive cone is closed in $W_0^{1,\mathcal{H}}(\Omega)$, the set Y^- is a complete metric space. Moreover, since $u \mapsto |u|$ preserves $\mathcal{N}_\lambda^-$ and $J_\lambda(|u|) = J_\lambda(u)$,

$$\inf_{u \in Y^-} J_\lambda(u) = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u).$$

By Ekeland's variational principle (see Theorem 2.3), applied to the complete metric space Y^-, there exists a minimizing sequence $\{v_n\}_{n\in\mathbb{N}} \subset Y^-$ such that

(1) $J_\lambda(v_n) < \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) + \frac{1}{n}$;
(2) $J_\lambda(u) \geq J_\lambda(v_n) - \frac{1}{n} \|u - v_n\|_{1,\mathcal{H},0}$, for every $u \in Y^-$.

The coercivity of J_λ on $\mathcal{N}_\lambda$ implies that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Thus, we can find a constant $M_2 > 0$ such that

$$\|v_n\|_{1,\mathcal{H},0} \leq M_2 \quad \text{for all } n \in \mathbb{N}.$$

Passing to a subsequence if necessary, we may therefore assume that

$$v_n \rightharpoonup v_\lambda \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad v_n \to v_\lambda \quad \text{in } L^{\alpha+1}(\Omega) \text{ and in } L^{1-\gamma}(\Omega).$$

Consequently

$$J_\lambda(v_\lambda) \leq \liminf_{n\to\infty} J_\lambda(v_n) = \inf_{\mathcal{N}_\lambda^-} J_\lambda.$$

We note that $v_n \in \mathcal{N}_\lambda^-$ implies

$$(p - 1 + \gamma)\|\nabla v_n\|_p^p + (q - 1 + \gamma)\|\nabla v_n\|_{q,\mu}^q - \lambda(\gamma + \alpha) \int_\Omega g(x)|v_n|^{\alpha+1} \, \mathrm{d}x < 0,$$

and therefore,

$$\liminf_{n\to\infty}\left[(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q\right]-\lambda(\gamma+\alpha)\int_\Omega g(x)|v_\lambda|^{\alpha+1}\,\mathrm{d}x\le 0.$$

Claim:

$$\liminf_{n\to\infty}\left[(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q\right]<\lambda(\gamma+\alpha)\int_\Omega g(x)|v_\lambda|^{\alpha+1}\,\mathrm{d}x. \tag{4.7}$$

To prove (4.7), we argue by contradiction and assume that

$$\liminf_{n\to\infty}\left[(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q\right]=\lambda(\gamma+\alpha)\int_\Omega g(x)|v_\lambda|^{\alpha+1}\,\mathrm{d}x. \tag{4.8}$$

Passing to a subsequence if necessary, we may suppose that the limit inferior is attained as a limit. Set

$$D_n=(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q-\lambda(\gamma+\alpha)\int_\Omega g(x)|v_n|^{\alpha+1}\,\mathrm{d}x. \tag{4.9}$$

Since $v_n\in\mathcal{N}_\lambda^-$, we have $D_n<0$. The strong convergence of v_n in $L^{\alpha+1}(\Omega)$, together with the contradiction assumption, yields $D_n\to 0$. Moreover, because $v_n\in\mathcal{N}_\lambda$,

$$\|\nabla v_n\|_p^p+\|\nabla v_n\|_{q,\mu}^q=\int_\Omega h(x)|v_n|^{1-\gamma}\,\mathrm{d}x+\lambda\int_\Omega g(x)|v_n|^{\alpha+1}\,\mathrm{d}x. \tag{4.10}$$

Combining (4.10) with the definition of D_n in (4.9), we obtain

$$\int_\Omega h(x)|v_n|^{1-\gamma}\,\mathrm{d}x=\frac{\alpha+1-p}{\gamma+\alpha}\|\nabla v_n\|_p^p+\frac{\alpha+1-q}{\gamma+\alpha}\|\nabla v_n\|_{q,\mu}^q+\frac{D_n}{\gamma+\alpha},$$

where D_n is defined in (4.9). We first observe that $v_\lambda\not\equiv 0$. Indeed, if $v_\lambda\equiv 0$, then (4.8) implies

$$(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q\to 0,$$

which contradicts the lower bound (3.4) for $v_n\in\mathcal{N}_\lambda^-$. Since $D_n\to 0$, the same argument as in the proof of Lemma 3.1, now applied asymptotically, gives

$$\begin{aligned}0&<E_\lambda\|\nabla v_\lambda\|_p^{\alpha+1}\\&\le\liminf_{n\to\infty}\left[\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)^{\frac{-p+1+\alpha}{p-1+\gamma}}\frac{\|\nabla v_n\|_p^{p\frac{\gamma+\alpha}{p-1+\gamma}}}{\left[\left(\frac{-p+1+\alpha}{\gamma+\alpha}\right)\|\nabla v_n\|_p^p\right]^{\frac{-p+1+\alpha}{p-1+\gamma}}}-\left(\frac{p-1+\gamma}{\gamma+\alpha}\right)\|\nabla v_n\|_p^p\right]\\&\le 0,\end{aligned}$$

which is impossible. This contradiction proves (4.7). Thus, (4.7), together with the strong convergence of v_n in $L^{\alpha+1}(\Omega)$, implies that

$$\limsup_{n\to\infty}\Big[(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q-\lambda(\gamma+\alpha)\int\limits_\Omega g(x)|v_n|^{\alpha+1}\,\mathrm{d}x\Big]<0.$$

Consequently, there exist constants $C_6>0$ and $n_0\in\mathbb{N}$ such that

$$(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q-\lambda(\gamma+\alpha)\int\limits_\Omega g(x)|v_n|^{\alpha+1}\,\mathrm{d}x\le -C_6$$

for every $n\ge n_0$. After discarding finitely many terms, we may assume that this estimate holds for all $n\in\mathbb{N}$.

Applying Proposition 4.2 with $u=v_n$ (for n sufficiently large so that $\frac{(1-\gamma)M_2}{n}<C_6$) and $v=t\varphi$, where $\varphi\in W_0^{1,\mathcal{H}}(\Omega)$ is fixed and nonnegative and $t>0$ is sufficiently small, we obtain a continuous function $f_n(t):=f_n(t\varphi)$ satisfying $f_n(0)=1$ and

$$f_n(t)(v_n+t\varphi)\in Y^-.$$

As in Step 1, the right-sided extended derivative

$$f_n'(0):=\lim_{t\to 0^+}\frac{f_n(t)-1}{t}\in[-\infty,+\infty]$$

exists. In the present case,

$$F_t(0,1)=(p-1+\gamma)\|\nabla v_n\|_p^p+(q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q-\lambda(\gamma+\alpha)\int\limits_\Omega g(x)v_n^{\alpha+1}\,\mathrm{d}x\le -C_6<0.$$

The identity used in Step 1 also shows that $\{f_n'(0)\}_{n\in\mathbb{N}}$ is bounded from above. Therefore, if $\{|f_n'(0)|\}_{n\in\mathbb{N}}$ were unbounded, then, after passing to a subsequence if necessary,

$$f_n'(0)\to-\infty.$$

In particular, for every sufficiently large n, we have $f_n(t)<1$ whenever $t>0$ is sufficiently small. Assertion (2) of Ekeland's variational principle yields

$$\frac{1}{n}\|f_n(t)(v_n+t\varphi)-v_n\|_{1,\mathcal{H},0}\ge J_\lambda(v_n)-J_\lambda\left(f_n(t)(v_n+t\varphi)\right).$$

Using the representation of J_λ on the Nehari manifold, we obtain

$$\begin{aligned}
&J_\lambda(v_n)-J_\lambda\left(f_n(t)(v_n+t\varphi)\right)\\
&=\frac{p-1+\gamma}{p(1-\gamma)}\left(\|\nabla(v_n+t\varphi)\|_p^p-\|\nabla v_n\|_p^p\right)+\frac{p-1+\gamma}{p(1-\gamma)}\left[f_n^p(t)-1\right]\|\nabla(v_n+t\varphi)\|_p^p\\
&\quad+\frac{q-1+\gamma}{q(1-\gamma)}\left(\|\nabla(v_n+t\varphi)\|_{q,\mu}^q-\|\nabla v_n\|_{q,\mu}^q\right)+\frac{q-1+\gamma}{q(1-\gamma)}\left[f_n^q(t)-1\right]\|\nabla(v_n+t\varphi)\|_{q,\mu}^q\\
&\quad-\lambda\frac{\gamma+\alpha}{(1-\gamma)(\alpha+1)}f_n^{\alpha+1}(t)\int\limits_\Omega g(x)\left[(v_n+t\varphi)^{\alpha+1}-v_n^{\alpha+1}\right]\mathrm{d}x\\
&\quad-\lambda\frac{\gamma+\alpha}{(1-\gamma)(\alpha+1)}\left[f_n^{\alpha+1}(t)-1\right]\int\limits_\Omega g(x)v_n^{\alpha+1}\,\mathrm{d}x.
\end{aligned}$$

Dividing by $t > 0$, rearranging the terms, and then letting $t \to 0^+$, we obtain, in the extended sense,

$$\frac{1}{n}\|\varphi\|_{1,\mathcal{H},0} \geq \frac{f_n'(0)}{1-\gamma}\left[(p-1+\gamma)\|\nabla v_n\|_p^p + (q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int\limits_\Omega g(x)v_n^{\alpha+1}\,\mathrm{d}x + \frac{1-\gamma}{n}\|v_n\|_{1,\mathcal{H},0}\right]$$
$$+ \frac{p-1+\gamma}{1-\gamma}\int\limits_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi\,\mathrm{d}x$$
$$+ \frac{q-1+\gamma}{1-\gamma}\int\limits_\Omega \mu(x)|\nabla v_n|^{q-2}\nabla v_n \cdot \nabla\varphi\,\mathrm{d}x$$
$$- \lambda\frac{\gamma+\alpha}{1-\gamma}\int\limits_\Omega g(x)v_n^{\alpha}\varphi\,\mathrm{d}x.$$

By the estimate above, the boundedness of $\{v_n\}_{n\in\mathbb{N}}$, and the choice of n, the expression in square brackets satisfies

$$(p-1+\gamma)\|\nabla v_n\|_p^p + (q-1+\gamma)\|\nabla v_n\|_{q,\mu}^q - \lambda(\gamma+\alpha)\int\limits_\Omega g(x)v_n^{\alpha+1}\,\mathrm{d}x$$
$$+ \frac{1-\gamma}{n}\|v_n\|_{1,\mathcal{H},0} \leq -C_6 + \frac{(1-\gamma)M_2}{n} < 0.$$

The remaining integral terms are uniformly bounded because $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Since $f_n'(0) \to -\infty$, the right-hand side tends to $+\infty$, whereas the left-hand side tends to zero, which is impossible. Thus, $\{f_n'(0)\}_{n\in\mathbb{N}}$ is bounded from below. Together with the upper bound established above, this proves that there exists a constant $C_7 > 0$, independent of n, such that

$$|f_n'(0)| \leq C_7.$$

Proceeding exactly as in Step 1 and using the (S_+)-property of A, we conclude that $v_\lambda \in \mathcal{N}_\lambda^-$ is a positive weak solution of problem (1.3) satisfying

$$J_\lambda(v_\lambda) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u).$$

Finally, the estimates (3.6), (3.7), (3.4), and (3.5) yield the bounds stated in the theorem. □

Finally, we present a concrete example illustrating the applicability of Theorem 1.1.

Example 4.3. Let $N = 3$, $\Omega = B(0,2) \subset \mathbb{R}^3$, $p = 2$, $q = 3$, $\gamma = 0.1$, $\alpha = 3$, $\mu(x) = |x|$, $h(x) = \frac{1}{2}|x|$, and $g(x) = \chi_{B(0,\frac{1}{2})}(x)$. The optimal Sobolev constant S (see Talenti [50]) and the Lebesgue measure of Ω are given by

$$S = \frac{3\pi^{\frac{4}{3}}}{4^{\frac{2}{3}}} \quad \text{and} \quad |\Omega| = \frac{4\pi}{3}\cdot 2^3,$$

respectively. Consider the boundary value problem

$$-\operatorname{div}\big(\nabla u + |x||\nabla u|\nabla u\big) = \frac{1}{2}|x|u^{-0.1} + \lambda\chi_{B(0,\frac{1}{2})}(x)u^3 \quad \text{in } B(0,2), \quad u = 0 \quad \text{on } \partial B(0,2). \tag{4.11}$$

Substituting these data into (1.5) yields

$$\Lambda = \left(\frac{2}{3.1}\right)^{\frac{2}{1.1}} \left(\frac{1.1}{3.1}\right) \cdot 1 \cdot 1^{\frac{2}{1.1}} \cdot \left[\frac{\frac{3\pi^{\frac{4}{3}}}{4^{\frac{2}{3}}}}{\left(\frac{4\pi}{3} \cdot 2^3\right)^{\frac{2}{3}}}\right]^{\frac{3.1}{1.1}} \approx 0.026.$$

Therefore, Theorem 1.1 implies that problem (4.11) admits at least two positive solutions for every $\lambda \in (0, \Lambda)$.

CRediT authorship contribution statement

The authors contributed equally to this work.

Ethical approval

Not applicable.

Declaration of competing interest

The authors declare that they have no competing interests.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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