Journal of Optimization Theory and Applications (2026) 210:28
https://doi.org/10.1007/s10957-026-03058-3

Variable Exponent Mixed Double Phase Inclusions Under Nonconvex Constraints

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Received: 4 February 2026 / Accepted: 27 June 2026

Abstract

This paper investigates a class of double phase inclusion problems with variable exponents and mixed boundary conditions on bounded Lipschitz domains. The right-hand side of the problem involves both Clarke subdifferentials, describing nonconvex constraints, and convex subdifferentials, corresponding to convex constraints. By constructing a total functional associated with the problem that combines convex and locally Lipschitz components, and by applying Ekeland's variational principle together with Yosida approximation techniques and subdifferential calculus, we establish the existence of nontrivial bounded weak solutions under suitable assumptions. Our results extend the ones established in [J. Differential Equations 316 (2022), 249-269].

Keywords Bounded weak solution · Clarke subdifferential · Convex subdifferential · Double phase inclusion · Ekeland's variational principle · Variable exponent

Mathematics Subject Classification 35J15 · 35J60 · 35M86 · 35R70 · 49J40

Communicated by Jafar Zafarani.

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Published online: 17 July 2026

1 Introduction

In this paper, we study the variable exponent double phase inclusion with mixed boundary conditions given by

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \in \partial f(x,u) - \partial_c h(u), & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_1, \\ -\dfrac{\partial u}{\partial n_{p,q}} \in \partial g(x,u), & \text{on } \Gamma_2, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $|\Gamma_1| > 0$ (with $|\Gamma_1|$ denoting the Lebesgue measure of Γ_1, see also [7, 8]). The exponents $p, q \in C(\overline{\Omega})$ satisfy $1 < p(x) < N$ and $p(x) \leq q(x)$ for all $x \in \overline{\Omega}$, while $\mu(\cdot) \in L^\infty(\Omega)$ with $\mu(x) \geq 0$ for a.a. $x \in \Omega$. Moreover, we set

$$\Omega_1 := \{x \in \Omega \colon p(x) < q(x)\} \subset \Omega_0 := \{x \in \Omega \colon \mu(x) = 0\}, \quad \Omega_1 \neq \Omega_0.$$

The functions $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $g \colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous with respect to the second variable, and $\partial f(x,\cdot)$ and $\partial g(x,\cdot)$ denote the Clarke subdifferentials of $f(x,\cdot)$ and $g(x,\cdot)$, respectively. Furthermore, h is a proper, convex and lower semicontinuous function, and $\partial_c h(x,\cdot)$ denotes its subdifferential. Finally, the conormal derivative is defined by

$$\frac{\partial \phi}{\partial n_{p,q}} := \left(|\nabla \phi|^{p(x)-2}\nabla\phi + \mu(x)|\nabla\phi|^{q(x)-2}\nabla\phi\right) \cdot \nu$$

where ν is the outward unit normal vector at Γ_2.

Double phase problems have emerged as a cornerstone in the study of nonlinear elliptic PDEs, owing to their ability to model inhomogeneous materials (for instance, composites with alternating mechanical properties) and to capture phenomena such as the Lavrentiev gap and homogenization in elasticity. The pioneering works in this direction are due to Zhikov [41, 42] and Marcellini [29, 30]. The classical double phase operator, defined by

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right),$$

exhibits unbalanced growth due to the coexistence of the p- and q-power terms, where the coefficient $\mu(\cdot)$ modulates the transition between two phases (regions dominated by p- or q-elasticity). Early breakthroughs focused on the constant-exponent case. Baroni–Colombo–Mingione [3–5] as well as Colombo–Mingione [12, 13] established regularity results for weak solutions to double phase problems, including local Hölder continuity of related minimizers.

A natural generalization of constant exponent double phase problems is the variable exponent setting, where the exponents p and q depend on the spatial variable x, that is, $p, q \in C(\overline{\Omega})$. This extension is motivated by real-world applications such as anisotropic heat conduction, where the conductivity varies with position, and image processing, where nonuniform regularization is required, see, for example, Colasuonno–Squassina [11] (existence of eigenvalues for double phase variational integrals), Crespo-Blanco–Gasiński–Harjulehto–Winkert [14], Ho–Winkert [23], and Liu–Dai [25]. Variable exponent double phase operators introduce new challenges: the loss of uniform growth conditions invalidates the classical Lebesgue and Sobolev space frameworks, thus requiring the use of Musielak-Orlicz spaces which is a class of function spaces that adapt to spatially varying growth via modular functions, see Harjulehto–Hästö [22]. Crespo-Blanco–Gasiński–Harjulehto–Winkert [14] were among the first to systematically study variable exponent double phase problems, proving existence and uniqueness results via the theory of pseudomonotone operators. Vetro–Zeng [35] first established the framework for double phase problems with logarithmic perturbations and constant exponents, showing existence and uniqueness results, while Arora–Crespo-Blanco–Winkert [1, 2] and Lu–Vetro–Zeng [28] extended these results to the case of variable exponents and logarithmic perturbations.

We also mention some papers concerning the study of double phase problems with mixed boundary conditions, for instance, Zeng–Gasiński–Winkert–Bai [36], Zeng–Migórski–Tarzia [38] and Zeng–Rădulescu–Winkert [40]. For more recent works focusing on double phase problems, the reader may refer to the papers by Crespo-Blanco–Gasiński–Winkert [15] (existence of sign-changing solutions for degenerate Kirchhoff double phase problems), Frisch–Winkert [16] (boundedness, existence and uniqueness results for coupled gradient dependent double phase problems), Gambera–Guarnotta–Papageorgiou [17] (continuous spectrum for the double phase operater), Gasiński–Papageorg-iou [19] (existence of constant sign and nodal solutions), Ghosh–Lakshmi–Zhang [20] (equivalence of weak and viscosity solutions to fractional double phase problems), Rădulescu–Stapenhorst–Winkert [32] (multiplicity results for logarithmic double phase problems) and Zeng–Papageorgiou–Winkert [39] (nonemptiness, boundedness and closedness of the solution set of variable exponent double phase obstacle problem).

We point out that the right-hand side of problem (1) involves two types of subdifferentials:

- **Clarke subdifferentials** (∂f, ∂g): capture nonconvex constraints (e.g., piecewise smooth potentials), defined for locally Lipschitz functions and characterized by generalized directional derivatives;
- **Convex subdifferentials** ($\partial_c h$): describe convex constraints (e.g., obstacle conditions), defined for proper, lower semicontinuous convex functions ($h \in \mathcal{K}$ defined in (4)).

Beyond single-valued equations, variational-hemivariational inequalities, combining convex and nonconvex unilateral constraints, have become essential tools for modeling contact mechanics, friction, and phase transitions, see, for example Clarke [10], Han [21], Liu–Motreanu–Zeng [26], Liu–Papageorgiou [27], Sofonea–Migórski [33] and Tam–Hung–Liu–Yao [34] for more details.

Early work on double phase variational hemivariational inequalities focused on the case of constant exponents. Liu–Papageorgiou [27] studied the problem

$$\begin{cases} -\Delta_a^p u - \Delta^q u \in \partial j(x,u) - \partial_c k(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the weighted p-Laplacian involves a weight function $a(x) \geq c > 0$ for all $x \in \overline{\Omega}$. They proved the existence of nontrivial bounded solutions by employing Ekeland's variational principle and Yosida approximation techniques. Motivated by their work, we extend the analysis to a variable exponent double phase problem with mixed boundary conditions and vanishing weight regions ($\mu(x) = 0$ on subsets of Ω), a setting that, to the best of our knowledge, has not been previously studied. Due to the presence of the multivalued nonconvex Clarke subdifferentials ∂f and ∂g, classical convex minimization techniques fail to handle problems of the form (1). To establish the existence of weak solutions, we apply Ekeland's variational principle together with the (S_+)-property of the double phase operator in (1) and the Yosida approximation for convex subdifferentials. For more works applying Yosida approximation we refer to Jourani–Vilches [24] and Narváez–Vilches [31].

This paper is organized as follows. In Section 2 we review Musielak-Orlicz (Sobolev) spaces associated with the double phase operator given in (1), including norm-modular relationships and embedding results. We also analyze key properties of the double phase operator and we recall basic theory about subdifferentials for Clarke and convex subdifferentials. In Section 3, we establish the existence of a nontrivial bounded solution using Ekeland's variational principle. Finally, Section 4 summarizes the main results and discusses possible directions for future research.

2 Preliminaries

In this section, we recall some basic results concerning variable exponent Lebesgue spaces and a class of Musielak-Orlicz (Sobolev) spaces, mainly following Harjulehto–Hästö [22] and Crespo-Blanco–Gasiński–Harjulehto–Winkert [14]. Moreover, we also recall the basic definitions of both Clarke subdifferential and convex subdifferential, see Clarke [10] and Gasiński–Papageorgiou [18] for more details.

We denote by $C_+(\overline{\Omega})$ the subset of $C(\overline{\Omega})$ consisting of functions that are uniformly larger than one, that is,

$$C_+(\overline{\Omega}) := \left\{ \tau \in C(\overline{\Omega}) \colon \tau(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}.$$

For any exponent $r \in C_+(\overline{\Omega})$, we define its minimum and maximum values by

$$r_- := \min_{x \in \overline{\Omega}} r(x) \quad \text{and} \quad r_+ := \max_{x \in \overline{\Omega}} r(x).$$

The conjugate exponent $r' \in C_+(\overline{\Omega})$ is defined pointwise by

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Let $M(\Omega)$ denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$, where two functions are identified if they differ only on a set of Lebesgue measure zero. For a given exponent $r \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined as

$$L^{r(\cdot)}(\Omega) = \left\{u \in M(\Omega)\colon \varrho_{r(\cdot)}(u) < \infty\right\},$$

where the corresponding modular function $\varrho_{r(\cdot)}$ is given by

$$\varrho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}\,\mathrm{d}x.$$

The space $L^{r(\cdot)}(\Omega)$, endowed with the Luxemburg norm

$$\|u\|_{r(\cdot),\Omega} = \inf\left\{\lambda > 0\colon \int_\Omega \left(\frac{|u|}{\lambda}\right)^{r(x)} \mathrm{d}x \le 1\right\},$$

is a separable and reflexive Banach space. For simplicity, we will write $\|u\|_{r(\cdot),\Omega}$ instead of $\|u\|_{r(\cdot)}$ in the sequel. Moreover, its dual space is isometrically isomorphic to $L^{r'(\cdot)}(\Omega)$.

We now introduce the function $\mathcal{B}\colon \Omega \times [0,\infty) \to [0,\infty)$ associated with the variable exponent double phase operator defined by

$$\mathcal{B}(x,t) = t^{p(x)} + \mu(x)t^{q(x)}. \tag{1}$$

Throughout this work, we operate under the following main hypothesis:

(H0) The functions $p, q \in C(\overline{\Omega})$ satisfy

$$\frac{2N}{N+2} \le p(x) < N \quad \text{and} \quad p(x) \le q(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega},$$

$0 \le \mu(\cdot) \in L^\infty(\Omega)$ and

$$\Omega_1 := \{x \in \Omega\colon p(x) < q(x)\} \subset \Omega_0 := \{x \in \Omega\colon \mu(x) = 0\}, \quad \Omega_1 \neq \Omega_0.$$

Remark 2.1 While the usual assumption on the exponent p is $1 < p(x) < N$ for all $x \in \overline{\Omega}$, we impose the stronger condition

$$\frac{2N}{N+2} \le p(x) < N \quad \text{for all } x \in \overline{\Omega}.$$

This ensures the continuous embedding $W^{1,\mathcal{B}}(\Omega) \hookrightarrow L^2(\Omega)$ (see (2) for the precise definition of $W^{1,\mathcal{B}}(\Omega)$), which is essential for applying the Yosida approximation technique.

Under assumption (H0), the function $\mathcal{B}$ defined in (1) is a locally integrable N-function, see Definitions 2.5 and 2.8 of [14]. Its associated modular is defined by

$$\rho_{\mathcal{B}}(u) = \int_\Omega \mathcal{B}(x, |u|)\,\mathrm{d}x$$

and the corresponding Musielak-Orlicz space is given by

$$L^{\mathcal{B}}(\Omega) = \{u \in M(\Omega) \colon \rho_{\mathcal{B}}(u) < +\infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{B}} = \inf\left\{\lambda > 0 \colon \rho_{\mathcal{B}}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

The corresponding Sobolev-Musielak-Orlicz space is

$$W^{1,\mathcal{B}}(\Omega) := \left\{u \in L^{\mathcal{B}}(\Omega) \colon |\nabla u| \in L^{\mathcal{B}}(\Omega)\right\}, \tag{2}$$

and $W_0^{1,\mathcal{B}}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{B}}(\Omega)$. Both spaces are equipped with the norm

$$\|u\|_{1,\mathcal{B}} = \|u\|_{\mathcal{B}} + \|\nabla u\|_{\mathcal{B}}.$$

The following proposition presents the relationship between the Luxemburg norm and its corresponding modular, see Theorem 2.13 in [14].

Proposition 2.1 *Let hypotheses* (H0) *be satisfied,* $u \in L^{\mathcal{B}}(\Omega)$ *and the modular is defined by*

$$\rho_{\mathcal{B}}(u) = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right)\mathrm{d}x \quad \text{for all } u \in L^{\mathcal{B}}(\Omega).$$

Then the following statements hold:

(i) $\|u\|_{\mathcal{B}} < 1 \Longrightarrow \|u\|_{\mathcal{B}}^{q_+} \le \rho_{\mathcal{B}}(u) \le \|u\|_{\mathcal{B}}^{p_-}$;
(ii) $\|u\|_{\mathcal{B}} > 1 \Longrightarrow \|u\|_{\mathcal{B}}^{p_-} \le \rho_{\mathcal{B}}(u) \le \|u\|_{\mathcal{B}}^{q_+}$.

A function $\tau \colon \overline{\Omega} \to \mathbb{R}$ is said to be log-Hölder continuous on $\overline{\Omega}$, denoted $C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, if there exists a constant $C > 0$ such that

$$|\tau(x) - \tau(y)| \le \frac{C}{|\log|x-y||} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } 0 < |x-y| < \frac{1}{2}.$$

The embedding results stated below are adapted from Theorem 2.16 in [14] Here and in what follows, the symbol $\hookrightarrow$ denotes a continuous embedding, while $\hookrightarrow\hookrightarrow$ denotes a compact embedding.

Proposition 2.2 *Let hypotheses* (H0) *be satisfied. Then the following hold:*

(i) *If* $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, *then* $W^{1,\mathcal{B}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{B}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *for all* $r \in C(\overline{\Omega})$ *such that* $1 \le r(x) \le p^*(x)$ *for all* $x \in \overline{\Omega}$*;*
(ii) $W^{1,\mathcal{B}}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega)$, $W_0^{1,\mathcal{B}}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega)$ *for any* $r \in C(\overline{\Omega})$ *satisfying* $1 \le r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*;*
(iii) *If* $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ *for some* $\gamma > N$, *then* $W^{1,\mathcal{B}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *and* $W_0^{1,\mathcal{B}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *for all* $r \in C(\overline{\Omega})$ *such that* $1 \le r(x) \le p_*(x)$ *for all* $x \in \overline{\Omega}$*;*
(iv) $W^{1,\mathcal{B}}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega)$, $W_0^{1,\mathcal{B}}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *for any* $r \in C(\overline{\Omega})$ *satisfying* $1 \le r(x) < p_*(x)$ *for all* $x \in \overline{\Omega}$.

Recall that $p^*(\cdot)$ and $p_*(\cdot)$ are defined by

$$p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{and} \quad p_*(x) = \frac{(N-1)p(x)}{N - p(x)} \quad \text{for all } x \in \overline{\Omega},$$

and represent the critical Sobolev exponents in the domain and on the boundary, respectively.

Remark 2.2 Proposition 2.2 implies the existence of positive constants $C_p, C_{p,\Gamma} > 0$ such that

$$\|u\|_{p_-} \le C_p \|u\|_{1,\mathcal{B}} \quad \text{and} \quad \|u\|_{p_-,\Gamma_2} \le C_{p,\Gamma} \|u\|_{1,\mathcal{B}}$$

for all $u \in W^{1,\mathcal{B}}(\Omega)$. Furthermore, according to [9], if we consider the functional

$$P(u) := \int_{\Gamma_1} |u| \, \mathrm{d}\sigma$$

for $|\Gamma_1| > 0$ in Theorem 2.1 of [9], then the norm $\|\cdot\|_{1,\mathcal{B}}$ is equivalent to $\|\nabla \cdot\|_{\mathcal{B}}$ for all $u \in W^{1,\mathcal{B}}(\Omega)$. Consequently, under the condition $|\Gamma_1| > 0$, the following Poincaré-type inequality holds:

$$\|u\|_{1,\mathcal{B}} \le C_{\mathcal{B}} \|u\|_{\mathcal{B}} \quad \text{for all } u \in W^{1,\mathcal{B}}(\Omega).$$

We now turn our attention to the properties of the nonlinear operator $A\colon W^{1,\mathcal{B}}(\Omega) \to W^{1,\mathcal{B}}(\Omega)^*$ defined by

$$\langle A(u), v\rangle := \int_\Omega \frac{\mathcal{B}'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, \mathrm{d}x \tag{3}$$

for all $u, v \in X$, where $\mathcal{B}'$ stands for the derivative of $\mathcal{B}$ with respect to the second variable, and $\langle \cdot, \cdot \rangle$ donates the duality pairing between $W^{1,\mathcal{B}}(\Omega)$ and its dual $W^{1,\mathcal{B}}(\Omega)^*$. According to Theorem 3.3 in [14], the operator A satisfies the following fundamental properties.

Proposition 2.3 *Let hypotheses* (H0) *be satisfied.. Then the operator A defined in* (3) *is continuous, bounded, strictly monotone (hence, maximal monotone), and it satisfies the* (S_+)*-property, that is,*

$$u_n \rightharpoonup u \quad \textit{in } W^{1,\mathcal{B}}(\Omega) \quad \textit{and} \quad \limsup_{n\to\infty} \langle Au_n, u_n - u \rangle \leq 0,$$

imply $u_n \to u$ in $W^{1,\mathcal{B}}(\Omega)$.

Next, we recall the basic definitions and notations for convex and Clarke subdifferentials. Let X be a Banach space. We introduce the following cone of proper convex functions:

$$\mathcal{K}(X) = \{F : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} : F \text{ is convex, lower semicontinuous and } F \not\equiv +\infty\}. \tag{4}$$

Given $F \in \mathcal{K}(X)$, the conjugate (or Fenchel conjugate) of $F(\cdot)$ is the function $F^* : X^* \to \overline{\mathbb{R}}$ defined by

$$F^*\left(x^*\right) = \sup\left\{\left\langle x^*, x \right\rangle - F(x) : x \in X\right\}.$$

It follows that $F^* \in \mathcal{K}(X^*)$. The convex subdifferential of F at a point $x \in X$ is given by

$$\partial_c F(x) = \left\{x^* \in X^* : \left\langle x^*, h \right\rangle \leq F(x+h) - F(x) \text{ for all } h \in X\right\}.$$

The effective domain of F is $\operatorname{dom} F = \{x \in X : F(x) < +\infty\}$. It is well known that the restriction of F to the interior of its domain, denoted by $F\big|_{\operatorname{int} \operatorname{dom} f}$, is locally Lipschitz continuous. If $F : X \to \mathbb{R}$ is continuous and convex, then it is locally Lipschitz continuous, and we have $\partial_c F(x) = \partial F(x)$ for all $x \in X$. For a comprehensive treatment of subdifferential calculus for nonsmooth functions, we refer to Clarke [10] and Gasiński–Papageorgiou [18].

Let X be a real Banach space and X^* its topological dual. A function $F : X \to \mathbb{R}$ is said to be locally Lipschitz continuous at $u \in X$ if there exist a constant $L_u > 0$ and a neighborhood $N(u)$ of u such that

$$|F(w) - F(v)| \leq L_u \|w - v\|_X \quad \text{for all } w, v \in N(u).$$

Definition 2.1 Let $F : X \to \mathbb{R}$ be a locally Lipschitz continuous function. The Clarke generalized directional derivative of F at the point $u \in X$ in the direction $v \in X$ is defined as

$$F^\circ(u; v) = \limsup_{w \to u, t \downarrow 0} \frac{F(w + tv) - F(w)}{t}.$$

The generalized gradient (or Clarke subdifferential) $\partial F : X \to 2^{X^*}$ of F is then given by

$$\partial F(u) = \left\{ \xi \in X^* : F^\circ(u; v) \geq \langle \xi, v \rangle_{X^* \times X} \text{ for all } v \in X \right\}, \quad u \in X.$$

It is worth noting that the generalized directional derivative $F^\circ(x; \cdot)$ serves as the support function of the set $\partial F(x)$, which is nonempty, convex, and weak*-compact.

3 Existence of Nontrivial Bounded Weak Solution

In this section, we aim to establish the existence of at least one nontrivial bounded weak solution to problem (1) by applying Ekeland's variational principle, together with the (S_+)-property of the operator defined in (3) and the Yosida approximation technique. The proof is mainly inspired by the work of Liu–Papageorgiou [27]. The main assumptions on the data are stated below.

(F_0) $h \in \mathcal{K}(\mathbb{R})$ such that $h(x) \geq 0$ for all $x \in \mathbb{R}$ and $h(0) = 0$.

(F_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ are measurable with respect to the first variable, and are locally Lipschitz with respect to the second variable, fulfilling $f(x, 0) = 0$ for a.a. $x \in \Omega$ and $g(x, 0) = 0$ for a.a. $x \in \Gamma_2$.

(F_2) For any $\eta \in \partial f(x, t)$ and any $\zeta \in \partial g(x, t)$, there exist $a_1, a_2 \geq 0$, $\alpha_1 \in L^{r_1'(\cdot)}(\Omega)$ with $r_1 \in C(\overline{\Omega})$, $1 < r_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$, and $\alpha_2 \in L^{r_2'(\cdot)}(\Omega)$ with $r_2 \in C(\Gamma_2)$, $1 < r_2(x) < (p_*)_-$ for all $x \in \Gamma_2$, such that

$$|\eta| \leq a_1 |t|^{r_1(x)-1} + \alpha_1(x),$$

for a.a. $x \in \overline{\Omega}$ and for all $t \in \mathbb{R}$, as well as

$$|\zeta| \leq a_2 |t|^{r_2(x)-1} + \alpha_2(x),$$

for a.a. $x \in \Gamma_2$ and for all $t \in \mathbb{R}$.

(F_3) For any $\eta \in \partial f(x, t)$ and any $\zeta \in \partial g(x, t)$, there exist $b_1, b_2 \geq 0$, $\beta_1 \in L^{r_1'(\cdot)}(\Omega)$ with $r_1 \in C(\overline{\Omega})$, $1 < r_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$, and $\beta_2 \in L^{r_2'(\cdot)}(\Omega)$ with $r_2 \in C(\Gamma_2)$, $1 < r_2(x) < (p_*)_-$ for all $x \in \Gamma_2$, such that

$$\eta t \leq b_1 |t|^{p_-} + \beta_1(x),$$

for a.a. $x \in \overline{\Omega}$ and for all $t \in \mathbb{R}$, as well as

$$\zeta t \leq b_2 |t|^{p_-} + \beta_2(x),$$

for a.a. $x \in \Gamma_2$ and for all $t \in \mathbb{R}$. Moreover, it holds that

$$\frac{1}{C_{\mathcal{B}} q_+} - \frac{b_1 C_p}{p_-} - \frac{b_2 C_{p,\Gamma}}{p_-} > 0. \tag{5}$$

(F$_4$) There exist $\delta > 0$, $\gamma \in (1, p_-)$ and $k > 0$ such that

$$k|t|^\gamma \leq f(x,t),$$

for a.a. $x \in \Omega$ and all $|t| \leq \delta$.

Remark 3.1 Among hypotheses (F$_2$), the condition $1 < r_2(x) < (p_*)_-$ is imposed to ensure the boundedness of weak solutions to problem (1), see [37] for details, where a complete proof is provided. The constants appearing in (5) are defined in Remark 2.2. Moreover, we refer to [27] for some typical examples of functions h as in (F$_0$).

By hypotheses (F$_0$)–(F$_2$), the definition of a weak solution to problem (1) is well-defined, as stated next.

Definition 3.1 We say that $u \in W^{1,\mathcal{B}}(\Omega)$ is a weak solution to problem (1), if there exist $r_1, r_2 \in C(\overline{\Omega})$ fulfilling $1 < r_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $1 < r_2(x) < p_*(x)$ for all $x \in \Gamma_2$, $\eta, \ell \in L^{r_1'(\cdot)}(\Omega)$, $\zeta \in L^{r_2'(\cdot)}(\Gamma_2)$ with $\zeta(x) \in \partial f(x, u(x))$, $\ell \in \partial_c h(u(x))$ for a.a. $x \in \Omega$ and $\zeta \in \partial g(x, u(x))$ for a.a. $x \in \Gamma_2$ such that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x \\
&= \int_\Omega \eta v \,\mathrm{d}x + \int_{\Gamma_2} \zeta v \,\mathrm{d}\varsigma - \int_\Omega \ell v \,\mathrm{d}x,
\end{aligned}$$

for all $v \in W^{1,\mathcal{B}}(\Omega)$.

Before presenting the proof of our main results, we introduce the functionals associated with problem (1). First, define $\mathcal{I}_1 \colon W^{1,\mathcal{B}}(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{I}_1(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right) \mathrm{d}x + \int_\Omega h(u)\,\mathrm{d}x \\ \quad \text{if } h(u(\cdot)) \in L^1(\Omega), \\ +\infty \quad \text{otherwise.} \end{cases}$$

By Theorem 4.5.2 in [18], we have $\mathcal{I}_1 \in \mathcal{K}(W^{1,\mathcal{B}}(\Omega))$. Next, we define $\mathcal{I}_2, \mathcal{I}_3 \colon W^{1,\mathcal{B}}(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_2(u) = -\int_\Omega f(x,u)\,\mathrm{d}x \quad \text{and} \quad \mathcal{I}_3(u) = -\int_{\Gamma_2} g(x,u)\,\mathrm{d}\varsigma \quad \text{for all } u \in W^{1,\mathcal{B}}(\Omega).$$

According to Theorem 2.7.5 in [10], both $\mathcal{I}_2$ and $\mathcal{I}_3$ are locally Lipschitz continuous on $W^{1,\mathcal{B}}(\Omega)$. Finally, we define the total functional $\mathcal{I}\colon W^{1,\mathcal{B}}(\Omega) \to \mathbb{R}$ by

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

We are now in a position to establish our main existence result.

Theorem 3.1 *Let hypotheses* (H0) *and* (F_0)–(F_4) *be satisfied. Then problem* (1) *admits a nontrivial solution* $u_0 \in W^{1,\mathcal{B}}(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\Gamma_2)$.

Proof First, we establish the coercivity of the functional $\mathcal{I}(\cdot)$. For any sequence satisfying $\|u\|_{1,\mathcal{B}} \to +\infty$, by Proposition 2.1, Remark 2.2, hypothesis (F_0), and inequality (5), we obtain the estimates

$$\begin{aligned}
\mathcal{I}(u) &= \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \right) \mathrm{d}x - \int_\Omega f(x,u)\,\mathrm{d}x \\
&\quad - \int_{\Gamma_2} g(x,u)\,\mathrm{d}\varsigma + \int_\Omega h(x,u)\,\mathrm{d}x \\
&\geq \frac{1}{q_+} \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x) |\nabla u|^{q(x)} \right) \mathrm{d}x - \frac{b_1}{p_-} \|u\|_{p_-}^{p_-} - \frac{b_2}{p_-} \|u\|_{p_-,\Gamma_2}^{p_-} - C \\
&\geq \frac{1}{C_{\mathcal{B}} q_+} \|u\|_{1,\mathcal{B}}^{p_-} - \frac{b_1 C_p}{p_-} \|u\|_{1,\mathcal{B}}^{p_-} - \frac{b_2 C_{p,\Gamma}}{p_-} \|u\|_{1,\mathcal{B}}^{p_-} - C \\
&\geq \left(\frac{1}{C_{\mathcal{B}} q_+} - \frac{b_1 C_p}{p_-} - \frac{b_2 C_{p,\Gamma}}{p_-} \right) \|u\|_{1,\mathcal{B}}^{p_-} - C \to +\infty.
\end{aligned}$$

Hence, $\mathcal{I}(\cdot)$ is coercive. Combining the coercivity of $\mathcal{I}(\cdot)$ with hypotheses (F_0) and (F_3), we further deduce that $\mathcal{I}(\cdot)$ is bounded from below. In particular, there exists a constant $c_m > -\infty$ such that

$$\inf_{u \in W^{1,\mathcal{B}}(\Omega)} \mathcal{I}(u) = c_m.$$

By applying Ekeland's variational principle, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{B}}(\Omega)$ such that

$$\mathcal{I}(u_n) \downarrow c_m = \inf_{u \in W^{1,\mathcal{B}}(\Omega)} \mathcal{I}(u), \tag{6}$$

and for all $v \in W^{1,\mathcal{B}}(\Omega)$,

$$\mathcal{I}(u_n) \leq \mathcal{I}(v) + \frac{1}{n} \|v - u_n\|_{1,\mathcal{B}}. \tag{7}$$

We define the integral functional $\widehat{\ell}\colon W^{1,\mathcal{B}}(\Omega) \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ by

$$\widehat{\ell}(u) = \begin{cases} \displaystyle\int_\Omega h(u)\,\mathrm{d}x & \text{if } h(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

According to Theorem 4.5.2 in [18], we have $\widehat{\ell} \in \mathcal{K}(W^{1,\mathcal{B}}(\Omega))$, where $\mathcal{K}$ denotes the class of convex, lower semicontinuous functionals that are not identically $+\infty$. For any $y \in W^{1,\mathcal{B}}(\Omega)$, set $v = u_n + \lambda(y - u_n)$ with $\lambda \in [0, 1]$ and $n \in \mathbb{N}$. From (7), we obtain

$$\begin{aligned}
-\frac{1}{n}\left\|y-u_n\right\|_{1,\mathcal{B}} \leq{}& \frac{1}{\lambda}\left(\mathcal{I}_2\left(u_n+\lambda\left(y-u_n\right)\right)-\mathcal{I}_2\left(u_n\right)\right)\\
&+\frac{1}{\lambda}\left(\mathcal{I}_3\left(u_n+\lambda\left(y-u_n\right)\right)-\mathcal{I}_3\left(u_n\right)\right)\\
&+\frac{1}{\lambda}\int_\Omega\left(\frac{1}{p(x)}|\nabla u_n+\lambda\nabla(y-u_n)|^{p(x)}-|\nabla u_n|^{p(x)}\right)\mathrm{d}x\\
&+\frac{1}{\lambda}\int_\Omega\left(\frac{\mu(x)}{q(x)}|\nabla u_n+\lambda\nabla(y-u_n)|^{q(x)}-|\nabla u_n|^{q(x)}\right)\mathrm{d}x\\
&+\int_\Omega\left(h(y)-h\left(u_n\right)\right)\,\mathrm{d}x,
\end{aligned}$$

where the coercivity of $h(\cdot)$ has been used. Passing to the limit as $\lambda \to 0^+$, we arrive at the inequality, for all $y \in W^{1,\mathcal{B}}(\Omega)$,

$$\begin{aligned}
-\frac{1}{n}\left\|y-u_n\right\|_{1,\mathcal{B}} \leq{}& \mathcal{I}_2^\circ\left(u_n; y-u_n\right)+\mathcal{I}_3^\circ\left(u_n; y-u_n\right)\\
&+\left\langle A\left(u_n\right), y-u_n\right\rangle+\int_\Omega h(y)\,\mathrm{d}x-\int_\Omega h(u_n)\,\mathrm{d}x.
\end{aligned} \tag{8}$$

Taking $y = 0$ in (8) yields, for all $n \in \mathbb{N}$,

$$-\frac{1}{n}\left\|u_n\right\|_{1,\mathcal{B}} \leq \mathcal{I}_2^\circ\left(u_n; -u_n\right)+\mathcal{I}_3^\circ\left(u_n; -u_n\right)+\left\langle A\left(u_n\right), -u_n\right\rangle-\int_\Omega h(u_n)\,\mathrm{d}x$$

for all $n \in \mathbb{N}$. Next, we choose $-\eta_n^* \in \partial\mathcal{I}_2(u_n)$ and $-\zeta_n^* \in \partial\mathcal{I}_3(u_n)$ such that, for all $n \in \mathbb{N}$,

$$\mathcal{I}_2^\circ\left(u_n; -u_n\right)=\left\langle\eta_n^*, u_n\right\rangle, \tag{9}$$

$$\mathcal{I}_3^\circ\left(u_n; -u_n\right)=\left\langle\zeta_n^*, u_n\right\rangle. \tag{10}$$

By Corollary 4.4.75 and Theorem 4.5.19 of [18], it follows that

$$\begin{aligned}
&\eta_n^* \in S^{r_1'}_{\partial f(\cdot,u_n(\cdot))}=\left\{j^* \in L^{r_1'(\cdot)}(\Omega)\colon\ j^*(x) \in \partial f\left(x, u_n(x)\right)\ \text{a.e. in }\Omega\right\}\\
&\left\langle\eta_n^*, \eta_n\right\rangle=\int_\Omega u_n^* u_n\,\mathrm{d}x\ \text{for all } n \in \mathbb{N},
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\zeta_n^* \in S^{r_2'}_{\partial g(\cdot, u_n(\cdot))} = \left\{ j^* \in L^{r_2'(\cdot)}(\Gamma_2) \colon \ j^*(x) \in \partial g\left(x, u_n(x)\right) \ \text{a.e. on } \Gamma_2 \right\} \\
&\left\langle \zeta_n^*, \zeta_n \right\rangle = \int_{\Gamma_2} u_n^* u_n \,\mathrm{d}\varsigma \ \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{12}$$

Combining (8), (9), (10), (11), (12), and noting that $h \geq 0$, we obtain

$$\langle A\left(u_n\right), u_n \rangle - \int_\Omega \eta_n^* u_n \,\mathrm{d}x - \int_{\Gamma_2} \zeta_n^* u_n \,\mathrm{d}\varsigma \leq \frac{1}{n} \left\| u_n \right\|_{1,\mathcal{B}} \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Furthermore, by Proposition 2.1, Remark 2.2, hypothesis (F_0), and (5), we deduce

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p(x)} + \mu(x) |\nabla u_n|^{q(x)} \right) \mathrm{d}x - \int_\Omega \eta_n^* u_n \,\mathrm{d}x - \int_{\Gamma_2} \zeta_n^* u_n \,\mathrm{d}\varsigma \\
&\geq \frac{1}{C_{\mathcal{B}}} \|u_n\|_{1,\mathcal{B}}^{p_-} - b_1 C_p \|u_n\|_{1,\mathcal{B}}^{p_-} - b_2 C_{p,\Gamma} \|u_n\|_{1,\mathcal{B}}^{p_-} - C \\
&\geq \left(\frac{1}{C_{\mathcal{B}}} - b_1 C_p - b_2 C_{p,\Gamma} \right) \|u_n\|_{1,\mathcal{B}}^{p_-} - C.
\end{aligned} \tag{14}$$

Combining (13) and (14), we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{B}}(\Omega)$ is bounded. Hence, up to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in } W^{1,\mathcal{B}}(\Omega) \quad \text{and} \quad u_n \to u_0 \quad \text{in } L^{r_1(\cdot)}(\Omega) \text{ and in } L^{r_2(\cdot)}(\Gamma_2). \tag{15}$$

Choose $\widehat{\eta}_n^* \in \partial \mathcal{I}_2(u_n)$ such that

$$\mathcal{I}_2^\circ\left(u_n, u_0 - u_n\right) = \left\langle \eta_n^*, u_0 - u_n \right\rangle.$$

Then, by (15),

$$\mathcal{I}_2^\circ\left(u_0, u_0 - u_n\right) = \int_\Omega \eta_n^* \left(u_0 - u_n\right) \,\mathrm{d}x \to 0. \tag{16}$$

Similarly, for some $\widehat{\zeta}_n^* \in \partial \mathcal{I}_3\left(u_n\right)$ satisfying

$$\mathcal{I}_3^\circ\left(u_n, u_0 - u_n\right) = \left\langle \zeta_n^*, u_0 - u_n \right\rangle,$$

we have

$$\mathcal{I}_3^\circ\left(u_0, u_0 - u_n\right) = \int_{\Gamma_2} \zeta_n^* \left(u_0 - u_n\right) \,\mathrm{d}\varsigma \to 0. \tag{17}$$

Since the integral functional $L^{r_1(\cdot)}(\Omega) \ni u \mapsto \int_\Omega h(u)\,\mathrm{d}x$ is lower semicontinuous, it follows that

$$\int_\Omega h(u_0)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega h(u_n)\,\mathrm{d}x. \tag{18}$$

Taking $y = u_0 \in W^{1,\mathcal{B}}(\Omega)$ in (8), letting $n \to \infty$, and using (16), (17) as well as (18), we obtain

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u_0\rangle \leq 0,$$

By the (S_+)-property of the operator A (see Proposition 2.3), this implies

$$u_n \to u_0 \quad \text{in } W^{1,\mathcal{B}}(\Omega). \tag{19}$$

Note that the mapping

$$u \mapsto \rho_{\mathcal{B}}(\nabla u) = \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \right) \mathrm{d}x$$

is convex. Combining this property with (8), we have, for all $n \in \mathbb{N}$,

$$-\frac{1}{n} \|y - u_n\|_{1,\mathcal{B}} \leq \mathcal{I}_2^\circ(u_n; y - u_n) + \mathcal{I}_3^\circ(u_n; y - u_n) + \mathcal{I}_1(y) - \mathcal{I}_1(u_n).$$

Letting $n \to \infty$ and using (19), the upper semicontinuity of the mappings $(u, j) \mapsto \mathcal{I}_2^\circ(u; j)$ and $(u, j) \mapsto \mathcal{I}_3^\circ(u; j)$ (see Proposition 2.1.1 in [10]), together with the lower semicontinuity of $\mathcal{I}_1(\cdot)$, yields

$$0 \leq \mathcal{I}_2^\circ(u_0; y - u_0) + \mathcal{I}_3^\circ(u_0; y - u_0) + \mathcal{I}_1(y) - \mathcal{I}_1(u_0). \tag{20}$$

Define the auxiliary functionals

$$\xi_1(y) = \mathcal{I}_1(y) - \mathcal{I}_1(u_0), \quad \xi_2(y) = \mathcal{I}_2^\circ(u_0; y - u_0), \quad \text{and} \quad \xi_3(y) = \mathcal{I}_3^\circ(u_0; y - u_0).$$

Then

$$\begin{aligned} &-\xi_1 \in \mathcal{K}\left(W^{1,\mathcal{B}}(\Omega)\right) \text{ and } \partial_c \xi_1(u_0) = \partial_c \mathcal{I}_1(u_0), \\ &-\xi_2(\cdot), -\xi_3(\cdot) \text{ are continuous, convex} \\ &\qquad \text{and } \partial_c \xi_2(u_0) = \partial \mathcal{I}_2(u_0), \partial_c \xi_3(u_0) = \partial \mathcal{I}_3(u_0). \end{aligned}$$

By the rules of convex subdifferential calculus (see Proposition 4.4.31 in [18]),

$$\begin{aligned} \partial_c(\xi_1 + \xi_2 + \xi_3)(u_0) &= \partial_c \xi_1(u_0) + \partial_c \xi_2(u_0) + \partial_c \xi_3(u_0) \\ &= \partial_c \mathcal{I}_1(u_0) + \partial \mathcal{I}_2(u_0) + \partial \mathcal{I}_3(u_0). \end{aligned}$$

By the definition of the convex subdifferential,

$$
\begin{aligned}
&\partial_c\left(\xi_1+\xi_2+\xi_3\right)\left(u_0\right)\\
&=\Big\{z^* \in W^{1,\mathcal{B}}(\Omega)^*\colon \left\langle z^*, y-u_0\right\rangle \leq \xi_1(y)-\xi_1\left(u_0\right)+\xi_2(y)+\xi_3(y)\\
&\qquad =\mathcal{I}_1(y)-\mathcal{I}_1\left(u_0\right)+\mathcal{I}_2^{\circ}\left(u_0; y-u_0\right)+\mathcal{I}_3^{\circ}\left(u_0; y-u_0\right) \text{ for all } y \in W^{1,\mathcal{B}}(\Omega)\Big\}.
\end{aligned}
$$

From (20), it follows that

$$
0 \in \partial_c\left(\xi_1+\xi_2+\xi_3\right)\left(u_0\right),
$$

which implies

$$
0 \in \partial_c \mathcal{I}_1\left(u_0\right)+\partial \mathcal{I}_2\left(u_0\right)+\partial \mathcal{I}_3\left(u_0\right).
$$

Consequently, there exist $z_1^* \in \partial \mathcal{I}_1(u_0)$, $z_2^* \in \partial \mathcal{I}_2(u_0)$, and $z_3^* \in \partial \mathcal{I}_3(u_0)$ such that

$$
z_1^*+z_2^*+z_3^*=0. \tag{21}
$$

By Proposition 2.3.1 and Theorem 2.7.5 in [10], we have

$$
\begin{aligned}
-z_2^* \in S_{\partial f(\cdot,u_0(\cdot))}^{r_1'} &= \left\{j^* \in L^{r_1'(\cdot)}(\Omega)\colon j^*(x) \in \partial f\left(x, u_0(x)\right) \text{ a.e. in } \Omega\right\},\\
-z_3^* \in S_{\partial g(\cdot,u_0(\cdot))}^{r_2'} &= \left\{j^* \in L^{r_2'(\cdot)}(\Gamma_2)\colon j^*(x) \in \partial g\left(x, u_0(x)\right) \text{ a.e. on } \Gamma_2\right\}.
\end{aligned}
$$

Next, we define

$$
J_h(u)=\int_{\Omega} h(u)\, \mathrm{d}x, \quad u \in W^{1,\mathcal{B}}(\Omega).
$$

Referring to the proof of Proposition 4.4.31 in [18], we obtain

$$
A\left(u_0\right)+\partial_c J_h\left(u_0\right) \subseteq \partial_c \mathcal{I}_1\left(u_0\right). \tag{22}
$$

Moreover, by hypothesis $(H0)$, we have $\frac{2N}{N+2} \leq p(x)$ for all $x \in \overline{\Omega}$. Combining this with Proposition 2.2, we deduce the continuous embeddings

$$
W^{1,\mathcal{B}}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega) \hookrightarrow L^2(\Omega).
$$

Define the functional $\widehat{\mathcal{I}}_1\colon L^2(\Omega) \to \overline{\mathbb{R}}=\mathbb{R} \cup\{+\infty\}$ by

$$
\widehat{\mathcal{I}}_1(u)=\begin{cases}
\int_{\Omega}\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right) \mathrm{d}x+J_h(u) \\
\quad \text{if } u \in W^{1,\mathcal{B}}(\Omega), h(u(\cdot)) \in L^1(\Omega), \\
+\infty \quad \text{otherwise.}
\end{cases}
$$

Then $\widehat{\mathcal{I}}_1 \in \mathcal{K}(L^2(\Omega))$, where $\mathcal{K}$ denotes the class of convex, lower semicontinuous functionals that are not identically $+\infty$. Furthermore, $\operatorname{dom}\widehat{\mathcal{I}}_1 = \operatorname{dom}\mathcal{I}_1$ and $\widehat{\mathcal{I}}_1|_{W^{1,\mathcal{B}}(\Omega)} = \mathcal{I}_1$. Let $i\colon W^{1,\mathcal{B}}(\Omega) \to L^2(\Omega)$ be the inclusion map. Then $\mathcal{I}_1 = \widehat{\mathcal{I}}_1 \circ i$. By Proposition 4.4.33 in [18], it follows that

$$\partial_c \mathcal{I}_1(u) \supseteq i^* \partial_c \widehat{\mathcal{I}}_1(u) \quad \text{for all } u \in W^{1,\mathcal{B}}(\Omega), \tag{23}$$

where i^* denotes the adjoint operator of i.

For $\lambda > 0$, let $(\partial J_h)_\lambda$ be the Yosida approximation of the maximal monotone operator $\partial J_h(\cdot)$ (see Definition 3.2.36 in [18]). Then,

$$\begin{aligned}\langle A(u), (\partial J_h)_\lambda (u)\rangle &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla (\partial J_h)_\lambda (u)\, \mathrm{d}x \\ &= \int_\Omega \mathcal{B}(x, \nabla u)\, (\partial J_h)'_\lambda (u)\, \mathrm{d}x \geq 0.\end{aligned}$$

By Proposition 2.17 and Theorem 4.4 in [6], the operator $u \mapsto A(u) + \partial J_h(u)$ is maximal monotone. Hence, from (22) and (23), we have

$$\partial_c \mathcal{I}_1 (u_0) = i^* \partial_c \widehat{\mathcal{I}}_1 (u_0)\,.$$

From (21), it follows that

$$z_1^* = -z_2^* - z_3^* \in L^{r_1'(\cdot)}(\Omega) \cup L^{r_2'(\cdot)}(\Gamma_2).$$

Therefore, there exist an operator $\widehat{A}$ with $\mathrm{Gr}\widehat{A} = \mathrm{Gr}A|_{W^{1,\mathcal{B}}(\Omega)\times L^2(\Omega)}$ and $\ell^* \in S^{r_1'(\cdot)}_{\partial_c h(u_0)}$ such that

$$z_1^* = \widehat{A}\,(u_0) + \ell^*.$$

Consequently,

$$\widehat{A}\,(u_0) + \ell^* + z_2^* + z_3^* = 0.$$

Take $\phi \in C_c^\infty(\Omega)$ and consider the duality pairing:

$$\begin{aligned}&\left\langle \widehat{A}\,(u_0)\,, \phi\right\rangle_{W^{1,\mathcal{B}}(\Omega)^* \times W^{1,\mathcal{B}}(\Omega)} \\ &= \left\langle -\ell^*, \phi\right\rangle_{L^{r_1'(\cdot)}(\Omega)\times L^{r_1(\cdot)}(\Omega)} + \left\langle -z_2^*, \phi\right\rangle_{L^{r_1'(\cdot)}(\Omega)\times L^{r_1(\cdot)}(\Omega)} \\ &\quad + \left\langle -z_3^*, \phi\right\rangle_{L^{r_2'(\cdot)}(\Gamma_2)\times L^{r_2(\cdot)}(\Gamma_2)}.\end{aligned}$$

Hence,

$$\langle A\,(u_0)\,, \phi\rangle = \int_\Omega \left(-\ell^*\right) \phi\, \mathrm{d}x + \int_\Omega \left(-z_2^*\right) \phi\, \mathrm{d}x + \int_{\Gamma_2} \left(-z_3^*\right) \phi\, \mathrm{d}\varsigma.$$

That is,

$$\int_{\Omega} \left(|\nabla u_0|^{p(x)-2} \nabla u_0 + \mu(x) |\nabla u_0|^{q(x)-2} \nabla u_0 \right) \cdot \nabla \phi \, \mathrm{d}x$$
$$= \int_{\Omega} \left(-\ell^* \right) \phi \, \mathrm{d}x + \int_{\Omega} \left(-z_2^* \right) \phi \, \mathrm{d}x + \int_{\Gamma_2} \left(-z_3^* \right) \phi \, \mathrm{d}\varsigma.$$

Since $C_c^\infty(\Omega)$ is dense in $W^{1,\mathcal{B}}(\Omega)$, we conclude that

$$- \operatorname{div} \left(|\nabla u_0|^{p(x)-2} \nabla u_0 + \mu(x) |\nabla u_0|^{q(x)-2} \nabla u_0 \right)$$
$$= -\ell^* - z_2^* - z_3^* \in -\partial_c h\,(u_0) + \partial f\,(x, u_0) + \partial g\,(x, u_0) \quad \text{in } \Omega.$$

Moreover, by (7) and (19),

$$\mathcal{I}\,(u_0) = \inf \left[\mathcal{I}(u) \colon u \in W^{1,\mathcal{B}}(\Omega) \right] = c_m.$$

From hypothesis (F_4) and the fact that $1 < \gamma < q$, it follows that $\mathcal{I}(tu) < 0$ for all $u \in C_0^1(\overline{\Omega})$ and sufficiently small $|t| < 1$. This implies

$$\mathcal{I}\,(u_0) = c_m < 0 = \mathcal{I}(0),$$

which shows that $u_0 \neq 0$. Finally, by Theorem 4.2 in [37], we have $u_0 \in L^\infty(\Omega) \cap L^\infty(\Gamma_2)$, completing the proof. □

4 Conclusions

This paper investigates a variable exponent double phase inclusion problem with mixed boundary conditions. The main contributions can be summarized as follows:

By constructing the total functional

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where $\mathcal{I}_1$ is convex and $\mathcal{I}_2, \mathcal{I}_3$ are locally Lipschitz functionals, and by combining Ekeland's variational principle (to obtain the minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$), Yosida approximation techniques (to deal with the multivalued nature of convex subdifferentials), and subdifferential calculus (to establish inclusion relationships), the existence of nontrivial bounded weak solution is proved under assumptions (H0) and (F_0)–(F_4) (see Theorem 3.3). Moreover, the obtained solution u_0 satisfies $u_0 \in W^{1,\mathcal{B}}(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\Gamma_2)$.

Building on the findings of this work, several natural extensions can be explored in future research:

- **Nonlocal double phase operators:** The present study addresses local operators of the form $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u)$. A promising direction is to extend the analysis to nonlocal double phase operators, such as the fractional variant $(-\Delta)^{\alpha}_{p(x),q(x)}u$.

- **Logarithmic perturbations in variable exponent settings:** The current framework does not include logarithmic terms such as $\mu(x)|\nabla u|^{q(x)}\log(1+|\nabla u|)$, which significantly modify the growth behavior of the operator. Incorporating such perturbations would generalize recent results published in [1, 2, 28].

Acknowledgements The authors are grateful to the referees for their careful reading and insightful comments, which have substantially enhanced the quality of the manuscript. This paper was partially supported by the National Natural Science Foundation of China under Grant No. 12371312, the Natural Science Foundation of Chongqing under Grant No. CSTB2024NSCQ-JQX0033, and the Science and Technology Research Program of Chongqing Municipal Education Commission No. KJZD-M202500502, and Startup Project of doctor Scientific Research of Chongqing Normal University No. 24XLB034.

Author Contributions The authors contributed equally to this paper.

Data Availability Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Competing interests There is no conflict of interest.

Ethical Statement This work is purely theoretical in nature and does not involve any human subjects, animal experiments, personal data, or any materials requiring ethical approval. All results are derived from abstract mathematical reasoning and do not raise any ethical concerns. The authors have adhered to the highest standards of research integrity, including those related to authorship, plagiarism, and data fabrication.

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