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Anisotropic nonlocal double phase problems with logarithmic perturbation: maximum principle and qualitative analysis of solutions

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Abstract

In this paper, we study multivalued nonlocal elliptic problems driven by the fractional double phase operator with variable exponents and ω-logarithmic perturbation formulated by

$$\begin{cases} (-\Delta)^s_{\mathcal{H}} u \in \mathcal{F}(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We are going to establish maximum principles for the fractional perturbed double phase operator and show the boundedness of weak solutions to the above problem. Finally, under appropriate assumptions we discuss the existence of infinitely many small (non-negative) weak solutions to a single-valued nonlocal double phase problem.

Keywords A priori bounds · De Giorgi's iteration · Fractional logarithmic double phase operator · Localization method · Maximum principle · Multivalued problem · Variational methods

Mathematics Subject Classification 35B50 · 35J15 · 35R11 · 35R70

1 Introduction

In the last years, problems involving fractional-order operators have been studied intensively due to their mathematical challenges and various real applications in fluid mechanics, relativistic quantum mechanics, conformal geometry, probability and molecular dynamics, see Bertoin [11], Cabré–Tan [14], Caffarelli–Vasseur [15] and Chen–Li–Ma [18] for more details. Particularly, the studies for problems involving fractional double phase operators have attracted much attention for their compelling theoretical framework and diverse practical applications. Recently, de Albuquerque–de Assis–Carvalho–Salort [23] established some abstract results on a new class of fractional Musielak–Sobolev spaces including uniformly convexity, Brézis–Lieb type Lemma and Radon-Riesz property to the modular function, (S_+)-property and monotonicity. In this paper, based on the results obtained by de Albuquerque–de

This paper is dedicated with esteem to Professor Shujie Li on the occasion of his 85th birthday

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Assis–Carvalho–Salort [23] for the solution space and the operator we deal with multivalued nonlinear problems with Dirichlet boundary condition of the form

$$\begin{cases} (-\Delta)^s_{\mathcal{H}} u \in \mathcal{F}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

for $u \in W_0^{s,\mathcal{H}}(\Omega)$ (see Sect. 2), where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $\mathcal{F}\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is multivalued function, the associated variable exponent fractional double phase operator with logarithmic perturbation is given as

$$\begin{aligned} (-\Delta)^s_{\mathcal{H}} u(x) &:= C_{N,s,p,q} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \mathcal{H}'\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{\mathrm{d}y}{|x-y|^{N+s}} \\ &= C_{N,s,p,q} \,\mathrm{PV} \int_{\mathbb{R}^N} \mathcal{H}'\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{\mathrm{d}y}{|x-y|^{N+s}} \end{aligned} \tag{1.2}$$

with $B_\varepsilon(x) := \{z \in \mathbb{R}^N : |z-x| < \varepsilon\}$, $s \in (0,1)$, $C_{N,s,p,q}$ is some constant depending on N, s, p, q while PV denotes the Cauchy principle value and $\mathcal{H}\colon \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is defined as

$$\mathcal{H}(x,y,t) = \left[t^{p(x,y)} + \mu(x,y)t^{q(x,y)}\right] \log(e + \omega t), \tag{1.3}$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and for all $t \geq 0$, where $\omega \geq 0$, $p, q \in C(\mathbb{R}^N \times \mathbb{R}^N)$ such that $p(x,y) = p(y,x)$, $q(x,y) = q(y,x)$ as well as $1 < p(x,y) < \frac{N}{s}$, $p(x,y) \leq q(x,y)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, and $0 \leq \mu(\cdot,\cdot) \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ satisfies $U_1 := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : p(x,y) < q(x,y)\} \nsubseteq U_0 := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \mu(x,y) = 0\}$ and $\mu(x,y) = \mu(y,x)$.

As we can see, problem (1.1) is driven by a type of fractional double phase operator, which is developed from the classical double phase operator given by

$$\operatorname{div}\left(|\nabla v|^{p-2}\nabla v + \mu(x)|\nabla v|^{q-2}\nabla v\right),$$

associated with the following energy functional

$$v \mapsto \int_\Omega \left(|\nabla v|^p + \mu(x)|\nabla v|^q\right) \mathrm{d}x. \tag{1.4}$$

This type of energy functional was introduced first by Zhikov in 1986 to describe the nature of certain phenomena occurring in the theory of elasticity, for example, it can describe the mathematical models of strongly anisotropic materials as well as the Lavrentiev phenomenon, see Zhikov [71, 72]. In fact, energy functionals with of the form (1.4) characterizes the phenomenon where the energy density varies its ellipticity and growth characteristics, contingent upon the specific location within the domain. It can also describe the geometric properties of a composite formed from distinct two materials characterized by the power hardening exponents p and q. Since the energy functional (1.4) exhibits ellipticity in the gradient of order q when the modulating coefficient $\mu(\cdot) \neq 0$ and exhibits ellipticity in the gradient of order p when the modulating coefficient $\mu(\cdot) = 0$, we call it double phase.

In recent years, the classical double phase operator has been extended to various new class of operators. Crespo-Blanco–Gasiński–Harjulehto–Winkert [20] considered the double phase operator with variable exponents defined by

$$\operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v + \mu(x)|\nabla v|^{q(x)-2}\nabla v\right),$$

and established some basic properties of this type of operator and the associated Musielak–Orlicz Sobolev spaces. Furthermore, Vetro–Zeng [60] studied a type of double phase energy functional with log L-perturbed p, q-growth defined by

$$\operatorname{div}\left(\frac{\mathcal{H}'_L(x, |\nabla v|)}{|\nabla v|}\nabla v\right) \quad \text{with} \quad \mathcal{H}_L = \left[t^p + \mu(x)t^q\right]\log(e+t).$$

They obtained the properties of the associated Musielak Orlicz-Sobolev space and then proved the existence and uniqueness results of weak solution for Dirichlet double phase problems, see also Lu–Vetro–Zeng [47] for detailed results concerning double phase energy operator with log L-perturbed $p(\cdot)$, $q(\cdot)$-growth. For more results involving the double phase type operator with logarithmic perturbation we refer to the recent work by Arora–Crespo-Blanco–Winkert [4] who focused on the operator

$$\operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v + \mu(x)\left[\log(e+|\nabla v|) + \frac{|\nabla v|}{q(x)(e+|\nabla v|)}\right]|\nabla v|^{q(x)-2}\nabla v\right),$$

and established the existence and multiplicity results to the related double phase problems. We also mention some recent existence results for double phase problems, see the works by Guarnotta–Livrea–Winkert [31] (variable exponent double phase systems), Liu–Dai [46] (existence and multiplicity results of double phase problems), Vetro–Zeng [60] (double phase Dirichlet problems), Zeng–Bai–Gasiński–Winkert [66] (multivalued double phase implicit obstacle problems), Zeng–Rădulescu–Winkert [67] (double phase implicit obstacle problems), and Zeng–Rădulescu–Winkert [68] (nonlocal double phase implicit obstacle problems). Finally, we refer to important works concerning the regularity of local minimizers of related double phase functionals, see Baroni–Colombo–Mingione [8], Beck–Mingione [9], Colombo–Mingione [19], Fuchs–Mingione [28] and Marcellini [48, 49], see also the references therein.

It is worth mentioning that more and more impressive studies on fractional double-phase problems have been carried out recently. To be more precise, by using variational and topological arguments, the existence of weak solutions to various fractional elliptic or parabolic double phase problems have been established by Ambrosio [2] (existence of a nontrivial non-negative solution), Ambrosio–Isernia [3] (existence of infinitely many solutions), Bhakta–Mukherjee [12] (existence of infinitely many nontrivial solutions), Xiang–Ma [65] (existence of normalized ground state solutions), Zhang–Zhang [69] (existence and concentration phenomena of positive solutions) and Zhang–Zhang–Rădulescu [70] (existence of positive ground state solutions). In the direction of Hölder continuity and boundedness of weak solutions for nonlocal double phase problems we refer to the papers by Byun–Ok–Song [13], Fang–Zhang [27] and Prasad–Tewary [54]. In terms of practical application, both integer and fractional double phase problems can be used in a variety of real-world problems, such as, obstacle problems, nonlinear Derrick's problem, transonic flow problems, optimization, finance and image processing. More details can be found in the works by Bahrouni–Rădulescu–Repovš [6] Benci–D'Avenia–Fortunato–Pisani [10] and Charkaoui–Ben-loghfyry [16]. For very recent advances regarding local and nonlocal double phase problems, we refer to Guo–Liang–Lin–Pucci [32], who established global bifurcation results for double phase problems; Liang–Pucci–Van-Nguyen [44], who obtained multiplicity and concentration results for certain fractional variable-exponent double phase Choquard equations; Pucci–Wang–Zhang [56], who demonstrated the multiplicity and stability of normalized solutions in nonlocal double phase problems; and Pucci–Xiang [57], who found multi-bump solutions for logarithmic double phase critical Schrödinger equations.

On the one hand, we are going to show the maximum principle for the perturbed fractional double phase operator. It is well known that the maximum principle is useful for investigating the uniqueness and continuous dependence of classical solutions for elliptic and parabolic boundary value problems, see Pucci–Serrin [55], Ladyzhenskaya–Solonnikov–Uralt́seva [42] and Vladimirov [64]. The general form of the maximum principle implies that the appropriate solution of the homogeneous equation attains its extreme values on the boundary of the domain and allows to derive an approximation for the maximum magnitude of the solution. Particularly, maximum principles can be applied to investigate the stability and convergence of the difference solution in a uniform norm, see for example Crouzeix–Thomée [22] and Thomée [58, 59]. Moreover, in Chen–Li [17] and Hu–Peng [40], the authors combined the maximum principle for anti-symmetric functions and the method of moving planes to establish the symmetry and monotonicity of positive solutions to nonlocal double phase problems. Motivated by these results, we will show the maximum principle for the nonlocal double phase operator with logarithmic perturbation in Sect. 3.

On the other hand, we are interested to get a priori bounds for weak solutions of problem (1.1) with subcritical and critical growth by utilizing De Giorgi's iteration (or De Giorgi–Nash–Moser theory) and a localization method. The beginning of research into the De Giorgi–Nash–Moser theory goes back to the works by De Giorgi [24], Nash [53] as well as Moser [51]. This theory is a powerful tool for proving local and global L^∞-bounds of weak solutions and establishing the Harnack inequality and the Hölder continuity for weak solutions. For more details we refer to the monographs of Gilbarg–Trudinger [30], Ladyženskaja–Solonnikov–Ural'ceva [42], Ladyženskaja–Ural'ceva [43] and Lieberman [45]. Our proofs for the boundedness of weak solutions of problem (1.1) are mainly inspired by the papers of Ho–Kim [35] (nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian), Ho–Kim–Winkert–Zhang [38] (quasilinear elliptic equations involving variable exponents critical growth), Ho–Winkert [39] (generalized double phase problems with critical and subcritical growth) and Winkert–Zacher [62, 63] (nonlinear elliptic equations with nonstandard growth). In addition, motivated by the works of Ho–Kim [35] and Wang [61], we will show the existence of infinitely many small solutions to the nonlinear problems driven by the operator given in (1.2) (see Sect. 5) by employing the boundedness of weak solutions obtained in Sect. 4. More works related to L^∞-bounds can be found in Barletta–Cianchi–Marino [7], Crespo-Blanco–Winkert [21], Frisch–Winkert [29], and Marino–Winkert [50].

To the best of our knowledge, the maximum principle for the perturbed nonlocal double phase operator (1.2) and the boundedness of weak solutions to problems driven by the fractional double phase operator with variable exponents and logarithmic perturbation have not been studied yet. Moreover, problem (1.1) contains many interesting special cases as follows:

(P1) Let $\omega = 0$, $\mu = 0$ in $\mathcal{H}$ (i.e. $\mathcal{H}(x, y, t) = t^{p(x,y)} =: \mathcal{H}_1(x, y, t)$). Moreover let $\mathcal{F}$ be a single-valued Carathéodory function f, then problem (1.1) becomes the nonlinear elliptic problem involving the fractional $p(\cdot)$-Laplacian

$$\begin{cases} (-\Delta)^s_{p(x)}\, u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega; \end{cases}$$

(P2) Let $\omega = 0$ (i.e. $\mathcal{H}(x, y, t) = t^{p(x,y)} + \mu(x, y)t^{q(x,y)} =: \mathcal{H}_2(x, y, t)$), then problem (1.1) becomes the nonlocal elliptic variable exponents double phase problem;

(P3) Let $1 < p(\cdot) \equiv p$ and $1 < q(\cdot) \equiv q$ (i.e. $\mathcal{H}(x, y, t) = \left[t^p + \mu(x, y)t^q\right]\log(e+\omega t) =: \mathcal{H}_3(x, y, t)$), then problem (1.1) becomes the perturbed nonlocal double phase problem with constant exponents.

(P4) Let $\omega = 0$ and $1 < p(\cdot) \equiv p, 1 < q(\cdot) \equiv q$ (i.e. $\mathcal{H}(x, y, t) = t^p + \mu(x, y)t^q =: \mathcal{H}_4(x, y, t)$), then problem (1.1) becomes nonlocal double phase problem.

This paper is organized as follows. In Sect. 2, we recall several basic definitions and notations of variable exponent Lebesgue spaces and Musielak–Orlicz spaces concerning the perturbed double phase function $\mathcal{H}$. Furthermore, we will give the definition and basic properties of the fractional Musielak–Sobolev space $W^{s,\mathcal{H}}(\Omega)$, which is the solution space of the considered problem. In Sect. 3, we establish the maximum principle for the fractional perturbed double phase operator (1.2) while in Sect. 4 we show the main results of this paper, that is, proving the boundedness of weak solutions to problem (1.1) by applying an appropriate version of De Giorgi's iteration along with the localization method. Finally, in Sect. 5, based on the L^∞-bounds of the solutions and the maximum principle we prove the existence of infinitely many small non-negative weak solutions to the single-valued nonlocal double phase problem (5.1).

2 Preliminaries

In this section, we recall some basic results concerning variable exponent Lebesgue spaces, the Musielak–Orlicz spaces and fractional Musielak–Sobolev spaces, see Diening–Harjulehto–Hästö–Růžička [25], Fan–Zhao [26], Harjulehto–Hästö [33], Kováčik–Rákosník [41], Lu–Vetro–Zeng [47] and de Albuquerque–de Assis–Carvalho–Salort [23] for more details. In the sequel let C be a constant that will change from line to line, and C_r means a constant depending on the parameter r.

First, we introduce the subset $C_+(\overline{\Omega})$ of $C(\overline{\Omega})$ given by

$$C_+(\overline{\Omega}) := \left\{ g \in C(\overline{\Omega}) \colon 1 < \inf_{x \in \overline{\Omega}} g(x) \text{ for all } x \in \overline{\Omega} \right\}.$$

For every $r \in C_+(\overline{\Omega})$ we define r^- and r^+ as

$$r^- := \inf_{x \in \overline{\Omega}} r(x) \quad \text{and} \quad r^+ := \sup_{x \in \overline{\Omega}} r(x),$$

and denote by $r' \in C_+(\overline{\Omega})$ the conjugate variable exponent of r, that is

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Let $M(\Omega)$ be the set of all measurable functions $u \colon \Omega \to \mathbb{R}$, where two functions are considered identical if they differ only on a Lebesgue-null set. Given a fixed $r \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is given by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) \colon \varrho_{r(\cdot)}(u) < \infty \right\},$$

where the corresponding modular function $\varrho_{r(\cdot)}$ is defined as

$$\varrho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)} \, \mathrm{d}x.$$

It is well known that $L^{r(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf \left\{ \lambda > 0 \colon \int_\Omega \left(\frac{|u|}{\lambda} \right)^{r(x)} \mathrm{d}x \le 1 \right\}$$

forms a separable and reflexive Banach space. Moreover, $L^{r'(\cdot)}(\Omega)$ is the dual space of $L^{r(\cdot)}(\Omega)$ and the following Hölder type inequality holds:

$$\int_\Omega |uv|\,\mathrm{d}x \le \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(\cdot)}\|v\|_{r'(\cdot)} \le 2\|u\|_{r(\cdot)}\|v\|_{r'(\cdot)}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and all $v \in L^{r'(\cdot)}(\Omega)$. Additionally, if $r_1, r_2 \in C_+(\overline{\Omega})$ satisfying $r_1(x) \le r_2(x)$ for all $x \in \overline{\Omega}$, then the following embedding is valid

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega).$$

Next, in order to introduce Musielak–Orlicz spaces, we give the definition of N-functions and generalized N-functions.

Definition 2.1 (i) A function $\varphi\colon [0,\infty) \to [0,\infty)$ is called a N-function if it possesses the following properties: it is continuous, convex with $\varphi(t) = 0$ if and only if $t = 0$, Additionally, it fulfills

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty.$$

(ii) A function $\varphi\colon \Omega \times \Omega \times [0,\infty) \to [0,\infty)$ is called a generalized N-function, denoted by $\varphi \in N(\Omega \times \Omega)$, if for all $t \ge 0$ $\varphi(\cdot,\cdot,t)$ is measurable. Additionally, $\varphi(x,x,\cdot)$ is a N-function for a.a. $(x,x) \in \Omega \times \Omega$. Similarly, we can give the definition of functions $\varphi \in N(\Omega)$.

Next, we recall some definitions related to N-functions and generalized N-functions.

Definition 2.2 (i) A function $\varphi\colon \Omega \times [0,\infty) \to [0,\infty)$ is locally integrable if for all $t > 0$, $\varphi(\cdot,t)$ belongs to $L^1(\Omega)$.

(ii) Let $\varphi, \psi \in \mathrm{N}(\Omega)$, we say that φ is weaker than ψ, denoted by $\varphi \prec \psi$, if there exist constants $c_1, c_2 > 0$ such that

$$\varphi(x,t) \le c_1\psi(x,c_2t) + g(x) \quad \text{for a.a.}\, x \in \Omega \text{ and for all } t \ge 0,$$

where $0 \le g(\cdot) \in L^1(\Omega)$. Furthermore, φ, ψ are equivalent, denoted by $\varphi \sim \psi$, if $\varphi \prec \psi$ and in the same time $\psi \prec \varphi$.

(iii) Let $\varphi, \psi \in N(\Omega)$, we say that φ increases essentially slower than ψ near infinity, denoted by $\varphi \ll \psi$, if for every $k > 0$ the limit

$$\lim_{t\to\infty} \frac{\varphi(x,kt)}{\psi(x,t)} = 0$$

holds uniformly for a.a. $x \in \Omega$.

Given $\varphi \in N(\Omega)$, we can define the associated modular function as

$$\rho_\varphi(u) = \int_\Omega \varphi(x,|u|)\,\mathrm{d}x,$$

and the corresponding Musielak–Orlicz space, denoted by $L^\varphi(\Omega)$, is given as

$$L^\varphi(\Omega) := \{u \in M(\Omega)\colon \text{ there exists } \lambda > 0 \text{ such that } \rho_\varphi(\lambda u) < +\infty\}.$$

This space is equipped with the Luxemburg norm given by

$$\|u\|_{\varphi,\Omega} := \inf\left\{\lambda > 0\colon \rho_\varphi\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

To simplify the notation, we may write the norm for the domain Ω as $\|u\|_{\varphi}$ instead of $\|u\|_{\varphi,\Omega}$.

The following useful embedding result can be found in Musielak [52, Theorem 8.5].

Proposition 2.3 *If $\varphi \in N(\Omega)$ and $\psi \in N(\Omega)$ satisfying $\varphi \prec \psi$, then $L^{\psi}(\Omega) \hookrightarrow L^{\varphi}(\Omega)$.*

Next, we introduce some basic definitions and notations for fractional Musielak–Sobolev spaces which are mainly taken from the work by de Albuquerque–de Assis–Carvalho–Salort [23].

In the remaining parts of this paper, we define

$$\mathcal{H}(x, y, t) = \int_0^t h(x, y, \tau)\, d\tau,$$

where $h\colon \Omega \times \Omega \times [0, \infty) \to [0, \infty)$. Moreover, we introduce the following assumptions:

(φ_1) $\lim_{t\to 0} \varphi(x, y, t) = 0$ and $\lim_{t\to\infty} \varphi(x, y, t) = +\infty$ with $t \mapsto \varphi(x, y, t)$ being continuous on the interval $(0, \infty)$ for all $(x, y) \in \Omega \times \Omega$;

(φ_2) $t \mapsto \varphi(\cdot, \cdot, t)$ is increasing on $(0, \infty)$;

(φ_3) there exist constants $1 < \ell \leq m < +\infty$ satisfying

$$\ell \leq \frac{h(x, y, t)}{\mathcal{H}(x, y, t)} \leq m,$$

for all $(x, y) \in \Omega \times \Omega$ and for all $t \in (0, \infty)$.

From de Albuquerque–de Assis–Carvalho–Salort [23], we know that if the function h satisfies conditions (φ_1)–(φ_3) and $h(\cdot, \cdot, t)$ is measurable for all $t \geq 0$, then $\mathcal{H}$ is a generalized N-function. Moreover, we consider the function $\widehat{\mathcal{H}}\colon \Omega \times [0, \infty) \to [0, \infty)$ given by

$$\widehat{\mathcal{H}}(x, t) := \int_0^t \hat{h}(x, \tau)\, d\tau,$$

where $\hat{h}(x, t) := h(x, x, t)$ for all $(x, t) \in \Omega \times [0, \infty)$.

Recall that

$$\mathcal{H}(x, y, t) = [t^{p(x,y)} + \mu(x, y)t^{q(x,y)}]\log(e + \omega t) \quad \text{for all } (x, y, t) \in \Omega \times \Omega \times [0, \infty).$$

Throughout this paper we will assume the following hypotheses:

(H1) $p, q \in C(\mathbb{R}^N \times \mathbb{R}^N)$ such that $1 < \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x, y) \leq \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} < \frac{N}{s}$ and $p(x, y) \leq q(x, y)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $U_1 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : p(x, y) < q(x, y)\} \nsubseteq U_0 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \mu(x, y) = 0\}$ and $p(x, y) = p(y, x), q(x, y) = q(y, x)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. $0 \leq \mu(\cdot, \cdot) \in L^{\infty}(\mathbb{R}^N \times \mathbb{R}^N)$ such that $\mu(x, y) = \mu(y, x)$ and $\mu(x) = 0 \implies \mu(x, y) = 0$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Note that

$$p_s^*(x, y) = \frac{Np(x, y)}{N - sp(x, y)}.$$

In the sequel, we use the notations

$$p^- := \inf_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) \quad \text{and} \quad q^- := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y).$$

Moreover, q^-, q^+ can be defined in the same way.

Under the hypotheses of (H1), we deduce from the argument in Section 5 of [23] that h satisfies assumptions (φ_1)–(φ_3) with $\ell = p^-$ and $m = q^+ + 1$.

Let (H1) hold true, it is easy to check that $\mathcal{H}$ given in (1.3) is a locally integrable N-function. Then the modular function related to $\widehat{\mathcal{H}}$ is given as

$$\rho_{\widehat{\mathcal{H}}}(u) = \int_\Omega \widehat{\mathcal{H}}(x, |u|)\,\mathrm{d}x$$

while the corresponding Musielak–Orlicz space is

$$L^{\widehat{\mathcal{H}}}(\Omega) = \{u \in M(\Omega)\colon \rho_{\widehat{\mathcal{H}}}(\lambda u) < +\infty, \text{ for some } \lambda > 0\},$$

endowed with the Luxemburg norm

$$\|u\|_{\widehat{\mathcal{H}}} = \inf\left\{\lambda > 0\colon \rho_{\widehat{\mathcal{H}}}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Furthermore, the fractional Musielak–Orlicz space $W^{s,\mathcal{H}}(\Omega)$ is defined as

$$W^{s,\mathcal{H}}(\Omega) := \left\{u \in L^{\widehat{\mathcal{H}}}(\Omega)\colon \rho_{s,\mathcal{H}}(\lambda u) < \infty \text{ for some } \lambda > 0\right\},$$

where

$$\rho_{s,\mathcal{H}}(u) := \int_\Omega \int_\Omega \mathcal{H}(x, y, |D_s u(x, y)|)\,\mathrm{d}\nu \quad \text{for } s \in (0, 1),$$

with

$$\mathrm{d}\nu : \frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|^N} \quad \text{and} \quad D_s u(x, y) := \frac{u(x) - u(y)}{|x - y|^s},$$

where $\mathrm{d}\nu$ is a regular Borel measure on $\Omega \times \Omega$. The Musielak–Sobolev space $W^{s,\mathcal{H}}(\Omega)$ is equipped with the norm

$$\|u\|_{s,\mathcal{H}} := \|u\|_{\widehat{\mathcal{H}}} + [u]_{s,\mathcal{H}},$$

where $[\,\cdot\,]_{s,\mathcal{H}}$ is called $(s, \mathcal{H})$-Gagliardo seminorm defined by

$$[u]_{s,\mathcal{H}} := \inf\left\{\lambda > 0\colon \rho_{s,\mathcal{H}}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Furthermore, we introduce the following closed subspace of $W^{s,\mathcal{H}}(\Omega)$ defined by

$$W_0^{s,\mathcal{H}}(\Omega) = \left\{u \in W^{s,\mathcal{H}}(\mathbb{R}^N)\colon u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

It is worth to note that since the function $\mathcal{H}$ fulfills assumptions (φ_1)–(φ_3), we infer from [23] that the corresponding Musielak–Orlicz Lebesgue space $L^{\widehat{\mathcal{H}}}(\Omega)$ and the fractional Musielak–Sobolev space $W_0^{s,\mathcal{H}}(\Omega)$ are separable and reflexive Banach spaces.

The following boundedness condition is used to established a generalized Poincaré type inequality.

Definition 2.4 Let $\mathcal{H} \in N(\Omega \times \Omega)$, then $\mathcal{H}$ is said to satisfy the fractional boundedness condition if there exist some constants $C_1, C_2 > 0$ such that

$$0 < C_1 \le \mathcal{H}(x, y, 1) \le C_2 \quad \text{for all } (x, y) \in \Omega \times \Omega. \tag{B_f}$$

It is easy to check that, if hypotheses (H1) hold, then hypotheses (B_f) is satisfied with $C_1 = 1$ and $C_2 = (1 + \|\mu\|_\infty)\log(e + \omega)$.

The next proposition can be found in the work by Azroul–Benkirane–Shimi–Srati [5, Theorem 2.3].

Proposition 2.5 *Let $s \in (0, 1)$, and let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. If* (H1) *hold, then one can find a positive constant C satisfying*

$$\|u\|_{\widehat{\mathcal{H}}} \le C[u]_{s,\mathcal{H}},$$

for all $u \in W_0^{s,\mathcal{H}}(\Omega)$.

By Proposition 2.5, for all $u \in W_0^{s,\mathcal{H}}(\Omega)$, we can find $\lambda_1 > 0$ such that

$$\int_\Omega \widehat{\mathcal{H}}(x, |u(x)|)\,\mathrm{d}x \le \lambda_1 \int_\Omega \int_\Omega \mathcal{H}(x, y, |D_s u(x, y)|)\,\mathrm{d}\nu.$$

Moreover, $[\,\cdot\,]_{s,\mathcal{H}}$ is an equivalent norm of $\|\cdot\|_{s,\mathcal{H}}$ on $W_0^{s,\mathcal{H}}(\Omega)$, that is

$$[u]_{s,\mathcal{H}} \le \|u\|_{s,\mathcal{H}} \le C'[u]_{s,\mathcal{H}} \quad \text{for all } u \in W_0^{s,\mathcal{H}}(\Omega), \tag{2.1}$$

with C' being a positive constant.

The following proposition gives the relation between the norm of the space $L^{\widehat{\mathcal{H}}}(\Omega)$ and its modular, the proof can be found in Theorem 2.21 of Lu–Vetro–Zeng [47].

Proposition 2.6 *Let hypotheses* (H1) *be satisfied, $u \in L^{\widehat{\mathcal{H}}}(\Omega)$ and the modular is defined by*

$$\rho_{\widehat{\mathcal{H}}}(u) = \int_\Omega \left[|u|^{p(x)} + \mu(x)|u|^{q(x)}\right]\log(e + \omega|u|)\,\mathrm{d}x \quad \textit{for all } u \in L^{\widehat{\mathcal{H}}}(\Omega).$$

Then for $\sigma > 0$, the following hold:

(i) $\|u\|_{\widehat{\mathcal{H}}} = \lambda \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(\frac{u}{\lambda}) = 1$ *with* $u \neq 0$*;*
(ii) $\|u\|_{\widehat{\mathcal{H}}} < 1$ *(resp.*$= 1, > 1$) $\Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) < 1$ *(resp.*$= 1, > 1$)*;*
(iii) *if* $\|u\|_{\widehat{\mathcal{H}}} < 1$*, then* $C_\sigma^{-1}\|u\|_{\widehat{\mathcal{H}}}^{q^+ + \sigma} \le \rho_{\widehat{\mathcal{H}}}(u) \le \|u\|_{\widehat{\mathcal{H}}}^{p^-}$*;*
(iv) *if* $\|u\|_{\widehat{\mathcal{H}}} > 1$*, then* $\|u\|_{\widehat{\mathcal{H}}}^{p^-} \le \rho_{\widehat{\mathcal{H}}}(u) \le C_\sigma\|u\|_{\widehat{\mathcal{H}}}^{q^+ + \sigma}$*;*
(v) $\|u\|_{\widehat{\mathcal{H}}} \to 0 \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) \to 0$*;*
(vi) $\|u\|_{\widehat{\mathcal{H}}} \to \infty \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) \to \infty$*;*
(vii) $\|u\|_{\widehat{\mathcal{H}}} \to 1 \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) \to 1$*;*
(viii) *if* $u_n \to u$ *in* $L^{\widehat{\mathcal{H}}}(\Omega)$ *then* $\rho_{\widehat{\mathcal{H}}}(u_n) \to \rho_{\widehat{\mathcal{H}}}(u)$.

Remark 2.7 For $\gamma > 0$, we consider the function $f_{\sigma'}\colon [0, \infty) \to [0, \infty)$ defined as

$$f_{\sigma'} = \frac{t^{\sigma'}}{\log^\gamma(e + \omega t)} \quad \text{with } \sigma', \gamma > 0 \text{ and } \omega \ge 0.$$

Obviously, one can find $\sigma^* > 0$ such that $f_{\sigma'} > 0$ is increasing for all $\sigma' \ge \sigma^*$. Also, for $0 < \sigma' < \sigma^*$, there exist points t_1, t_2 such that the following hold: if $0 < t < t_1$ and $t > t_2$, then $f_{\sigma'}$ is increasing, conversely, $f_{\sigma'}$ is decreasing for $t_1 \le t \le t_2$. So that for any $0 < a \le b$, we have $f_{\sigma'}(a) \le C_{\sigma'} \cdot f_{\sigma'}(b)$ with $C_{\sigma'} = \frac{f_{\sigma'}(t_1)}{f_{\sigma'}(t_2)} > 1$. Hence, as done in the proof of Proposition 2.21 of [47], we can get the same conclusions given in Proposition 2.6 with

$$\rho_{\widehat{\mathcal{H}}}(u) := \int_\Omega \left[|u|^{p(x)} + \mu(x)|u|^{q(x)}\right]\log^\gamma(e + \omega|u|)\,\mathrm{d}x,$$

where $\gamma > 0$.

Similar to Proposition 2.6, we deduce the following relations between the semi-modular $\rho_{s,\mathcal{H}}(\cdot)$ and the $(s,\mathcal{H})$-Gagliardo seminorm $[\cdot]_{s,\mathcal{H}}$.

Proposition 2.8 *Let* (H1) *be satisfied and* $u \in W^{s,\mathcal{H}}(\Omega)$*. Then, for* $\sigma > 0$*, the following hold:*

(i) *if* $[u]_{s,\mathcal{H}} < 1$*, then* $C_\sigma^{-1}[u]_{s,\mathcal{H}}^{q^+ + \sigma} \le \rho_{s,\mathcal{H}}(u) \le [u]_{s,\mathcal{H}}^{p^-}$*;*
(ii) *if* $[u]_{s,\mathcal{H}} > 1$*, then* $[u]_{s,\mathcal{H}}^{p^-} \le \rho_{s,\mathcal{H}}(u) \le C_\sigma [u]_{s,\mathcal{H}}^{q^+ + \sigma}$*.*

Under conditions (φ_1)–(φ_3) we see that $\widehat{\mathcal{H}}\colon [0,+\infty) \to [0,+\infty)$ is an increasing homeomorphism. Next, we introduce the inverse function of $\widehat{\mathcal{H}}$ denoted by $\widehat{\mathcal{H}}^{-1}$ satisfying the following conditions:

$$\int_0^1 \frac{\widehat{\mathcal{H}}^{-1}(x,\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau < \infty \quad \text{and} \quad \int_1^\infty \frac{\widehat{\mathcal{H}}^{-1}(x,\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau = \infty \quad \text{for all } x \in \Omega.$$

We denote by $\widehat{\mathcal{H}}_s^*$ the Musielak–Sobolev conjugate function of $\widehat{\mathcal{H}}$ and the inverse function of $\widehat{\mathcal{H}}_s^*$ is defined by

$$(\widehat{\mathcal{H}}_s^*)^{-1}(x,t) = \int_0^t \frac{\widehat{\mathcal{H}}^{-1}(x,\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau \quad \text{for all } x \in \Omega \text{ and for all } t \ge 0.$$

In the sequel, we denote by $X \hookrightarrow Y$ the continuous embedding from the space X into the space Y. Also, denote by $X \hookrightarrow\hookrightarrow Y$ the compact embedding from X into Y. The next result is due to Azroul–Benkirane–Shimi–Srati [5, Lemma 2.3].

Lemma 2.9 *Let* $0 < s' < s < 1$*,* Ω *be a bounded domain in* $\mathbb{R}^N$ *and suppose* (H1)*. Then there exists holds the continuous embedding* $W^{s,\mathcal{H}}(\Omega) \hookrightarrow W^{s',r}(\Omega)$ *with* $r \in [1, p^-)$*.*

Next, we give the definition of a Young function.

Definition 2.10 A function $\varphi\colon [0,\infty) \to [0,\infty]$ is called a Young function if it is convex, continuous, non-constant, $\varphi(0) = 0$ and $\varphi(t) = \int_0^t a(\tau)\,\mathrm{d}\tau$, where $a\colon [0,\infty) \to [0,\infty]$ is a non-decreasing function. Moreover, we denote the left-continuous inverse of φ by $\varphi^{-1}\colon [0,\infty) \to [0,\infty)$ given by

$$\varphi^{-1}(t) = \inf\{\tau \ge 0\colon \varphi(\tau) \ge t\}$$

for $t \ge 0$.

Let H be a Young function such that

$$\int^\infty \left(\frac{t}{H(t)}\right)^{\frac{s}{N-s}}\,\mathrm{d}t = \infty \quad \text{and} \quad \int_0 \left(\frac{t}{H(t)}\right)^{\frac{s}{N-s}}\,\mathrm{d}t < \infty. \tag{2.2}$$

Then the corresponding Orlicz target is defined as

$$H_{\frac{N}{s}}(t) = H(T^{-1}(t)) \tag{2.3}$$

for all $t \ge 0$, where

$$T(t) = \left(\int_0^t \left(\frac{\tau}{H(\tau)}\right)^{\frac{s}{N-s}}\,\mathrm{d}\tau\right)^{\frac{N-s}{N}}$$

for all $t \ge 0$.

The following continuous embedding with respect to the fractional Orlicz-Sobolev space $W^{s,H}(\Omega)$ is taken from Alberico–Cianchi–Pick–Slavíková [1, Theorem 8.1].

Theorem 2.11 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and let $s \in (0,1)$. If H is a Young function satisfying conditions (2.2) and $H_{\frac{N}{s}}$ is given by (2.3), then there holds*

$$W^{s,H}(\Omega) \hookrightarrow L^{H_{\frac{N}{s}}}(\Omega),$$

and the embedding is optimal.

By the definition of $W_0^{s,H}(\Omega)$, under the hypotheses of Theorem2.11, we deduce that $W_0^{s,H}(\Omega) \hookrightarrow W^{s,H}(\Omega) \hookrightarrow L^{H_{\frac{N}{s}}}(\Omega)$. Referring to Example 8.3 by Alberico–Cianchi–Pick–Slavíková [1], we see that if we set

$$H := t^{p^-}\log(e+\omega t) + \mu(x)t^{q^-}\log(e+\omega t),$$

then

$$H_{\frac{N}{s}} \sim H^* := t^{(p^-)^*_s}\log^{\frac{(p^-)^*_s}{N}}(e+\omega t) + \mu(x)^{\gamma}t^{(q^-)^*_s}\log^{\frac{(q^-)^*_s}{N}}(e+\omega t),$$

for $1 \le p^-, q^- < \frac{N}{s}$, for all $t \ge 0$ and $\gamma > 0$. Furthermore, we introduce that following function:

$$\mathcal{B}(x,t) = t^{\varsigma(x)}\log^{\frac{\varsigma(x)}{N}}(e+\omega t) + \mu(x)^{\gamma}t^{\tau(x)}\log^{\frac{\tau(x)}{N}}(e+\omega t)$$

for all $\gamma > 0$, for all $x \in \overline{\Omega}$, and for all $t \in [0,\infty)$ with $\varsigma, \tau \in C(\overline{\Omega})$ such that $1 < \varsigma(x) \le (p^-)^*_s$ and $1 < \tau(x) \le (q^-)^*_s$ for all $x \in \overline{\Omega}$. It is not hard to see that $H \prec \mathcal{H}$ as well as $\mathcal{B} \prec H_{\frac{N}{s}}$, so we conclude that

$$W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow W_0^{s,H} \hookrightarrow L^{H_{\frac{N}{s}}}(\Omega) \hookrightarrow L^{\mathcal{B}}(\Omega). \tag{2.4}$$

According to Theorem 9.1 by Alberico–Cianchi–Pick–Slavíková [1], we get that following compact embedding theorem.

Proposition 2.12 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, and let $s \in (0,1)$. Assume that H is a Young function satisfying conditions (2.2) and $H_{\frac{N}{s}}$ is given by (2.3). If G is a Young function such that $G \ll H_{\frac{N}{s}}$, then there holds*

$$W^{s,H}(\Omega) \hookrightarrow\hookrightarrow L^G(\Omega).$$

Hence, it follows that $W_0^{s,H}(\Omega) \hookrightarrow W^{s,H}(\Omega) \hookrightarrow\hookrightarrow L^G(\Omega)$.

Finally, we recall some background from the theory of operators of monotone type.

Definition 2.13 Let X be a reflexive Banach space with X^* being the corresponding dual space, the duality pairing is denoted by $\langle\cdot,\cdot\rangle$ and $A\colon X \to X^*$.

(i) A satisfies the (S_+)-property if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$ imply $u_n \to u$ in X;
(ii) A is monotone (strictly monotone) if $\langle Au - Av, u - v\rangle \ge 0$ (> 0) for all $u, v \in X$ such that $u \ne v$;
(iii) A is coercive if there exists a function $g\colon [0,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty} g(t) = \infty$ such that

$$\frac{\langle Au, u\rangle}{\|u\|_X} \ge g(\|u\|_X) \quad \text{for all } u \in X.$$

According to Lemma 3.10 of [23], we have the following properties of the functional

$$I_{s,\mathcal{H}} = \rho_{s,\mathcal{H}}(u) := \int_\Omega \int_\Omega \mathcal{H}(x, y, |D_s u(x, y)|)\, \mathrm{d}\nu$$

and its Gâteaux derivative.

Proposition 2.14 *Let* (H1) *be satisfied. Then* $I_{s,\mathcal{H}} \in C^1(W_0^{s,\mathcal{H}}(\Omega), \mathbb{R})$ *and the Gâteaux derivative of* $I_{s,\mathcal{H}}$ *is given by*

$$\langle A(u), v\rangle = \int_\Omega \int_\Omega \mathcal{H}'(x, y, |D_s u(x, y)|) D_s v(x, y)\, \mathrm{d}\nu,$$

for all $u, v \in W_0^{s,\mathcal{H}}(\Omega)$. *Moreover, A satisfies the* (S_+)*-property.*

We end this section with the following iteration lemma, which is the important tool for the proof of the boundedness results of solutions, see Ho–Kim [36, 37, Lemma 4.3].

Lemma 2.15 *Let* $\{Z_n\}, n = 0, 1, 2, \dots,$ *be a sequence of positive numbers satisfying the recursive inequality*

$$Z_{n+1} \le M k^n \left(Z_n^{1+\gamma_1} + Z_n^{1+\gamma_2}\right), \quad n = 0, 1, 2, \dots,$$

for some $k > 1$, $M > 0$ *and* $\gamma_2 \ge \gamma_1 > 0$. *If*

$$Z_0 \le \min\left(1, (2M)^{-\frac{1}{\gamma_1}} k^{-\frac{1}{\gamma_1^2}}\right)$$

or

$$Z_0 \le \min\left((2M)^{-\frac{1}{\gamma_1}} k^{-\frac{1}{\gamma_1^2}}, (2M)^{-\frac{1}{\gamma_2}} k^{-\frac{1}{\gamma_1\gamma_2} - \frac{\gamma_2-\gamma_1}{\gamma_2^2}}\right),$$

then $Z_n \le 1$ *for some* $n \in \mathbb{N} \cup \{0\}$. *Furthermore,*

$$Z_n \le \min\left(1, (2M)^{-\frac{1}{\gamma_1}} k^{-\frac{1}{\gamma_1^2}} k^{-\frac{n}{\gamma_1}}\right), \quad \textit{for all } n \ge n_0,$$

with n_0 *being the smallest* $n \in \mathbb{N} \cup \{0\}$ *fulfilling* $Z_n \le 1$. *In particular,* $Z_n \to 0$ *as* $n \to \infty$.

3 Maximum principle

In this section, we establish the maximum principle for functions $u \in W^{s,\mathcal{H}}(\Omega)$. The proof is inspired by Chen–Li [17].

Theorem 3.1 *Let* (H1) *be satisfied and* Ω *be a bounded domain in* $\mathbb{R}^N$. *Let* $u \in W^{s,\mathcal{H}}(\Omega)$ *be lower semi-continuous on* $\overline{\Omega}$ *such that*

$$\begin{cases} (-\Delta)^s_{\mathcal{H}} u(x) \ge 0, & x \in \Omega, \\ u(x) \ge 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

then

$$u(x) \ge 0 \quad \text{in } \Omega. \tag{3.2}$$

Moreover, if there exists some point $x_0 \in \Omega$ *such that* $u(x_0) = 0$*, then* $u(x) = 0$ *for a.a.* $x \in \mathbb{R}^N$*. In addition, if we assume that*

$$\underline{\lim}_{|x|\to\infty} u(x) \geq 0,$$

then we have the same conclusions for Ω *being unbounded.*

Proof Suppose that (3.2) fails, then the lower semi-continuity of u on $\overline{\Omega}$ implies that there exists $x^* \in \Omega$ such that

$$u(x^*) = \min_{\Omega} u < 0.$$

Taking $u(x) \geq 0$ for $x \in \mathbb{R}^N \setminus \Omega$ into account, we calculate that

$$\begin{aligned}
(-\Delta)^s_{\mathcal{H}} u(x^*) &= C_{N,s,p,q}\, \mathrm{PV} \int_{\mathbb{R}^N} \\
&\quad \left(\frac{|u(x^*) - u(y)|^{p(x^*,y)-2}(u(x^*) - u(y))}{|x^* - y|^{N+sp(x^*,y)}} \log\left(e + \omega \frac{|u(x^*) - u(y)|}{|x^* - y|^s}\right) \right. \\
&\quad + \frac{\omega |u(x^*) - u(y)|^{p(x^*,y)-1}(u(x^*) - u(y))}{|x^* - y|^{N+s(p(x^*,y)+1)} \left(e + \omega \frac{|u(x^*)-u(y)|}{|x^*-y|^s}\right)} \\
&\quad + \mu(x^*, y) \frac{|u(x^*) - u(y)|^{q(x^*,y)-2}(u(x^*) - u(y))}{|x^* - y|^{N+sq(x^*,y)}} \log\left(e + \omega \frac{|u(x^*) - u(y)|}{|x^* - y|^s}\right) \\
&\quad \left. + \mu(x^*, y) \frac{\omega |u(x^*) - u(y)|^{q(x^*,y)-1}(u(x^*) - u(y))}{|x^* - y|^{N+s(q(x^*,y)+1)} \left(e + \omega \frac{|u(x^*)-u(y)|}{|x^*-y|^s}\right)} \right) \mathrm{d}y \\
&\leq C_{N,s,p,q} \int_{\mathbb{R}^N \setminus \Omega} \left(\frac{|u(x^*) - u(y)|^{p(x^*,y)-2}(u(x^*) - u(y))}{|x^* - y|^{sp(x^*,y)}} \log\left(e + \omega \frac{|u(x^*)-u(y)|}{|x^* - y|^s}\right) \right. \\
&\quad + \frac{\omega |u(x^*) - u(y)|^{p(x^*,y)-1}(u(x^*) - u(y))}{|x^* - y|^{N+s(p(x^*,y)+1)} \left(e + \omega \frac{|u(x^*)-u(y)|}{|x^*-y|^s}\right)} \\
&\quad + \mu(x^*, y) \frac{|u(x^*) - u(y)|^{q(x^*,y)-2}(u(x^*) - u(y))}{|x^* - y|^{sq(x^*,y)}} \log\left(e + \omega \frac{|u(x^*) - u(y)|}{|x^* - y|^s}\right) \\
&\quad \left. + \mu(x^*, y) \frac{\omega |u(x^*) - u(y)|^{q(x^*,y)-1}(u(x^*) - u(y))}{|x^* - y|^{N+s(q(x^*,y)+1)} \left(e + \omega \frac{|u(x^*)-u(y)|}{|x^*-y|^s}\right)} \right) \mathrm{d}y \\
&< 0.
\end{aligned}$$

The above inequality contradicts to the first inequality in (3.1), thus, (3.2) holds true.

On the other hand, if there exists some point $x_0 \in \Omega$ such that $u(x_0) = 0$, then we have

$$\begin{aligned}
(-\Delta)^s_{\mathcal{H}} u(x_0) = C_{N,s,p,q}\, \mathrm{PV} \int_{\mathbb{R}} & \left(\frac{|u(y)|^{p(x_0,y)-2}(-u(y))}{|x_0 - y|^{N+sp(x_0,y)}} \log\left(e + \omega \frac{|u(y)|}{|x_0 - y|^s}\right) \right. \\
& + \frac{\omega |u(y)|^{p(x_0,y)-1}(-u(y))}{|x_0 - y|^{N+s(p(x_0,y)+1)} \left(e + \omega \frac{|u(y)|}{|x_0-y|^s}\right)} \\
& + \mu(x_0, y) \frac{|u(y)|^{q(x_0,y)-2}(-u(y))}{|x_0 - y|^{sq(x_0,y)}} \log\left(e + \omega \frac{|u(y)|}{|x_0 - y|^s}\right)
\end{aligned}$$

$$+\mu(x_0,y)\frac{\omega|u(y)|^{q(x_0,y)-1}(-u(y))}{|x_0-y|^{N+s(q(x_0,y)+1)}\left(e+\omega\frac{|u(y)|}{|x_0-y|^s}\right)}\Bigg)\,\mathrm{d}y$$
$$\leq 0,$$

Combining this with the first inequality in (3.1) implies that the above integral must be zero. Note that we have proved that $u \geq 0$ in $\mathbb{R}^N$, thus $u(x) = 0$ for a.a. $x \in \mathbb{R}^N$.

Suppose now Ω is unbounded. Then, since $\underline{\lim}_{|x|\to\infty} u(x) \geq 0$ and u is lower semi-continuous, if $u(x) \geq 0$ in Ω, we can find $x^* \in \Omega$ such that $u(x^*) = \min_\Omega u < 0$. As done in the above proof we can show the remaining conclusions. □

The following corollary can be directly derived since $\mathcal{H}_3$ given in (P3) is a special case of $\mathcal{H}$.

Corollary 3.2 *Let* (H1) *be satisfied with* $1 < p(\cdot,\cdot) \equiv p$, $1 < q(\cdot,\cdot) \equiv q$ *and let* Ω *be a bounded domain in* $\mathbb{R}^N$. *Let* $u \in W^{s,\mathcal{H}_3}(\Omega)$ *be lower semi-continuous on* $\overline{\Omega}$ *such that*

$$\begin{cases}(-\Delta)^s_{\mathcal{H}_3} u(x) \geq 0, & x \in \Omega,\\ u(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega,\end{cases}$$

then

$$u(x) \geq 0 \quad in\ \Omega.$$

Moreover, if there exists some point $x_0 \in \Omega$ *such that* $u(x_0) = 0$, *then* $u(x) = 0$ *for a.a.* $x \in \mathbb{R}^N$. *In addition, if we assume that*

$$\underline{\lim}_{|x|\to\infty} u(x) \geq 0,$$

then we have the same conclusions for Ω *being unbounded.*

In particular, if $\omega = 0$, i.e. $\mathcal{H}(x, y, t) = t^{p(x,y)} + \mu(x, y)t^{q(x,y)} = \mathcal{H}_2(x, y, t)$ for $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ and for $t \in [0,\infty)$. Due to the homogeneity of $t^{p(\cdot,\cdot)}$ and $t^{q(\cdot,\cdot)}$, we can establish the maximum principle for anti-symmetric functions, which is essential for applying the method of moving planes to investigating symmetry and monotonicity of solutions, see for example Chen–Li [17] and Hu–Peng [40]. To this end, we introduce the following notations. First, we define the moving planes as

$$T_\lambda = \left\{x \in \mathbb{R}^N : x_1 = \lambda \text{ for some } \lambda \in \mathbb{R}^N\right\},$$

and define the left region of the plane T_λ as

$$\Sigma = \left\{x \in \mathbb{R}^N : x_1 < \lambda\right\}.$$

Moreover, we denote the reflection of x of the plane T_λ by x^λ, that is

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_N),$$

and let

$$w = u_\lambda(x) - u(x) = u(x^\lambda) - u(x).$$

Theorem 3.3 *Let* (H1) *be satisfied and suppose that* $\omega = 0$. *Let* Ω *be a bounded domain in* Σ *and* $u \in W^{s,\mathcal{H}_2}(\Omega)$ *be lower semi-continuous on* $\overline{\Omega}$ *such that*

$$\begin{cases}(-\Delta)^s_{\mathcal{H}_2} u_\lambda(x) - (-\Delta)^s_{\mathcal{H}_2} u(x) \geq 0, & x \in \Omega,\\ w(x) \geq 0, & x \in \Sigma \setminus \Omega,\end{cases} \tag{3.3}$$

then

$$w(x) \geq 0 \text{ in } \Omega. \tag{3.4}$$

Moreover, if there exists some point $x_0 \in \Omega$ such that $w(x_0) = 0$, then $w(x) = 0$ for a.a. $x \in \mathbb{R}^N$. In addition, if we assume that

$$\underline{\lim}_{|x|\to\infty} w(x) \geq 0,$$

then we have the same conclusions for Ω being unbounded.

Proof Suppose that (3.4) is not true, then we can find a point $x^* \in \Omega$ such that

$$w(x^*) = \min_{\Omega} w < 0.$$

We set $G^1(x, y, t) := |t|^{p(x,y)-2}t$ and $G^2(x, y, t) := |t|^{q(x,y)-2}t$. It is not hard to see that $t \mapsto G^1(x, y, t)$ and $t \mapsto G^2(x, y, t)$ are strictly increasing functions with

$$(G^1)'(x, y, t) = (p(x, y) - 1)|t|^{p(x,y)-2} \geq 0,$$
$$(G^2)'(x, y, t) = (q(x, y) - 1)|t|^{q(x,y)-2} \geq 0.$$

The following inequalities hold

$$\begin{aligned}
&(-\Delta)^s_{\mathcal{H}_2} u_\lambda(x^*) - (-\Delta)^s_{\mathcal{H}_2} u(x^*) \\
&\quad = C_{N,s,p,q}\,\mathrm{PV}\int_{\mathbb{R}^N} \frac{G^1\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^1\left[x^*, y, u(x^*) - u(y)\right]}{|x^* - y|^{N+sp(x^*,y)}}\,\mathrm{d}y \\
&\qquad + C_{N,s,p,q}\,\mathrm{PV}\int_{\mathbb{R}^N} \frac{G^2\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^2\left[x^*, y, u(x^*) - u(y)\right]}{|x^* - y|^{N+sq(x^*,y)}}\mu(x^*, y)\,\mathrm{d}y \\
&\quad \leq C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \frac{G^1\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^1\left[x^*, y, u(x^*) - u(y)\right]}{|x^* - y|^{N+sp(x^*,y)}}\,\mathrm{d}y \\
&\qquad + C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \frac{G^1\left[x^*, y, u_\lambda(x^*) - u(y)\right] - G^1\left[x^*, y, u(x^*) - u_\lambda(y)\right]}{|x^* - y^\lambda|^{N+sp(x^*,y)}}\,\mathrm{d}y \\
&\qquad + C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \frac{G^2\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^2\left[x^*, y, u(x^*) - u(y)\right]}{|x^* - y|^{N+sq(x^*,y)}}\mu(x^*, y)\,\mathrm{d}y \\
&\qquad + C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \frac{G^2\left[x^*, y, u_\lambda(x^*) - u(y)\right] - G^2\left[x^*, y, u(x^*) - u_\lambda(y)\right]}{|x^* - y^\lambda|^{N+sq(x^*,y)}}\mu(x^*, y)\,\mathrm{d}y \\
&\quad \leq C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \left[\frac{1}{|x^* - y|^{N+sp(x^*,y)}} - \frac{1}{|x^* - y^\lambda|^{N+sp(x^*,y)}}\right] \\
&\qquad \times \left[G^1\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^1\left[x^*, y, u(x^*) - u(y)\right]\right]\mathrm{d}y \\
&\qquad + C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \Big[G^1\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^1\left[x^*, y, u(x^*) - u(y)\right] \\
&\qquad + G^1\left[x^*, y, u_\lambda(x^*) - u(y)\right] - G^1\left[x^*, y, u(x^*) - u_\lambda(y)\right]\Big] \frac{\mathrm{d}y}{|x^* - y^\lambda|^{N+sp(x^*,y)}} \\
&\qquad + C_{N,s,p,q}\,\mathrm{PV}\int_{\Sigma} \left[\frac{1}{|x^* - y|^{N+sq(x^*,y)}} - \frac{1}{|x^* - y^\lambda|^{N+sq(x^*,y)}}\right] \\
&\qquad \times \left[G^2\left[x^*, y, u_\lambda(x^*) - u_\lambda(y)\right] - G^2\left[x^*, y, u(x^*) - u(y)\right]\right]\mu(x^*, y)\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&+C_{N,s,p,q}\,\mathrm{PV}\int_\Sigma \Big[G^2\left[x^*, y, u_\lambda(x^*)-u_\lambda(y)\right]-G^2\left[x^*, y, u(x^*)-u(y)\right]\\
&+G^2\left[x^*, y, u_\lambda(x^*)-u(y)\right]-G^2\left[x^*, y, u(x^*)-u_\lambda(y)\right]\Big]\mu(x^*, y)\\
&\frac{\mathrm{d}y}{|x^*-y^\lambda|^{N+sq(x^*,y)}}\\
&=C_{N,s,p,q}\,\mathrm{PV}(I_1+I_2+I_3+I_4).
\end{aligned}
\tag{3.5}
$$

Moreover, since

$$
\frac{1}{|x^*-y|}>\frac{1}{|x^*-y^\lambda|}>0
$$

for any $x^*, y\in\Sigma$, and by the monotonicity of G^1, G^2 along with $[u_\lambda(x^*)-u_\lambda(y)]-[u(x^*)-u(y)]=w(x^*)-w(y)\le 0$ but not equal to zero, we have $I_1<0$, and similarly, taking $\mu\ge 0$ into account, we deduce that $I_3\le 0$.

On the other hand, by applying the mean value theorem we get

$$
\begin{aligned}
I_2&=\int_\Sigma \Big[G^1\left[x^*, y, u_\lambda(x^*)-u_\lambda(y)\right]-G^1\left[x^*, y, u(x^*)-u_\lambda(y)\right]\\
&\quad+G^1\left[x^*, y, u_\lambda(x^*)-u(y)\right]-G^1\left[x^*, y, u(x^*)-u(y)\right]\Big]\frac{\mathrm{d}y}{|x^*-y^\lambda|^{N+sp(x^*,y)}}\\
&=w(x^*)\int_\Sigma\left[\left(G^1\right)'(\xi(y))+\left(G^1\right)'(\zeta(y))\right]\frac{\mathrm{d}y}{|x^*-y^\lambda|^{N+sp(x^*,y)}}\le 0,
\end{aligned}
$$

where $\xi(y)\in(u_\lambda(x^*)-u_\lambda(y), u(x^*)-u_\lambda(y))$ and $\zeta(y)\in(u_\lambda(x^*)-u(y), u(x^*)-u(y))$. Thus $I_2\le 0$, and analogously we get $I_4\le 0$ (note that $\mu\ge 0$). Recall that $I_1<0$ and $I_3\le 0$, applying (3.5) we conclude that

$$
(-\Delta)^s_{\mathcal{H}_2}u_\lambda(x^*)-(-\Delta)^s_{\mathcal{H}_2}u(x^*)<0,
$$

which contradicts (3.3). Hence, it must hold $w(x^*)\ge 0$.

Moreover, if we assume that $w(x_0)=0$ at some point $x_0\in\Omega$, then x_0 is a minimum of w in Ω, which indicates $I_2=I_4=0$. So, (3.3) implies $I_1, I_3\ge 0$. However, since $[u_\lambda(x_0)-u_\lambda(y)]-[u(x_0)-u(y)]=w(x_0)-w(y)=-w(y)\le 0$, it holds $I_1, I_3\le 0$. Hence, we conclude that $I_1=I_3=0$, thus

$$
w(y)=0\quad\text{for a.a. } y\in\Sigma,
$$

and by the antisymmetry of w we get

$$
w(y)=0\quad\text{for a.a. } y\in\mathbb{R}.
$$

Similarly, we get the conclusion for the case that Ω is unbounded. □

Moreover, since $\mathcal{H}_4$ given in (P4) is a special case of $\mathcal{H}_2$ given in (P2), we have the following corollary.

Corollary 3.4 *Let* (H1) *be satisfied with* $\omega=0$ *and* $1<p(\cdot,\cdot)\equiv p$, $1<q(\cdot,\cdot)\equiv q$. *Let* Ω *be a bounded domain in* Σ *and* $u\in W^{s,\mathcal{H}_4}(\Omega)$ *be lower semi-continuous on* $\overline{\Omega}$ *such that*

$$
\begin{cases}
(-\Delta)^s_{\mathcal{H}_4}u_\lambda(x)-(-\Delta)^s_{\mathcal{H}_4}u(x)\ge 0, & x\in\Omega,\\
w(x)\ge 0, & x\in\Sigma\setminus\Omega,
\end{cases}
$$

then

$$w(x) \geq 0 \quad \text{in } \Omega.$$

Moreover, if there exists some point $x_0 \in \Omega$ *such that* $w(x_0) = 0$*, then* $w(x) = 0$ *for a.a.* $x \in \mathbb{R}^N$*. In addition, if we assume that*

$$\underline{\lim}_{|x| \to \infty} w(x) \geq 0,$$

then we have the same conclusions for Ω *being unbounded.*

4 Boundedness of weak solutions

The aim of this section is to obtain a priori bounds for solutions to problem (1.1) with subcritical and critical growth. The proofs are mainly inspired by Ho–Kim [35], Ho–Kim–Winkert–Zhang [38], Ho–Winkert [39], and Winkert–Zacher [62, 63] using De Giorgi's iteration along with the localization method. In this section, we denote by $C_i\,(i \in \mathbb{N})$ positive constants.

Given a fixed $u \in M(\Omega)$ we define

$$\mathcal{F}(u) = \left\{\xi \in M(\Omega) \colon \xi(x) \in f(x, u(x)) \text{ for a.a. } x \text{ in } \Omega\right\},$$

which is the measurable selection of $f(\cdot, u)$.

First, we introduce the following definition of weak solutions to problem (1.1), which are well defined under the hypotheses given in this section.

Definition 4.1 A function $u \in W_0^{s,\mathcal{H}}(\Omega)$ is said to be a weak solution of problem (1.1), if there exist $\xi(x) \in f(x, u(x))$ for a.a. $x \in \Omega$ satisfying

$$\int_\Omega \int_\Omega \mathcal{H}'(x, y, |D_s u(x, y)|) D_s v(x, y) \cdot \mathrm{d}\nu = \int_\Omega \xi v \,\mathrm{d}x \tag{4.1}$$

for all $v \in W_0^{s,\mathcal{H}}(\Omega)$.

4.1 Subcritical growth

First, we consider the subcritical case and suppose appropriate growth conditions on f that guarantee that the set $\mathcal{F}(u)$ given above is not empty.

(H2) (i) Assume $f \colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is graph measurable and $f(x, \cdot) \colon \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is upper semicontinuous for a.a. $x \in \Omega$.

(ii) Let $\gamma > 0$, $\varsigma, \tau \in C(\overline{\Omega})$ such that $p^+ < \varsigma(x) < p_s^*(x)$ and $q^+ < \tau(x) < q_s^*(x)$ for all $x \in \overline{\Omega}$. Suppose that there exists a constant $\beta > 0$ satisfying

$$\begin{aligned} &\sup\{|\xi| \colon \xi \in f(x, t)\} \\ &\leq \beta \left[|t|^{\varsigma(x)-1} \log^{\frac{\varsigma(x)}{N}}(e + \omega|t|) + \mu(x)^\gamma |t|^{\tau(x)-1} \log^{\frac{\tau(x)}{N}}(e + \omega|t|) + 1 \right] \end{aligned}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.

The next theorem is one of our main results in this section.

Theorem 4.2 *Let hypotheses* (H1) *and* (H2) *be satisfied. Then, for any weak solution* $u \in W_0^{s,\mathcal{H}}(\Omega)$ *of problem* (1.1)*, it holds that* $u \in L^\infty(\Omega)$ *and*

$$\|u\|_{\infty,\Omega} \le C \max\left\{\|u\|_{\mathcal{B},\Omega}^{\ell_1}, \|u\|_{\mathcal{B},\Omega}^{\ell_2}\right\}, \tag{4.2}$$

where the positive constants C, ℓ_1, ℓ_2 *are independent of* u.

Proof Assume that $u \in W_0^{s,\mathcal{H}}(\Omega)$ is a weak solution of problem (1.1). Our proof is divided into several steps.

Step 1. Constructing the iteration sequence and developing basic estimates.

For any $n \in \mathbb{N}_0$ we define

$$\begin{aligned} Z_n := \int_{A_{\psi_n}} \Big[& (u-\psi_n)^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega(u-\psi_n)) \\ & + \mu(x)^\gamma \, (u-\psi_n)^{\tau(x)} \log^{\frac{\tau(x)}{N}}(e+\omega(u-\psi_n))\Big]\,\mathrm{d}x, \end{aligned} \tag{4.3}$$

with

$$A_\psi := \{x \in \Omega \colon u(x) > \psi\}, \quad \psi \in \mathbb{R}. \tag{4.4}$$

Moreover, for $n \in \mathbb{N}_0$, ψ_n is defined by

$$\psi_n := \psi_*\left(2 - \frac{1}{2^n}\right), \tag{4.5}$$

where $\psi_* > 0$ will be specified later. Obviously, for all $n \in \mathbb{N}_0$, we have

$$\begin{aligned} &\psi_n \nearrow 2\psi_* \quad \text{and} \quad \psi_* \le \psi_n < 2\psi_*, \\ &A_{\psi_{n+1}} \subset A_{\psi_n} \quad \text{and} \quad Z_{n+1} \le Z_n. \end{aligned}$$

By the definition of ψ_n, we obtain

$$u(x) - \psi_n \ge u(x)\left(1 - \frac{\psi_n}{\psi_{n+1}}\right) = \frac{u(x)}{2^{n+2}-1} \quad \text{for a.a.}\ x \in A_{\psi_{n+1}}$$

and

$$\begin{aligned} \left|A_{\psi_{n+1}}\right| &\le \int_{A_{\psi_{n+1}}} \left(\frac{u-\psi_n}{\psi_{n+1}-\psi_n}\right)^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega(u-\psi_n))\,\mathrm{d}x \\ &\le \int_{A_{\psi_n}} \frac{2^{\varsigma(x)(n+1)}}{\psi_*^{\varsigma(x)}} (u-\psi_n)^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega(u-\psi_n))\,\mathrm{d}x. \end{aligned}$$

This implies

$$u(x) \le \left(2^{n+2}-1\right)(u(x)-\psi_n) \quad \text{for a.a. } x \in A_{\psi_{n+1}} \text{ and for all } n \in \mathbb{N}_0, \tag{4.6}$$

$$\left|A_{\psi_{n+1}}\right| \le \left(\psi_*^{-\varsigma^-} + \psi_*^{-\varsigma^+}\right) 2^{(n+1)\varsigma^+} Z_n \le 2\left(1+\psi_*^{-\varsigma^+}\right) 2^{(n+1)\varsigma^+} Z_n \quad \text{for all } n \in \mathbb{N}_0. \tag{4.7}$$

Let $u_n := (u-\psi_{n+1})_+$ for $n \in \mathbb{N}_0$. We claim that

$$\begin{aligned}\int_\Omega\int_\Omega\Bigg(&\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\&+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\mathrm{d}\nu\\&\le C_1\left(1+\psi_*^{-\varsigma^+}\right)2^{n(\alpha_0+\frac{\alpha_0}{N})}Z_n,\end{aligned}\tag{4.8}$$

where $\alpha_0 := \max\{\varsigma^+, \tau^+\}$. Now, we are going to verify (4.8). To this end, we take $u_n = (u-\psi_{n+1})_+ \in W_0^{s,\mathcal{H}}(\Omega)$ as test function in (4.1) and obtain

$$\begin{aligned}\int_\Omega\int_\Omega\Bigg(&\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(u_n(x)-u_n(y))}{|x-y|^{sp(x,y)}}\log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\\&+\frac{\omega|u(x)-u(y)|^{p(x,y)-1}(u(x)-u(y))(u_n(x)-u_n(y))}{|x-y|^{s(p(x,y)+1)}\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)}\\&+\mu(x,y)\frac{|u(x)-u(y)|^{q(x,y)-2}(u(x)-u(y))(u_n(x)-u_n(y))}{|x-y|^{sq(x,y)}}\log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\\&+\mu(x,y)\frac{\omega|u(x)-u(y)|^{q(x,y)-1}(u(x)-u(y))(u_n(x)-u_n(y))}{|x-y|^{s(q(x,y)+1)}\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)}\Bigg)\mathrm{d}\nu\\&=\int_\Omega \xi u_n(x)\,\mathrm{d}x.\end{aligned}$$

Since $(u(x)-u(y))(u_n(x)-u_n(y)) \ge (u_n(x)-u_n(y))^2$ and $|u(x)-u(y)| \ge |u_n(x)-u_n(y)|$, also, $u \ge u-\psi_{n+1} \ge 0$ on $A_{\psi_{n+1}}$, by the above equality, we calculate that

$$\begin{aligned}\int_\Omega\int_\Omega\Bigg(&\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\&+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\mathrm{d}\nu\\&\le\int_\Omega\beta\left[|u|^{\varsigma(x)-1}\log^{\frac{\varsigma(x)}{N}}(e+\omega|u|)+\mu(x)^\gamma|u|^{\tau(x)-1}\log^{\frac{\tau(x)}{N}}(e+\omega|u|)+1\right]u_n(x)\,\mathrm{d}x\\&\le 2\beta\int_{A_{\psi_{n+1}}}\left[u^{\varsigma(x)}\log^{\frac{\varsigma(x)}{N}}(e+\omega u)+\mu(x)^\gamma u^{\tau(x)}\log^{\frac{\tau(x)}{N}}(e+\omega u)+1\right]\mathrm{d}x\\&\le C_2\int_{A_{\psi_{n+1}}}\Bigg(\left[\left(2^{n+2}-1\right)(u-\psi_n)\right]^{\varsigma(x)}\log^{\frac{\varsigma(x)}{N}}\left[e+\omega\left(2^{n+2}-1\right)(u-\psi_n)\right]\\&+\mu(x)^\gamma\left[\left(2^{n+2}-1\right)(u-\psi_n)\right]^{\tau(x)}\\&\log^{\frac{\tau(x)}{N}}\left[e+\omega\left(2^{n+2}-1\right)(u-\psi_n)\right]\Bigg)\mathrm{d}x+C_2\left|A_{\psi_{n+1}}\right|\\&\le C_1\left(1+\psi_*^{-\varsigma^+}\right)2^{n(\alpha_0+\frac{\alpha_0}{N})}Z_n,\end{aligned}\tag{4.9}$$

which associated (4.7) indicates (4.8).

Step 2. Localization and estimating Z_{n+1} by Z_n.

Let $B_i \subset \mathbb{R}^N$ be open balls of radius R with $i \in \mathcal{I} := \{1, \cdots, m\}$ and let $\{B_i\}_{i\in\mathcal{I}}$ be a finite open covering of $\overline{\Omega}$ such that $\Omega_i := B_i \cap \Omega$ for $i \in \mathcal{I}$ are Lipschitz domains. For any $i \in \mathcal{I}$, we choose R small enough such that

$$|\Omega_i| < 1, \quad p_i^+ := \sup_{(x,y)\in B_i\times B_i} p(x,y) < \varsigma_i^- := \inf_{x\in B_i\cap\Omega} \varsigma(x) \le \varsigma_i^+ := \sup_{x\in B_i\cap\Omega} \varsigma(x) < \left(p_i^-\right)_s^* \tag{4.10}$$

$$\text{and} \quad q_i^+ := \sup_{(x,y)\in B_i\times B_i} q(x,y) < \tau_i^- := \inf_{x\in B_i\cap\Omega} \tau(x) \le \tau_i^+ := \sup_{x\in B_i\cap\Omega} \tau(x) < \left(q_i^-\right)_s^*. \tag{4.11}$$

Let $\{\eta_i\}_{i=1}^m$ be a partition of unity of $\overline{\Omega}$ with respect to $\{B_i\}_{i=1}^m$, namely, for each $i \in \mathcal{I}$, we have

$$\eta_i \in C_c^\infty(\mathbb{R}^N), \quad \operatorname{supp}(\eta_i) \subset B_i, \quad 0 \le \eta_i \le 1 \quad \text{and} \quad \sum_{i=1}^m \eta_i = 1 \quad \text{on } \overline{\Omega}. \tag{4.12}$$

By applying Jensen's inequality and the following interpolation inequality

$$t^{\alpha_2} \le t^{\alpha_1} + t^{\alpha_3} \quad \text{for all } t \ge 0 \text{ and for all } \alpha_1, \alpha_2, \alpha_3 \text{ with } 0 < \alpha_1 \le \alpha_2 \le \alpha_3, \tag{4.13}$$

we get

$$\begin{aligned} Z_{n+1} &= \int_{A_{\psi_{n+1}}} \left(u_n^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega u_n) + \mu(x)^\gamma u_n^{\tau(x)} \log^{\frac{\tau(x)}{N}}(e+\omega u_n) \right) \mathrm{d}x \\ &\le m^{\max\{\varsigma^+,\tau^+\}} \sum_{i=1}^m \int_{A_{\psi_{n+1}}\cap\Omega_i} \Big(|u_n\eta_i|^{\varsigma_i(x)} \log^{\frac{\varsigma_i(x)}{N}}(e+\omega|u_n\eta_i|) \\ &\quad + \mu(x)^\gamma |u_n\eta_i|^{\tau_i(x)} \log^{\frac{\tau_i(x)}{N}}(e+\omega|u_n\eta_i|) \Big) \mathrm{d}x \\ &\le m^{\max\{\varsigma^+,\tau^+\}} \sum_{i=1}^m \int_{A_{\psi_{n+1}}\cap\Omega_i} \Big(|u_n\eta_i|^{\varsigma_i^+} \log^{\frac{\varsigma_i^+}{N}}(e+\omega|u_n\eta_i|) \\ &\quad + \mu(x)^\gamma |u_n\eta_i|^{\tau_i^+} \log^{\frac{\tau_i^+}{N}}(e+\omega|u_n\eta_i|) + |u_n\eta_i|^{\varsigma_i^-} \log^{\frac{\varsigma_i^-}{N}}(e+\omega|u_n\eta_i|) \\ &\quad + \mu(x)^\gamma |u_n\eta_i|^{\tau_i^-} \log^{\frac{\tau_i^-}{N}}(e+\omega|u_n\eta_i|) \Big) \mathrm{d}x. \end{aligned} \tag{4.14}$$

For any $i \in \mathcal{I}$, $r_1 > 0$, and $r_2 > 0$, we define

$$L_{n,i}(r_1,r_2) := \int_{A_{\psi_{n+1}}\cap\Omega_i} \left[|u_n\eta_i|^{r_1} \log^{\frac{r_1}{N}}(e+\omega|u_n\eta_i|) + \mu(x)^\gamma |u_n\eta_i|^{r_2} \log^{\frac{r_2}{N}}(e+\omega|u_n\eta_i|) \right] \mathrm{d}x. \tag{4.15}$$

Then, from (4.14) and (4.15) it follows that

$$Z_{n+1} \le m^{\max\{\varsigma^+,\tau^+\}} \sum_{i=1}^m \left[L_{n,i}(\varsigma_i^-, \tau_i^-), L_{n,i}(\varsigma_i^+, \tau_i^+) \right]. \tag{4.16}$$

Let $\star \in \{+,-\}$ for $i \in \mathcal{I}$. Using (4.10) and Hölder's inequality for $\varepsilon > 0$ satisfying $\varsigma^\star + \varepsilon < (p_i^-)_s^*$ and $\tau^\star + \varepsilon < (q_i^-)_s^*$ we arrive at

$$
\begin{aligned}
L_{n,i}(\varsigma_i^\star, \tau_i^\star) &= \int_{A_{\psi_{n+1}} \cap \Omega_i} \left[|u_n\eta_i|^{\varsigma_i^\star} \log^{\frac{\varsigma_i^\star}{N}}(e+\omega|u_n\eta_i|) + \mu(x)^\gamma |u_n\eta_i|^{\tau_i^\star} \log^{\frac{\tau_i^\star}{N}}(e+\omega|u_n\eta_i|) \right] \mathrm{d}x \\
&\le \left(\int_\Omega |u_n\eta_i|^{\varsigma_i^\star+\varepsilon} \log^{\frac{\varsigma_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x \right)^{\frac{\varsigma_i^\star}{\varsigma_i^\star+\varepsilon}} |A_{\psi_{n+1}} \cap \Omega_i|^{\frac{\varepsilon}{\varsigma_i^\star+\varepsilon}} \\
&\quad + \left(\int_\Omega \mu(x)^\gamma |u_n\eta_i|^{\tau_i^\star+\varepsilon} \log^{\frac{\tau_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x \right)^{\frac{\tau_i^\star}{\tau_i^\star+\varepsilon}} |A_{\psi_{n+1}} \cap \Omega_i|^{\frac{\varepsilon}{\tau_i^\star+\varepsilon}} \\
&\le |A_{\psi_{n+1}} \cap \Omega_i|^{\frac{\varepsilon}{\varsigma^+ + \tau^+ + \varepsilon}} \left[\left(\int_\Omega |u_n\eta_i|^{\varsigma_i^\star+\varepsilon} \log^{\frac{\varsigma_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x \right)^{\frac{\varsigma_i^\star}{\varsigma_i^\star+\varepsilon}} \right. \\
&\quad \left. + \left(\int_\Omega \mu(x)^\gamma |u_n\eta_i|^{\tau_i^\star+\varepsilon} \log^{\frac{\tau_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x \right)^{\frac{\tau_i^\star}{\tau_i^\star+\varepsilon}} \right].
\end{aligned}
\tag{4.17}
$$

Next, we denote

$$
\widetilde{\mathcal{B}}(x,t) := t^{\varsigma_i^\star+\varepsilon} \log^{\frac{\varsigma_i^\star+\varepsilon}{N}}(e+\omega t) + \mu(x)^\gamma t^{\tau_i^\star+\varepsilon} \log^{\frac{\tau_i^\star+\varepsilon}{N}}(e+\omega t). \tag{4.18}
$$

By Proposition 2.12, we see that

$$
W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow L^{\widetilde{\mathcal{B}}}(\Omega). \tag{4.19}
$$

Note that for $s, t \ge 0$ and $r \ge 1$,

$$
\begin{aligned}
(s+t)^r \log(e+s+t) &\le (2s)^r \log(e+2s) + (2t)^r \log(e+2t) \\
&\le 2^{r+1} s^r \log(e+s) + 2^{r+1} t^r \log(e+t),
\end{aligned}
\tag{4.20}
$$

and for all $t \ge 0$, $C \ge 1$

$$
\log(e+Ct) \le C \log(e+t). \tag{4.21}
$$

Invoking the above inequalities, Remark 2.7, (4.10) and the continuous embedding (4.19) we see that there exist $\sigma > 0$ such that

$$
\sigma < \min\{\varsigma_i^- - p_i^+, \tau_i^- - q_i^+\} \quad \text{for } i \in \mathcal{I}
$$

satisfying

$$
\begin{aligned}
&\left(\int_\Omega |u_n\eta_i|^{\varsigma_i^\star+\varepsilon} \log^{\frac{\varsigma_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x \right)^{\frac{\varsigma_i^\star}{\varsigma_i^\star+\varepsilon}} \\
&\le \|u_n\eta_i\|_{\widetilde{\mathcal{B}}|_{\mu\equiv 0},\Omega}^{\tilde{\varsigma}_i^\star} \le C_3 \, [u_n\eta_i]_{s,\mathcal{H}|_{\mu\equiv 0},\Omega}^{\tilde{\varsigma}_i^\star} \le C_4 \left(S_{n,i}^{\frac{\tilde{\varsigma}_i^\star}{p_i^-}} + S_{n,i}^{\frac{\tilde{\varsigma}_i^\star}{p_i^+ + \sigma}} \right),
\end{aligned}
\tag{4.22}
$$

where

$$
\tilde{\varsigma}_i^\star = \begin{cases} \varsigma_i^\star & \text{if } \|u_n\eta_i\|_{\widetilde{\mathcal{B}},\Omega} \le 1, \\ \varsigma_i^\star + \frac{\varsigma_i^\star}{N} & \text{if } \|u_n\eta_i\|_{\widetilde{\mathcal{B}},\Omega} > 1, \end{cases} \tag{4.23}
$$

and

$$S_{n,i} = \int_\Omega \int_\Omega \left(\frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)}} \log\left(e + \omega \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|}{|x-y|^s}\right) + \mu(x,y) \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|^{q_i(x,y)}}{|x-y|^{sq_i(x,y)}} \log\left(e + \omega \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|}{|x-y|^s}\right) \right) \mathrm{d}\nu. \tag{4.24}$$

Analogously, by Remark 2.7, (4.11) and the continuous embedding (4.19) we obtain

$$\begin{aligned} &\left(\int_\Omega \mu(x)^\gamma |u_n\eta_i|^{\tau_i^\star+\varepsilon} \log^{\frac{\tau_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x \right)^{\frac{\tau_i^\star}{\tau_i^\star+\varepsilon}} \\ &\leq \|u_n\eta_i\|_{\widetilde{\mathcal{B}},\Omega}^{\tilde{\tau}_i^\star} \leq C_5\,[u_n\eta_i]_{s,\mathcal{H},\Omega}^{\tilde{\tau}_i^\star} \leq C_6 \left(S_{n,i}^{\frac{\tilde{\tau}_i^\star}{p_i^-}} + S_{n,i}^{\frac{\tilde{\tau}_i^\star}{q_i^++\sigma}} \right), \end{aligned} \tag{4.25}$$

with

$$\tilde{\tau}_i^\star = \begin{cases} \tau_i^\star & \text{if } \|u_n\eta_i\|_{\widetilde{\mathcal{B}},\Omega} \leq 1, \\ \tau_i^\star + \frac{\tau_i^\star}{N} & \text{if } \|u_n\eta_i\|_{\widetilde{\mathcal{B}},\Omega} > 1. \end{cases} \tag{4.26}$$

From the inequalities (4.16), (4.17), (4.22) and (4.25), we get

$$Z_{n+1} \leq C_7 \left| A_{\psi_{n+1}} \cap \Omega_i \right|^{\frac{\varepsilon}{\varsigma^+ + \tau^+ + \varepsilon}} \left(S_{n,i}^{\frac{\tilde{\varsigma}_i^\star}{p_i^-}} + S_{n,i}^{\frac{\tilde{\varsigma}_i^\star}{p_i^++\sigma}} + S_{n,i}^{\frac{\tilde{\tau}_i^\star}{p_i^-}} + S_{n,i}^{\frac{\tilde{\tau}_i^\star}{q_i^++\sigma}} \right).$$

Combining this and (4.13) we infer

$$Z_{n+1} \leq C_8 \left| A_{\psi_{n+1}} \right|^{\frac{\varepsilon}{\varsigma^+ + \tau^+ + \varepsilon}} \left(S_{n,i}^{1+\theta_1} + S_{n,i}^{1+\theta_2} \right), \tag{4.27}$$

with

$$0 < \theta_1 := \min_{1\leq i\leq m} \min\left\{ \frac{\varsigma_i^-}{p_i^+ + \sigma}, \frac{\tau_i^-}{q_i^+ + \sigma} \right\} - 1 \leq \theta_2 := \max_{1\leq i\leq m} \max\left\{ \frac{\varsigma_i^+ + \frac{\varsigma_i^+}{N}}{p_i^-}, \frac{\tau_i^+ + \frac{\tau_i^+}{N}}{p_i^-} \right\} - 1.$$

Next, let

$$S_{n,i} = J_1 + 2J_2, \tag{4.28}$$

where

$$\begin{aligned} J_1 = \int_{B_i} \int_{B_i} \Bigg(&\frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)}} \log\left(e + \omega \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|}{|x-y|^s}\right) \\ &+ \mu(x,y) \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|^{q_i(x,y)}}{|x-y|^{sq_i(x,y)}} \\ &\times \log\left(e + \omega \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|}{|x-y|^s}\right) \Bigg) \mathrm{d}\nu, \end{aligned}$$

and

$$J_2 = \int_{\Omega\setminus B_i} \int_{B_i} \left(\frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)}} \log\left(e + \omega \frac{|u_n(x)\eta_i(x) - u_n(y)\eta_i(y)|}{|x-y|^s}\right) \right.$$

$$+\mu(x,y)\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{q_i(x,y)}}{|x-y|^{sq_i(x,y)}}$$
$$\times\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\bigg)\,d\nu.$$

Next, we introduce the indicator function χ_μ satisfying $\chi_\mu(x)=1$ if $\mu(x)>0$ and $\chi_\mu(x)=0$ if $\mu(x)=0$. Applying inequalities (4.20), (4.21) and the interpolation inequality (4.13) we see that

$$\begin{aligned}
J_1 &= \int_{B_i}\int_{B_i}\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{q_i(x,y)}}{|x-y|^{N+sq_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)dx\,dy\\
&\le 2^{p_i^++1}\int_{B_i}\int_{B_i}\frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)dx\,dy\\
&\quad+2^{p_i^++1}\max\left\{\|\nabla\eta_i\|_\infty^{p_i^-},\|\nabla\eta_i\|_\infty^{p_i^++1}\right\}\\
&\quad\times\int_{B_i}\left(\int_{B_i}\frac{dx}{|x-y|^{N+(s-1)p_i^-}}+\int_{B_i}\frac{dx}{|x-y|^{N+(s-1)(p_i^++1)}}\right)\\
&\quad\times\left(|u_n(y)|^{p_i^-}+|u_n(y)|^{p_i^+}\right)\log(e+\omega|u_n(y)|)\,dy\\
&\quad+2^{q_i^++1}\int_{B_i}\int_{B_i}\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i(x,y)}}{|x-y|^{N+sq_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)dx\,dy\\
&\quad+2^{q_i^++1}\|\mu\|_\infty\max\left\{\|\nabla\eta_i\|_\infty^{q_i^-},\|\nabla\eta_i\|_\infty^{q_i^++1}\right\}\\
&\quad\times\int_{B_i}\left(\int_{B_i}\frac{dx}{|x-y|^{N+(s-1)q_i^-}}+\int_{B_i}\frac{dx}{|x-y|^{N+(s-1)(q_i^++1)}}\right)\\
&\quad\times\chi_\mu(y)\left(|u_n(y)|^{q_i^-}+|u_n(y)|^{q_i^+}\right)\log(e+\omega|u_n(y)|)\,dy\\
&\le C_9\int_{B_i}\int_{B_i}\frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i(x,y)}}{|x-y|^{sq_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)d\nu\\
&\quad+C_{10}\int_{B_i}|u_n(y)|^{p_i^+}\log(e+\omega|u_n(y)|)+\mu(y)|u_n(y)|^{q_i^+}\log(e+\omega|u_n(y)|)\,dy\\
&\quad+C_{10}\int_{B_i}|u_n(y)|^{p_i^-}\log(e+\omega|u_n(y)|)+\mu(y)|u_n(y)|^{q_i^-}\log(e+\omega|u_n(y)|)\,dy,
\end{aligned}\tag{4.29}$$

where we have used

$$\int_{B_i}\frac{dx}{|x-y|^{N+(s-1)r}}\le\int_{B_{R'}(0)}\frac{dz}{|z|^{N+(s-1)r}}=\frac{\omega_N(R')^{(1-s)r}}{(1-s)r}\tag{4.30}$$

for $r>0$ and $R'>1$ satisfying $B_i\subset B_{R'}(0)$ for all $i\in\mathcal{I}$. Since $u\ge u-\psi_{n+1}\ge 0$ on $A_{\psi_{n+1}}$, associating (4.9) and (4.13) we calculate that for $\hat{p}_i\in\{p_i^-,p_i^+,p^-,p^+\}$ and $\hat{q}_i\in\{q_i^-,q_i^+,q^-,q^+\}$, there hold

$$\begin{aligned}
&\int_{B_i} |u_n(y)|^{\hat{p}_i} \log(e+\omega|u_n(y)|) + \mu(y)|u_n(y)|^{\hat{q}_i} \log(e+\omega|u_n(y)|)\,\mathrm{d}y \\
&\quad \leq \int_{A_{\psi_{n+1}}\cap B_i} |u(y)|^{\hat{p}_i} \log(e+\omega|u(y)|) + \mu(y)|u(y)|^{\hat{q}_i} \log(e+\omega|u(y)|)\,\mathrm{d}y \\
&\quad \leq C_{11} \int_{A_{\psi_{n+1}}\cap B_i} \left[u(x)^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega u(x)) + \mu(x)^{\gamma} u(x)^{\tau(x)} \log^{\frac{\tau(x)}{N}}(e+\omega u(x))\right] \mathrm{d}x \\
&\qquad + C\left|A_{\psi_{n+1}}\right| \\
&\quad \leq C_{12}\left(1+\psi_*^{-\varsigma^+}\right) 2^{n(\alpha_0+\frac{\alpha_0}{N})} Z_n.
\end{aligned} \tag{4.31}$$

Combining (4.29)–(4.31) we get

$$J_1 \leq C_{13}\left(1+\psi_*^{-\varsigma^+}\right) 2^{n(\alpha_0+\frac{\alpha_0}{N})} Z_n.$$

Similarly, by inequalities (4.13), (4.20), (4.21) and (4.31) we have

$$\begin{aligned}
J_2 &= \int_{\Omega\setminus B_i} \left[\int_{B_i} \frac{|u_n(x)\eta_i(x)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}} \log\left(e+\omega\frac{|u_n(x)\eta_i(x)|}{|x-y|^s}\right)\right. \\
&\qquad \left. +\mu(x,y)\frac{|u_n(x)\eta_i(x)|^{q_i(x,y)}}{|x-y|^{N+sq_i(x,y)}} \log\left(e+\omega\frac{|u_n(x)\eta_i(x)|}{|x-y|^s}\right)\mathrm{d}x\right]\mathrm{d}y \\
&\leq \int_{\operatorname{supp}(\eta_i)\cap A_{\psi_{n+1}}} \left(\int_{\Omega\setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+sp^-}} + \int_{\Omega\setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+s(p^++1)}}\right) \\
&\qquad \times\left(|u_n(y)|^{p^-}+|u_n(y)|^{p^+}\right)\log(e+\omega|u_n(y)|)\,\mathrm{d}x \\
&\qquad + \|\mu\|_\infty \int_{\operatorname{supp}(\eta_i)\cap A_{\psi_{n+1}}} \left(\int_{\Omega\setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+sq^-}} + \int_{\Omega\setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+s(q^++1)}}\right) \\
&\qquad \times \chi_\mu(x)\left(|u_n(y)|^{q^-}+|u_n(y)|^{q^+}\right)\log(e+\omega|u_n(y)|)\,\mathrm{d}x \\
&\leq C_{14}\int_{A_{\psi_{n+1}}\cap B_i} |u(x)|^{p^-}\log(e+\omega|u(x)|)+\mu(x)|u(x)|^{q^-}\log(e+\omega|u(x)|)\,\mathrm{d}x \\
&\qquad + C_{14}\int_{A_{\psi_{n+1}}\cap B_i} |u(x)|^{p^+}\log(e+\omega|u(x)|)+\mu(x)|u(x)|^{q^+}\log(e+\omega|u(x)|)\,\mathrm{d}x \\
&\leq C_{15}\left(1+\psi_*^{-\varsigma^+}\right) 2^{n(\alpha_0+\frac{\alpha_0}{N})} Z_n,
\end{aligned} \tag{4.32}$$

where we have used that

$$\sup_{x\in\operatorname{supp}(\eta_i)} \int_{\Omega\setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+sr}} \leq \int_{|z|\geq d_i} \frac{\mathrm{d}y}{|z|^{N+sr}} = \frac{\omega_N}{srd_i^{sr}},$$

with $d_i := \operatorname{dist}(\Omega\setminus B_i, \operatorname{supp}(\eta_i)) > 0$ and $r > 0$.

Inequality (4.8) and (4.28)-(4.32) lead to

$$S_n \leq C_{16}\left(1+\psi_*^{-\varsigma^+}\right) 2^{n(\alpha_0+\frac{\alpha_0}{N})} Z_n \quad \text{for all } n\in\mathbb{N}_0.$$

Therefore, we get

$$S_n^{1+\theta_1}+S_n^{1+\theta_2} \leq C_{17}\left(1+\psi_*^{-\varsigma^+(1+\theta_2)}\right) 2^{n(\alpha_0+\frac{\alpha_0}{N})(1+\theta_2)}\left(Z_n^{1+\theta_1}+Z_n^{1+\theta_2}\right). \tag{4.33}$$

Moreover, (4.7) yields

$$\left|A_{\psi_{n+1}}\right|^{\frac{\varepsilon}{\varsigma^++\tau^++\varepsilon}} \le C_{18}\left(\psi_*^{-\frac{\varepsilon\varsigma^-}{\varsigma^++\tau^++\varepsilon}} + \psi_*^{-\frac{\varepsilon\varsigma^+}{\varsigma^++\tau^++\varepsilon}}\right) 2^{\frac{\varepsilon\varsigma^+}{\varsigma^++\tau^++\varepsilon}} Z_n^{\frac{\varepsilon}{\varsigma^++\tau^++\varepsilon}}.$$

Taking this and (4.27) as well as (4.33) into account, we get

$$Z_{n+1} \le C_{19}\left(\psi_*^{-\rho_1} + \psi_*^{-\rho_2}\right) k^n \left(Z_n^{1+\gamma_1} + Z_n^{1+\gamma_2}\right) \quad \text{for all } n \in \mathbb{N}_0, \tag{4.34}$$

where

$$\begin{aligned}
0 < \rho_1 &:= \frac{\varepsilon\varsigma^-}{\varsigma^++\tau^++\varepsilon} < \rho_2 := \varsigma^+(1+\theta_2) + \frac{\varepsilon\varsigma^+}{\varsigma^++\tau^++\varepsilon}\\
1 < k &:= 2^{(\alpha_0+\frac{\alpha_0}{N})(1+\theta_2)\frac{\varepsilon\varsigma^+}{\varsigma^++\tau^++\varepsilon}},\\
0 < \gamma_1 &:= \theta_1 + \frac{\varepsilon}{\varsigma^++\tau^++\varepsilon} \le \gamma_2 := \theta_2 + \frac{\varepsilon}{\varsigma^++\tau^++\varepsilon}.
\end{aligned}$$

Recall that $\alpha_0 = \max\{\varsigma^+, \tau^+\}$.

Step 3. A priori bounds

Referring to Lemma 2.15, we see that (4.34) yield

$$Z_n \to 0 \quad \text{as } n \to \infty, \tag{4.35}$$

provided that

$$Z_0 \le \min\left\{\left(2C_{19}\left(\psi_*^{-\rho_1} + \psi_*^{-\rho_2}\right)\right)^{-\frac{1}{\gamma_1}} k^{-\frac{1}{\gamma_1^2}}, \left(2C_{19}\left(\psi_*^{-\rho_1} + \psi_*^{-\rho_2}\right)\right)^{-\frac{1}{\gamma_2}} k^{-\frac{1}{\gamma_1\gamma_2}-\frac{\gamma_2-\gamma_1}{\gamma_2^2}}\right\}.$$

Note that

$$\begin{aligned}
Z_0 &= \int_\Omega \Big[(u-\psi_*)_+^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega(u-\psi_*)_+)\\
&\quad +\mu(x)^\gamma (u-\psi_*)_+^{\tau(x)} \log^{\frac{\tau(x)}{N}}(e+\omega(u-\psi_*)_+)\Big]\,\mathrm{d}x\\
&\le \int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x.
\end{aligned}$$

We also see that

$$\begin{aligned}
\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x &\le \left(2C_{19}\left(\psi_*^{-\rho_1} + \psi_*^{-\rho_2}\right)\right)^{-\frac{1}{\gamma_1}} k^{-\frac{1}{\gamma_1^2}},\\
\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x &\le \left(2C_{19}\left(\psi_*^{-\rho_1} + \psi_*^{-\rho_2}\right)\right)^{-\frac{1}{\gamma_2}} k^{-\frac{1}{\gamma_1\gamma_2}-\frac{\gamma_2-\gamma_1}{\gamma_2^2}}
\end{aligned}$$

is equivalent to

$$\begin{aligned}
\psi_*^{-\rho_1} + \psi_*^{-\rho_2} &\le (2C_{19})^{-1} k^{-\frac{1}{\gamma_1}} \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{-\gamma_1},\\
\psi_*^{-\rho_1} + \psi_*^{-\rho_2} &\le (2C_{19})^{-1} k^{-\frac{1}{\gamma_1}-\frac{\gamma_2-\gamma_1}{\gamma_2}} \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{-\gamma_2}.
\end{aligned}$$

Moreover,

$$2\psi_*^{-\rho_1} \leq (2C_{19})^{-1} k^{-\frac{1}{\gamma_1}-\frac{\gamma_2-\gamma_1}{\gamma_2}} \min\left\{\left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{-\gamma_1}, \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{-\gamma_2}\right\},$$

$$2\psi_*^{-\rho_2} \leq (2C_{19})^{-1} k^{-\frac{1}{\gamma_1}-\frac{\gamma_2-\gamma_1}{\gamma_2}} \min\left\{\left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{-\gamma_1}, \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{-\gamma_2}\right\},$$

is equivalent to

$$\psi_* \geq (4C_{19})^{\frac{1}{\rho_1}} k^{\frac{1}{\rho_1}(\frac{1}{\gamma_1}+\frac{\gamma_2-\gamma_1}{\gamma_2})} \max\left\{\left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{\frac{\gamma_1}{\rho_1}}, \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{\frac{\gamma_2}{\rho_1}}\right\},$$

$$\psi_* \geq (4C_{19})^{\frac{1}{\rho_2}} k^{\frac{1}{\rho_2}(\frac{1}{\gamma_1}+\frac{\gamma_2-\gamma_1}{\gamma_2})} \max\left\{\left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{\frac{\gamma_1}{\rho_2}}, \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{\frac{\gamma_2}{\rho_2}}\right\}.$$

Hence, if we take

$$\begin{aligned}\psi_* = \max&\left\{(4C_{19})^{\frac{1}{\rho_1}}, (4C_{19})^{\frac{1}{\rho_2}}\right\} k^{\frac{1}{\rho_1}\left(\frac{1}{\gamma_1}+\frac{\gamma_2-\gamma_1}{\gamma_2}\right)} \\ &\cdot \max\left\{\left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{\frac{\gamma_1}{\rho_2}}, \left(\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x\right)^{\frac{\gamma_2}{\rho_1}}\right\},\end{aligned}$$

(4.35) holds true, by applying Lebesgue's dominated convergence theorem we have

$$\begin{aligned} Z_n &= \int_\Omega \Big[(u-\psi_n)_+^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega(u-\psi_n)_+) \\ &\quad +\mu(x)^\gamma (u-\psi_n)_+^{\tau(x)} \log^{\frac{\tau(x)}{N}}(e+\omega(u-\psi_n)_+)\Big]\,\mathrm{d}x \\ &\to \int_\Omega \Big[(u-2\psi_*)_+^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}}(e+\omega(u-2\psi_*)_+) \\ &\quad +\mu(x)^\gamma (u-2\psi_*)_+^{\tau(x)} \log^{\frac{\tau(x)}{N}}(e+\omega(u-2\psi_*)_+)\Big]\,\mathrm{d}x \to 0, \end{aligned}$$

as $n\to\infty$. This implies that

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) \leq 2\psi_*.$$

Analogously, by replacing u with $-u$, we get

$$\operatorname*{ess\,sup}_{x\in\Omega} (-u)(x) \leq 2\psi_*.$$

Therefore,

$$\|u\|_{\infty,\Omega} \leq C\max\left\{\int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x^{\ell_1}, \int_\Omega \mathcal{B}(x,|u|)\,\mathrm{d}x^{\ell_2}\right\}, \tag{4.36}$$

with C, ℓ_1, ℓ_2 being positive constants independent of u. Finally, from (4.36) and Remark 2.7, we obtain (4.2). □

In addition, motivated by Ho–Kim [35] we can expend the range of ς and τ given in (H2)(ii) by strengthening the restrictive conditions on p and q (see (H2')(iii)). For this purpose, we consider the following assumptions:

(H2') (i) Assume $f: \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is graph measurable and $f(x, \cdot): \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is upper semicontinuous for a.a. $x \in \Omega$.

(ii) Let $\varsigma, \tau \in C(\overline{\Omega})$ such that $p(x) < \varsigma(x) < p_s^*(x)$ and $q(x) < \tau(x) < q_s^*(x)$ for all $x \in \overline{\Omega}$. Suppose that there exists a constant $\beta > 0$ satisfying

$$\begin{aligned} &\sup\{|\xi|: \xi \in f(x,t)\} \\ &\leq \beta\left[|t|^{\varsigma(x)-1} \log^{\frac{\varsigma(x)}{N}}(e+\omega|t|) + \mu(x)^{\gamma}|t|^{\tau(x)-1} \log^{\frac{\tau(x)}{N}}(e+\omega|t|) + 1\right] \end{aligned}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.

(iii) For $r \in \{p, q\}$, the following hypotheses hold

$$\inf_{R>0} \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ 0<|x-y|<1/2}} \left|r(x,y) - r^-_{B_R(x,y)}\right| \log \frac{1}{|x-y|} < \infty, \tag{4.37}$$

with $r^-_{B_R(x,y)} := \inf_{(\bar{x},\bar{y})\in B_R(x,y)} r(\bar{x},\bar{y})$.

Remark 4.3 A example for $r \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying the hypotheses (H2')(iii) was given by Ho–Kim [35, Example 4.3].

Theorem 4.4 *Let hypotheses* (H1) *and* (H2') *be satisfied. Then, for any weak solution* $u \in W_0^{s,\mathcal{H}}(\Omega)$ *of problem* (1.1)*, it holds that* $u \in L^\infty(\Omega)$ *and*

$$\|u\|_{\infty,\Omega} \leq C \max\left\{\|u\|_{\mathcal{B},\Omega}^{\tilde{\ell}_1}, \|u\|_{\mathcal{B},\Omega}^{\tilde{\ell}_2}\right\},$$

where the positive constants $C, \tilde{\ell}_1, \tilde{\ell}_2$ *are independent of* u.

Proof First, we repeat **Step 1** of the proof for Theorem 4.2, namely, assume that (4.3)-(4.8) hold.

(a): Localization

Let $B_i \subset \mathbb{R}^N$ be open balls of radius R with $i \in \mathcal{I} := \{1, \cdots, m\}$ and let $\{B_i\}_{i\in\mathcal{I}}$ be a finite open covering of $\overline{\Omega}$ such that $\Omega_i := B_i \cap \Omega$ for $i \in \mathcal{I}$ are Lipschitz domains. For any $i \in \mathcal{I}$, we choose R small enough such that (4.10) and (4.11) are fulfilled. According to the continuity of p, q given by (4.37), there exists $R \in (0, 1/4)$ small enough such that there exist $C_{20}, C_{21} > 0$ satisfying

$$-\left|p(x,y) - p^-_{B_{4R}(x,y)}\right| \log|x-y| \leq C_{20}, \tag{4.38}$$

$$-\left|q(x,y) - q^-_{B_{4R}(x,y)}\right| \log|x-y| \leq C_{21} \tag{4.39}$$

for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfying $|x-y| < \frac{1}{2}$. As done before, let $\{\eta_i\}_{i=1}^m$ be a partition of unity of $\overline{\Omega}$ satisfying (4.12). Let $p_i^- = p(x', y')$ for some $(x', y') \in \overline{B_i} \times \overline{B_i}$. Thus

$$|(x', y') - (x, y)| = |x' - x| + |y' - y| < 4R \quad \text{for all } (x, y) \in B_i \times B_i,$$

so $(x', y') \in B_{4R}(x, y)$ for all $(x, y) \in B_i \times B_i$. Also, we see that $|x - y| < 2R < 1/2$ for all $(x, y) \in B_i \times B_i$. Combining these conclusions and (4.38) we get

$$\begin{aligned} &-(p(x,y) - p_i^-)\log|x-y| \leq -\left(p(x,y) - p^-_{B_{4R}(x,y)}\right)\log|x-y| \\ &\leq C_{19} \quad \text{for all } (x,y) \in B_i \times B_i, \end{aligned}$$

which implies

$$|x-y|^{s(p(x,y)-p_i^-)} = e^{s(p(x,y)-p_i^-)\log|x-y|} \geq C_{22} \quad \text{for all } (x,y) \in B_i \times B_i. \tag{4.40}$$

Similarly, (4.39) implies

$$|x-y|^{s(q(x,y)-q_i^-)} = e^{s(q(x,y)-q_i^-)\log|x-y|} \geq C_{23} \quad \text{for all } (x,y) \in B_i \times B_i. \tag{4.41}$$

We claim that

$$\begin{aligned}
&\int_{B_i}\int_{B_i}\Bigg(\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&\leq C_{24}\Bigg[\int_{B_i}\int_{B_i}\Bigg(\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y+\left|A_{\psi_{n+1}}\right|\Bigg],
\end{aligned} \tag{4.42}$$

which associates (4.7) and (4.8) implies

$$\begin{aligned}
&\int_{B_i}\int_{B_i}\Bigg(\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&\leq C_{25}\left(1+\psi_*^{-\varsigma^+}\right)2^{\alpha_0+\frac{\alpha_0}{N}}Z_n,
\end{aligned}$$

for all $i \in \mathcal{I}$ and all $n \in \mathbb{N}$, where we recall that $\alpha_0 \in \max\{\varsigma^+, \tau^+\}$. Now, we are going to prove the claim. We have

$$\begin{aligned}
&\int_{B_i}\int_{B_i}\Bigg(\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\cap A_{\psi_{n+1}}}\Bigg(\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+2\int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\setminus A_{\psi_{n+1}}}\Bigg(\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&=: T_1+2T_2.
\end{aligned} \tag{4.43}$$

Invoking (4.40) and (4.41) we get

$$
\begin{aligned}
T_1 = & \int_{B_i \cap A_{\psi_{n+1}}} \int_{B_i \cap A_{\psi_{n+1}}} \left(\left| \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} \right|^{p(x,y)} \cdot \frac{1}{|x-y|^{N-sp_i^-}} \right. \\
& \times \frac{1}{|x-y|^{-s(p(x,y)-p_i^-)}} \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \\
& + \mu(x,y) \left| \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} \right|^{q(x,y)} \cdot \frac{1}{|x-y|^{N-sq_i^-}} \\
& \left. \times \frac{1}{|x-y|^{-s(q(x,y)-q_i^-)}} \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \right) \mathrm{d}x\, \mathrm{d}y \\
\geq & \int_{B_i \cap A_{\psi_{n+1}}} \int_{B_i \cap A_{\psi_{n+1}}} \left(C_{22} \left| \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} \right|^{p(x,y)} \right. \\
& \times \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{1}{|x-y|^{N-sp_i^-}} \\
& + C_{23} \mu(x,y) \left| \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} \right|^{q(x,y)} \\
& \left. \times \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{1}{|x-y|^{N-sq_i^-}} \right) \mathrm{d}x\, \mathrm{d}y.
\end{aligned}
$$

Furthermore, if $\left| \frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{2s}} \right| < 1$, it follows that

$$
\begin{aligned}
T_1 \geq & \int_{B_i \cap A_{\psi_{n+1}}} \int_{B_i \cap A_{\psi_{n+1}}} \left(C_{22} \left(\left| \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} \right|^{p_i^-} - 1 \right) \right. \\
& \times \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{1}{|x-y|^{N-sp_i^-}} \\
& + C_{23} \mu(x,y) \left(\left| \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} \right|^{q_i^-} - 1 \right) \\
& \left. \times \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{1}{|x-y|^{N-sq_i^-}} \right) \mathrm{d}x\, \mathrm{d}y \\
\geq & \int_{B_i \cap A_{\psi_{n+1}}} \int_{B_i \cap A_{\psi_{n+1}}} \left(C_{22} \frac{|u_n(x) - u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}} \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \right. \\
& - C_{22} \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} |x-y|^s \right) \frac{1}{|x-y|^{N-sp_i^-}} \\
& + C_{23} \mu(x,y) \frac{|u_n(x) - u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}} \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \\
& \left. - C_{23} \|\mu\|_\infty \log \left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^{2s}} |x-y|^s \right) \frac{1}{|x-y|^{N-sq_i^-}} \right) \mathrm{d}x\, \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\cap A_{\psi_{n+1}}}\left(C_{22}\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right.\\
&\quad +C_{23}\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad -C_{26}\log\left(e+\omega(2R)^s\right)\int_{A_{\psi_{n+1}}}\left(\int_{B_i}\left(\frac{1}{|x-y|^{N-sp_i^-}}+\frac{1}{|x-y|^{N-sq_i^-}}\right)\mathrm{d}x\right)\mathrm{d}y,
\end{aligned}
\tag{4.44}
$$

and if $\left|\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{2s}}\right| \geq 1$, it follows that

$$
\begin{aligned}
T_1 &\geq \int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\cap A_{\psi_{n+1}}}\left(C_{22}\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right.\\
&\quad \left.+\,C_{23}\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right)\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
\tag{4.45}
$$

Furthermore, we choose $\tilde{R} > 1$ such that $\Omega\times\Omega \subset B_{\tilde{R}-1}(0)$. Hence, for any $i \in \mathcal{I}$ and $r > 0$, it holds that

$$
\int_{B_i}\frac{1}{|x-y|^{N-sr}}\,\mathrm{d}x \leq \int_{B_{\tilde{R}}(0)}\frac{1}{|z|^{N-sr}}\,\mathrm{d}z = \frac{\omega_N\tilde{R}^{sr}}{sr} \quad \text{for all } y\in\Omega. \tag{4.46}
$$

From the above inequality we get

$$
\int_{B_i}\frac{1}{|x-y|^{N-sp_i^-}} \leq \frac{\omega_N\tilde{R}^{sp_i^-}}{sp_i^-} \leq \frac{\omega_N\tilde{R}^{sp^+}}{sp^-} \quad \text{and} \quad \int_{B_i}\frac{1}{|x-y|^{N-sq_i^-}} \leq \frac{\omega_N\tilde{R}^{sq^+}}{sq^-}.
$$

Utilizing the last two inequalities along with (4.44) and (4.45) we arrive at

$$
\begin{aligned}
T_1 &\geq C_{27}\int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\cap A_{\psi_{n+1}}}\left(\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right.\\
&\quad \left.+\,\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right)\mathrm{d}x\,\mathrm{d}y - C_{28}\left|A_{\psi_{n+1}}\right|.
\end{aligned}
\tag{4.47}
$$

Similarly, applying (4.40), (4.41) and (4.46) again, we have

$$
\begin{aligned}
T_2 &= \int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\setminus A_{\psi_{n+1}}}\left(\left|\frac{|u_n(x)-u_n(y)|}{|x-y|^{2s}}\right|^{p(x,y)}\cdot\frac{1}{|x-y|^{N-sp_i^-}}\right.\\
&\quad \times\frac{1}{|x-y|^{-s(p(x,y)-p_i^-)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\\
&\quad +\mu(x,y)\left|\frac{|u_n(x)-u_n(y)|}{|x-y|^{2s}}\right|^{q(x,y)}\cdot\frac{1}{|x-y|^{N-sq_i^-}}\\
&\quad \left.\times\frac{1}{|x-y|^{-s(q(x,y)-q_i^-)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right)\mathrm{d}x\,\mathrm{d}y\\
&\geq \int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\setminus A_{\psi_{n+1}}}\left(C_{22}\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ C_{23}\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y \\
&- C_{26}\log\left(e+\omega(2R)^s\right)\int_{A_{\psi_{n+1}}}\left(\int_{B_i}\frac{1}{|x-y|^{N-sp_i^-}}+\frac{1}{|x-y|^{N-sq_i^-}}\,\mathrm{d}x\right)\mathrm{d}y \\
\geq{}& C_{27}\int_{B_i\cap A_{\psi_{n+1}}}\int_{B_i\setminus A_{\psi_{n+1}}}\left(\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right. \\
&\left.+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right)\mathrm{d}x\,\mathrm{d}y - C_{28}\left|A_{\psi_{n+1}}\right|.
\end{aligned}
\tag{4.48}
$$

This along with (4.43), (4.47) and (4.48) yield (4.42), and the claim is proved.

(b): Estimating Z_{n+1} by Z_n.

Recall that $L_{n,i}$ and $\tilde{\mathcal{B}}$ are defined by (4.17) and (4.18), respectively, for $\star \in \{+,-\}$ and $i \in \mathcal{I}$. According to inequalities (4.20), (4.21), Remark 2.7, (4.10) and the continuous embedding (2.4) we see that there exist $\tilde{\sigma} > 0$ such that

$$
\tilde{\sigma} < \min\{\varsigma_i^- - p_i^-, \tau_i^- - q_i^-\} \quad \text{for } i \in \mathcal{I}
$$

satisfying

$$
\begin{aligned}
&\left(\int_\Omega |u_n\eta_i|^{\varsigma_i^\star+\varepsilon}\log^{\frac{\varsigma_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x\right)^{\frac{\varsigma_i^\star}{\varsigma_i^\star+\varepsilon}} \\
&\leq \|u_n\eta_i\|_{\tilde{\mathcal{B}}|_{\mu\equiv0},\Omega}^{\tilde{\varsigma}_i^\star} \leq C_{29}\,[u_n\eta_i]_{s,\mathcal{H}|_{\mu\equiv0},\Omega}^{\tilde{\varsigma}_i^\star} \leq C_{30}\left(\widetilde{S}_{n,i}^{\frac{\tilde{\varsigma}_i^\star}{p_i^-}}+\widetilde{S}_{n,i}^{\frac{\tilde{\varsigma}_i^\star}{p_i^-+\tilde{\sigma}}}\right),
\end{aligned}
$$

with $\tilde{\varsigma}_i^\star$ given by (4.23) and

$$
\begin{aligned}
\widetilde{S}_{n,i} = \int_\Omega\int_\Omega &\left(\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right. \\
&+\mu(x,y)\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{q_i^-}}{|x-y|^{sq_i^-}} \\
&\left.\times\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right)\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Analogously, inequalities (4.20), (4.21), Remark 2.7, (4.11) and the continuous embedding (2.4) yield

$$
\begin{aligned}
&\left(\int_\Omega \mu(x)^\gamma|u_n\eta_i|^{\tau_i^\star+\varepsilon}\log^{\frac{\tau_i^\star+\varepsilon}{N}}(e+\omega|u_n\eta_i|)\,\mathrm{d}x\right)^{\frac{\tau_i^\star}{\tau_i^\star+\varepsilon}} \\
&\leq \|u_n\eta_i\|_{\tilde{\mathcal{B}},\Omega}^{\tilde{\tau}_i^\star} \leq C_{31}\,[u_n\eta_i]_{s,\mathcal{H},\Omega}^{\tilde{\tau}_i^\star} \leq C_{32}\left(\widetilde{S}_{n,i}^{\frac{\tilde{\tau}_i^\star}{p_i^-}}+\widetilde{S}_{n,i}^{\frac{\tilde{\tau}_i^\star}{q_i^-+\tilde{\sigma}}}\right),
\end{aligned}
$$

with $\tilde{\tau}_i^\star$ given by (4.26). Similar to (4.27) one has

$$
Z_{n+1} \leq C_{33}\left|A_{\psi_{n+1}}\right|^{\frac{\varepsilon}{\varsigma^++\tau^++\varepsilon}}\left(\widetilde{S}_{n,i}^{1+\tilde{\theta}_1}+\widetilde{S}_{n,i}^{1+\tilde{\theta}_2}\right), \tag{4.49}
$$

with

$$0<\tilde{\theta}_1:=\min_{1\le i\le m}\min\left\{\frac{\varsigma_i^-}{p_i^-+\tilde{\sigma}},\frac{\tau_i^-}{q_i^-+\tilde{\sigma}}\right\}-1\le\tilde{\theta}_2:=\max_{1\le i\le m}\max\left\{\frac{\varsigma_i^++\frac{\varsigma_i^+}{N}}{p_i^-},\frac{\tau_i^++\frac{\tau_i^+}{N}}{p_i^-}\right\}-1.$$

Let

$$\widetilde{S}_{n,i}=\widetilde{J}_1+2\widetilde{J}_2, \tag{4.50}$$

where

$$\begin{aligned}\widetilde{J}_1&=\int_{B_i}\int_{B_i}\left(\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{p_i^-}}{|x-y|^{sp_i^-}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right.\\&\quad+\mu(x,y)\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{q_i^-}}{|x-y|^{sq_i^-}}\\&\quad\left.\times\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right)\mathrm{d}\nu,\end{aligned}$$

and

$$\begin{aligned}\widetilde{J}_2&=\int_{\Omega\setminus B_i}\int_{B_i}\left(\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{p_i^-}}{|x-y|^{sp_i^-}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right.\\&\quad+\mu(x,y)\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{q_i^-}}{|x-y|^{sq_i^-}}\\&\quad\left.\times\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right)\mathrm{d}\nu.\end{aligned}$$

By (4.9), (4.13) and (4.30) we get

$$\begin{aligned}\widetilde{J}_1&=\int_{B_i}\int_{B_i}\left(\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right.\\&\quad\left.+\mu(x,y)\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)\eta_i(x)-u_n(y)\eta_i(y)|}{|x-y|^s}\right)\right)\mathrm{d}x\,\mathrm{d}y\\&\le 2^{p_i^++1}\int_{B_i}\int_{B_i}\frac{|u_n(x)-u_n(y)|^{p_i^-}}{|x-y|^{N+sp_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\mathrm{d}x\,\mathrm{d}y\\&\quad+2^{p_i^++1}\max\{\|\nabla\eta_i\|_\infty^{p_i^-},\|\nabla\eta_i\|_\infty^{p_i^-+1}\}\\&\quad\times\int_{B_i}\left(\int_{B_i}\frac{\mathrm{d}x}{|x-y|^{N+(s-1)(p_i^-+1)}}\right)|u_n(y)|^{p_i^-}\log(e+\omega|u_n(y)|)\,\mathrm{d}y\\&\quad+2^{q_i^++1}\int_{B_i}\int_{B_i}\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i^-}}{|x-y|^{N+sq_i^-}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\mathrm{d}x\,\mathrm{d}y\\&\quad+2^{q_i^++1}\|\mu\|_\infty\max\{\|\nabla\eta_i\|_\infty^{q_i^-},\|\nabla\eta_i\|_\infty^{q_i^-+1}\}\\&\quad\times\int_{B_i}\left(\int_{B_i}\frac{\mathrm{d}x}{|x-y|^{N+(s-1)(q_i^-+1)}}\right)\chi_\mu(y)|u_n(y)|^{q_i^-}\log(e+\omega|u_n(y)|)\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\le C_{34} \int_{B_i} \int_{B_i} \left(\frac{|u_n(x) - u_n(y)|^{p_i^-}}{|x-y|^{sp_i^-}} \log\left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s}\right) \right.\\
&\quad \left. + \mu(x,y) \frac{|u_n(x) - u_n(y)|^{q_i^-}}{|x-y|^{sq_i^-}} \log\left(e + \omega \frac{|u_n(x) - u_n(y)|}{|x-y|^s}\right) \right) \mathrm{d}\nu\\
&\quad + C_{35} \int_{B_i} |u_n(y)|^{p_i^-} \log(e + \omega |u_n(y)|) + \mu(y) |u_n(y)|^{q_i^-} \log(e + \omega |u_n(y)|) \, \mathrm{d}y\\
&\le C_{36} \int_{A_{\psi_{n+1}} \cap B_i} \left[u(x)^{\varsigma(x)} \log^{\frac{\varsigma(x)}{N}} (e + \omega u(x)) + \mu(x)^{\gamma} u(x)^{\tau(x)} \log^{\frac{\tau(x)}{N}} (e + \omega u(x)) \right] \mathrm{d}x\\
&\quad + C \left| A_{\psi_{n+1}} \right|\\
&\le C_{37} \left(1 + \psi_*^{-\varsigma^+} \right) 2^{n(\alpha_0 + \frac{\alpha_0}{N})} Z_n,
\end{aligned} \tag{4.51}
$$

and

$$
\begin{aligned}
\widetilde{J}_2 &= \int_{\Omega \setminus B_i} \left(\int_{B_i} \frac{|u_n(x)\eta_i(x)|^{p_i^-}}{|x-y|^{N+sp_i^-}} \log\left(e + \omega \frac{|u_n(x)\eta_i(x)|}{|x-y|^s}\right) \right.\\
&\quad \left. + \mu(x,y) \frac{|u_n(x)\eta_i(x)|^{q_i^-}}{|x-y|^{N+sq_i^-}} \log\left(e + \omega \frac{|u_n(x)\eta_i(x)|}{|x-y|^s}\right) \mathrm{d}x \right) \mathrm{d}y\\
&\le \int_{\operatorname{supp}(\eta_i) \cap A_{\psi_{n+1}}} \left(\int_{\Omega \setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+sp_i^-}} + \int_{\Omega \setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+s(p_i^-+1)}} \right)\\
&\quad \times |u_n(y)|^{p_i^-} \log(e + \omega |u_n(y)|) \, \mathrm{d}x\\
&\quad + \|\mu\|_\infty \int_{\operatorname{supp}(\eta_i) \cap A_{\psi_{n+1}}} \left(\int_{\Omega \setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+sq_i^-}} + \int_{\Omega \setminus B_i} \frac{\mathrm{d}y}{|x-y|^{N+s(q_i^-+1)}} \right)\\
&\quad \times \chi_\mu(x) |u_n(y)|^{q_i^-} \log(e + \omega |u_n(y)|) \, \mathrm{d}x\\
&\le C_{38} \int_{A_{\psi_{n+1}} \cap B_i} |u(x)|^{p_i^-} \log(e + \omega |u(x)|) + \mu(x) |u(x)|^{q_i^-} \log(e + \omega |u(x)|) \, \mathrm{d}x\\
&\le C_{39} \left(1 + \psi_*^{-\varsigma^+} \right) 2^{n(\alpha_0 + \frac{\alpha_0}{N})} Z_n,
\end{aligned} \tag{4.52}
$$

Inequality (4.8) and (4.50), (4.51), as well as (4.52) imply

$$
\widetilde{S}_n^{1+\tilde{\theta}_1} + \widetilde{S}_n^{1+\tilde{\theta}_2} \le C_{40} \left(1 + \psi_*^{-\varsigma^+(1+\tilde{\theta}_2)} \right) 2^{n(\alpha_0 + \frac{\alpha_0}{N})\left(1+\tilde{\theta}_2\right)} \left(Z_n^{1+\tilde{\theta}_1} + Z_n^{1+\tilde{\theta}_2} \right), \tag{4.53}
$$

which along with (4.7), (4.49), (4.53) gives

$$
Z_{n+1} \le C_{41} \left(\psi_*^{-\tilde{\rho}_1} + \psi_*^{-\tilde{\rho}_2} \right) \tilde{k}^n \left(Z_n^{1+\tilde{\theta}_1} + Z_n^{1+\tilde{\theta}_2} \right) \quad \text{for all } n \in \mathbb{N}_0,
$$

where

$$
\begin{aligned}
0 < \tilde{\rho}_1 &:= \frac{\varepsilon \varsigma^-}{\varsigma^+ + \tau^+ + \varepsilon} < \tilde{\rho}_2 := \varsigma^+ \left(1 + \tilde{\theta}_2 \right) + \frac{\varepsilon \varsigma^+}{\varsigma^+ + \tau^+ + \varepsilon}\\
1 < \tilde{k} &:= 2^{(\alpha_0 + \frac{\alpha_0}{N})\left(1+\tilde{\theta}_2\right) \frac{\varepsilon \varsigma^+}{\varsigma^+ + \tau^+ + \varepsilon}},\\
0 < \tilde{\gamma}_1 &:= \tilde{\theta}_1 + \frac{\varepsilon}{\varsigma^+ + \tau^+ + \varepsilon} \le \tilde{\gamma}_2 := \tilde{\theta}_2 + \frac{\varepsilon}{\varsigma^+ + \tau^+ + \varepsilon}.
\end{aligned}
$$

Finally, repeating the arguments of **Step 3** in the proof of Theorem 4.2, gives the assertion. □

4.2 Critical growth

In this subsection we discuss the critical case. Recall that in Sect. 4.1, to apply the Hölder inequality in (4.17) we require that there exists $\varepsilon > 0$ such that $\varsigma^\star + \varepsilon < (p_i^-)_s^*$ and $\tau^\star + \varepsilon < (q_i^-)_s^*$ with $\star \in \{-, +\}$. However, in this subsection, we assume that $\varsigma(x) = (p^-)_s^*$ and $\tau(x) = (q^-)_s^*$ for all $x \in \overline{\Omega}$, so we cannot find $\varepsilon > 0$ satisfying the above conditions anymore. Hence, we consider a different argument to show the boundedness of weak solutions to problem (1.1), and under this argument, the inequality (4.2) is invalid. Now, we state our hypotheses on the data.

(H3) (i) Assume $f: \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is graph measurable and $f(x, \cdot): \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is upper semicontinuous for a.a. $x \in \Omega$.

(ii) Let $\varsigma, \tau \in C(\overline{\Omega})$ such that $p^+ < \varsigma(x) = (p^-)_s^*$ and $q^+ < \tau(x) = (q^-)_s^*$ for all $x \in \overline{\Omega}$. Suppose that there exists a constant $\beta > 0$ satisfying

$$\begin{aligned}&\sup\{|\xi| : \xi \in f(x,t)\}\\&\quad \leq \beta\left[|t|^{(p^-)_s^*-1}\log^{\frac{\varsigma(x)}{N}}(e+\omega|t|) + \mu(x)^\gamma|t|^{\tau(x)-1}\log^{\frac{(q^-)_s^*}{N}}(e+\omega|t|)+1\right]\end{aligned}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.

Theorem 4.5 *Let hypotheses* (H1) *and* (H3) *be satisfied. Then, for any weak solution* $u \in W_0^{s,\mathcal{H}}(\Omega)$ *of problem* (1.1) *is bounded, that is* $u \in L^\infty(\Omega)$.

Proof As done in Sect. 4.1, let $B_i \subset \mathbb{R}^N$ be open balls of radius R with $i \in \mathcal{I} := \{1, \cdots, m\}$ and let $\{B_i\}_{i=1}^m$ be a finite open covering of $\overline{\Omega}$ such that $\Omega_i := B_i \cap \Omega$ for $i \in \mathcal{I}$ are Lipschitz domains. For any $i \in \mathcal{I}$, we choose R small enough such that

$$q_i^+ < \left(p^-\right)_s^* \quad \text{for all } i \in \mathcal{I}.$$

Let A_ψ still be defined by (4.4), suppose $u \in W_0^{s,\mathcal{H}}(\Omega)$ is a weak solution of problem (1.1) in the sense of definition 4.1, and choose $\psi_* \geq 1$ large enough such that

$$\begin{aligned}&\int_\Omega\int_\Omega\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{sp(x,y)}}\log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\right.\\&\quad\left.+\mu(x,y)\frac{|u(x)-u(y)|^{q(x,y)}}{|x-y|^{sq(x,y)}}\log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\right)\mathrm{d}\nu+\int_{A_{\psi_*}}\mathcal{B}^*(x,|u|)\,\mathrm{d}x<1,\end{aligned} \tag{4.54}$$

with

$$\mathcal{B}^*(x,t) := t^{(p^-)_s^*}\log^{\frac{(p^-)_s^*}{N}}(e+\omega t) + \mu(x)^\gamma t^{(q^-)_s^*}\log^{\frac{(q^-)_s^*}{N}}(e+\omega t),$$

for all $x \in \overline{\Omega}$ and for all $t \geq 0$. Note that for any $n \in \mathbb{N}_0$, ψ_n is still given by (4.5).

In the sequel, for any $n \in \mathbb{N}_0$ we define $\overline{Z}_n$ by

$$\begin{aligned}&\overline{Z}_n := \int_{A_{\psi_n}}\\&\left[(u-\psi_n)^{(p^-)_s^*}\log^{\frac{(p^-)_s^*}{N}}(e+\omega(u-\psi_n)) + \mu(x)^\gamma\,(u-\psi_n)^{(q^-)_s^*}\log^{\frac{(q^-)_s^*}{N}}(e+\omega(u-\psi_n))\right]\mathrm{d}x.\end{aligned}$$

Note that $u \geq u - \psi_{n+1} \geq 0$ and $u > \psi_{n+1} \geq 1$ on $A_{\psi_{n+1}}$, similar to the proof of Theorem 4.2, we have

$$\begin{aligned}
&\int_\Omega \int_\Omega \Bigg(\frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{sp(x,y)}} \log\left(e + \omega \frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right) \\
&\quad + \mu(x,y) \frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{sq(x,y)}} \log\left(e + \omega \frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\Bigg) \mathrm{d}\nu \\
&\leq \int_\Omega \beta \Big[|u|^{(p^-)^*_s - 1} \log^{\frac{(p^-)^*_s}{N}}(e + \omega|u|) + \mu(x)^\gamma |u|^{(q^-)^*_s - 1} \log^{\frac{(q^-)^*_s}{N}}(e+\omega|u|) + 1 \Big] u_n(x)\, \mathrm{d}x \\
&\leq 2\beta \int_{A_{\psi_{n+1}}} \left[u^{(p^-)^*_s} \log^{\frac{(p^-)^*_s}{N}}(e+\omega u) + \mu(x)^\gamma u^{(q^-)^*_s} \log^{\frac{(q^-)^*_s}{N}}(e+\omega u)\right] \mathrm{d}x \\
&\leq C_{42} \int_{A_{\psi_{n+1}}} \Bigg(\left[\left(2^{n+2}-1\right)(u-\psi_n)\right]^{(p^-)^*_s} \log^{\frac{(p^-)^*_s}{N}} \left[e + \omega\left(2^{n+2}-1\right)(u-\psi_n)\right] \\
&\quad + \mu(x)^\gamma \left[\left(2^{n+2}-1\right)(u-\psi_n)\right]^{(q^-)^*_s} \log^{\frac{(q^-)^*_s}{N}} \left[e + \omega\left(2^{n+2}-1\right)(u-\psi_n)\right]\Bigg) \mathrm{d}x \\
&\leq C_{43} 2^{n\left((q^-)^*_s + \frac{(q^-)^*_s}{N}\right)} \overline{Z}_n.
\end{aligned}$$

Let $\{\eta_i\}_{i=1}^m$ be a partition of unity of $\overline{\Omega}$ with respect to $\{B_i\}_{i=1}^m$, namely, for each $i \in \mathcal{I}$, $\eta_i \in C_c^\infty(\mathbb{R}^N)$, $\operatorname{supp}(\eta_i) \subset B_i$, $0 \leq \eta_i \leq 1$, and

$$\sum_{i=1}^m \eta_i = 1 \quad \text{on } \overline{\Omega}.$$

By applying Jensen's inequality we get

$$\begin{aligned}
\overline{Z}_{n+1} &= \int_{A_{\psi_{n+1}}} \left[u_n^{(p^-)^*_s} \log^{\frac{(p^-)^*_s}{N}}(e+\omega u_n) + \mu(x)^\gamma u_n^{(q^-)^*_s} \log^{\frac{(q^-)^*_s}{N}}(e+\omega u_n)\right] \mathrm{d}x \\
&\leq m^{\max\{(p^-)^*_s, (q^-)^*_s\}} \sum_{i=1}^m \Bigg[\int_{A_{\psi_{n+1}}} |u_n \eta_i|^{(p^-)^*_s} \log^{\frac{(p^-)^*_s}{N}}(e + \omega|u_n\eta_i|) \\
&\quad + \mu(x)^\gamma |u_n\eta_i|^{(q^-)^*_s} \log^{\frac{(q^-)^*_s}{N}}(e+\omega|u_n\eta_i|)\, \mathrm{d}x \Bigg].
\end{aligned}$$

By Proposition 2.12, we see that

$$W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{B}^*}(\Omega),$$

then

$$\overline{Z}_{n+1} \leq m^{\max\{\iota^+, \pi^+\}} \sum_{i=1}^m \int_\Omega \mathcal{B}^*(x, |u_n\eta_i|)\, \mathrm{d}x.$$

From assumption (4.54) we have

$$\int_\Omega \mathcal{B}^*(x, |u_n\eta_i|)\, \mathrm{d}x \leq \|u_n\eta_i\|^{(p^-)^*_s}_{\mathcal{B}^*,\Omega} \leq C_{44}[u_n\eta_i]^{(p^-)^*_s}_{s,\mathcal{H},\Omega} \leq C_{45}(S_{n,i})^{\frac{(p^-)^*_s}{q_i^+ + \sigma}},$$

where $S_{n,i}$ is given by (4.24). So, we get

$$\overline{Z}_{n+1} \le C_{46}\left(S_{n,i}^{1+\vartheta_1} + S_{n,i}^{1+\vartheta_2}\right) \quad \text{for all } n \in \mathbb{N}_0 \tag{4.55}$$

with

$$0 < \vartheta_1 := \min_{1\le i\le m} \frac{(p^-)_s^*}{q_i^+ + \sigma} - 1 \le \vartheta_2 := \max_{1\le i\le m} \frac{(p^-)_s^*}{q_i^+ + \sigma} - 1,$$

where $\sigma > 0$ satisfies

$$\sigma < (p^-)_s^* - q_i^+ \quad \text{for } i \in \mathcal{I}.$$

Recalling (4.28) we make the similar estimation of J_1 and J_2, that is

$$\begin{aligned}
J_1 \le\ & C_{47}\int_{B_i}\int_{B_i}\left(\frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right.\\
&\left.+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q_i(x,y)}}{|x-y|^{sq_i(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right)\mathrm{d}\nu\\
&+C_{48}\int_{B_i}|u_n(y)|^{p_i^+}\log(e+\omega|u_n(y)|)+\mu(y)|u_n(y)|^{q_i^+}\log(e+\omega|u_n(y)|)\,\mathrm{d}y\\
&+C_{48}\int_{B_i}|u_n(y)|^{p_i^-}\log(e+\omega|u_n(y)|)+\mu(y)|u_n(y)|^{q_i^-}\log(e+\omega|u_n(y)|)\,\mathrm{d}y\\
\le\ & \int_{\Omega}\int_{\Omega}\left(\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{sp(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right.\\
&\left.+\mu(x,y)\frac{|u_n(x)-u_n(y)|^{q(x,y)}}{|x-y|^{sq(x,y)}}\log\left(e+\omega\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\right)\mathrm{d}\nu\\
&+C_{49}\int_{A_{\psi_{n+1}}\cap B_i}\left[u(x)^{(p^-)_s^*}\log^{\frac{(p^-)_s^*}{N}}(e+\omega u(x))+\mu(x)^{\gamma}u(x)^{(q^-)_s^*}\log^{\frac{(q^-)_s^*}{N}}(e+\omega u(x))\right]\mathrm{d}x,\\
\le\ & C_{50}2^{n\left((q^-)_s^*+\frac{(q^-)_s^*}{N}\right)}\overline{Z}_n,
\end{aligned}$$

and $J_2 \le C_{51}2^{n\left((q^-)_s^*+\frac{(q^-)_s^*}{N}\right)}\overline{Z}_n$. Hence

$$S_{n,i} \le C_{52}2^{n\left((q^-)_s^*+\frac{(q^-)_s^*}{N}\right)}\overline{Z}_n \quad \text{for all } n \in \mathbb{N}_0.$$

Therefore, we get

$$S_{n,i}^{1+\vartheta_1} + S_n^{1+\vartheta_2} \le C_{53}2^{n\left((q^-)_s^*+\frac{(q^-)_s^*}{N}\right)(1+\vartheta_2)}\left(\overline{Z}_n^{1+\vartheta_1} + \overline{Z}_n^{1+\vartheta_2}\right). \tag{4.56}$$

Taking (4.55) and (4.56) into account, we arrive at

$$\overline{Z}_{n+1} \le C_{54}\bar{k}^n\left(\overline{Z}_n^{1+\vartheta_1} + \overline{Z}_n^{1+\vartheta_2}\right) \quad \text{for all } n \in \mathbb{N}_0, \tag{4.57}$$

where

$$1 < \bar{k} := 2^{\left((q^-)_s^*+\frac{(q^-)_s^*}{N}\right)(1+\vartheta_2)}.$$

Using Lemma 2.15, we see that (4.57) yields

$$\overline{Z}_n \to 0 \quad \text{as } n \to \infty, \tag{4.58}$$

if we choose $\psi_* > 1$ large enough such that

$$\begin{aligned}\overline{Z}_0 = \int_\Omega \Big[&(u-\psi_*)_+^{(p^-)_s^*} \log^{\frac{(p^-)_s^*}{N}}(e+\omega(u-\psi_*)_+) \\ &+\mu(x)^\gamma (u-\psi_*)_+^{(q^-)_s^*} \log^{\frac{(q^-)_s^*}{N}}(e+\omega(u-\psi_*)_+)\Big]\,\mathrm{d}x \\ &\le \min\left\{(2C_{54})^{-\frac{1}{\vartheta_1}}\,\bar{k}^{-\frac{1}{\vartheta_1^2}},\,(2C_{54})^{-\frac{1}{\vartheta_2}}\,\bar{k}^{-\frac{1}{\vartheta_1\vartheta_2}-\frac{\vartheta_2-\vartheta_1}{\vartheta_2^2}}\right\}.\end{aligned}$$

Thus by (4.58) and Lebesgue's dominated convergence theorem we arrive at

$$\begin{aligned}\overline{Z}_n = \int_\Omega \Big[&(u-\psi_n)_+^{(p^-)_s^*} \log^{\frac{(p^-)_s^*}{N}}(e+\omega(u-\psi_n)_+) \\ &+\mu(x)^\gamma (u-\psi_n)_+^{(q^-)_s^*} \log^{\frac{(q^-)_s^*}{N}}(e+\omega(u-\psi_n)_+)\Big]\,\mathrm{d}x \\ \to \int_\Omega \Big[&(u-2\psi_*)_+^{(p^-)_s^*} \log^{\frac{(p^-)_s^*}{N}}(e+\omega(u-2\psi_*)_+) \\ &+\mu(x)^\gamma (u-2\psi_*)_+^{(q^-)_s^*} \log^{\frac{(q^-)_s^*}{N}}(e+\omega(u-2\psi_*)_+)\Big]\,\mathrm{d}x \to 0,\end{aligned}$$

as $n \to \infty$. This implies that

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) \le 2\psi_*.$$

Similarly, replacing u with $-u$, it can be shown that

$$\operatorname*{ess\,sup}_{x\in\Omega} (-u)(x) \le 2\psi_*.$$

Therefore,

$$\|u\|_{\infty,\Omega} \le 2\psi_*,$$

with $\psi_* \in \mathbb{R}$. □

Since problem (P2) and (P3) are special cases of problem (1.1), we obtain the following corollaries.

Corollary 4.6 *Let hypotheses* (H1) *and* (H2) (or (H2')) *be satisfied with* $\omega = 0$. *Then every weak solution* $u \in W_0^{s,\mathcal{H}_2}(\Omega)$ *of problem* (P2) *belongs to* $L^\infty(\Omega)$ *and it holds*

$$\|u\|_{\infty,\Omega} \le C \max\left\{\|u\|_{\mathcal{B},\Omega}^{\ell_1}, \|u\|_{\mathcal{B},\Omega}^{\ell_2}\right\},$$

with C, ℓ_1, ℓ_2 *being positive constants independent of* u. *Moreover, if hypotheses* (H1) *and* (H3) *hold, then any weak solution of problem* (P2) *belongs to* $L^\infty(\Omega)$.

Corollary 4.7 *Let hypotheses* (H1) *and* (H2) (or (H2')) *be satisfied with* $\omega = 0$. *Then every weak solution* $u \in W_0^{s,\mathcal{H}_3}(\Omega)$ *of problem* (P3) *belongs to* $L^\infty(\Omega)$ *and it holds*

$$\|u\|_{\infty,\Omega} \le C \max\left\{\|u\|_{\mathcal{B},\Omega}^{\ell_1}, \|u\|_{\mathcal{B},\Omega}^{\ell_2}\right\},$$

with C, ℓ_1, ℓ_2 being positive constants independent of u. Moreover, if hypotheses (H1) *and* (H3) *hold, then any weak solution of problem* (P3) *belongs to $L^\infty(\Omega)$.*

5 Application

In this section, we consider the existence of weak solutions to the following single valued elliptic problem driven by the fractional double phase operator with variable exponents and logarithmic perturbation:

$$\begin{cases} (-\Delta)^s_{\mathcal{H}} u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1}$$

where Ω, s, and p satisfy (H1). Furthermore, based on the priori bounds we obtained in Sect. 4, we will show the existence of infinitely many small weak solutions of (5.1) with the modified functional method applied by Ho–Kim [35] and Wang [61]. Moreover, under appropriate conditions, we show that the solutions are non-negative by applying the maximum principle established in Sect. 3. We will use a variational argument to establish the existence results, and the proof is mainly based on the following lemma, see Heinz [34] for more details.

Lemma 5.1 *Let X be a Banach space. Assume that $I \in C^1(X,\mathbb{R})$ and I is even, bounded from below and satisfies the* (PS)*-condition with $I(0) = 0$. If for any $n \in \mathbb{N}$, there exist an n-dimensional subspace X_n and $r_n > 0$ satisfying*

$$\sup_{X_n \cap S_{r_n}} I < 0,$$

where $S_r := \{u \in X : \|u\|_X = r\}$, then I has a sequence of critical values $c_n < 0$ such that $c_n \to 0$ as $n \to \infty$.

We suppose the following assumptions on the nonlinearity f:

(F1) The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x,t)| \le C(1 + |t|^{r(x)-1})$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$, for some constant C and $r \in C(\overline{\Omega})$ with $1 < r(x) \le p^-$.

(F2) There exists a constant $a > 0$ such that

$$f \text{ is odd in } t \quad \text{and} \quad p^- F(x,t) - f(x,t)t > 0,$$

for a.a. $x \in \Omega$ and for all $0 < |t| < a$, where $F(x,t) := \int_0^t f(x,\tau)\,\mathrm{d}\tau$.

(F3) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{r(x)-2}t} = +\infty$ uniformly for a.a. $x \in \Omega$.

(F4) $f(x,t) \ge 0$ for a.a. $x \in \Omega$ and $t \in \mathbb{R}$.

Next, we prove the existence of infinitely many small solutions to problem (5.1). The proof is divided into several steps, see also Ho–Kim [35] and Wang [61], in the following way:

(i) Modify the function f to $\tilde{f}$ and then construct a modified functional $\tilde{E}$.

(ii) Prove that the modified functional $\tilde{E}$ satisfies the conditions of Lemma 5.1 to get a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ such that $\tilde{E}(u_n) \to 0$ as $n \to \infty$.

(iii) Show that $u_n \to 0$ in $W_0^{s,\mathcal{H}}(\Omega)$ and apply Theorem 4.2 to get $\|u_n\|_{\infty,\Omega} \to 0$ as $n \to \infty$. Finally, we verify that u_n are solutions of the original problem (5.1).

Our existence result read as follows.

Theorem 5.2 *Let hypotheses* (H1) *and* (F1)–(F3) *be satisfied. Then problem* (5.1) *has a sequence of weak solutions* $\{u_n\}_{n\in\mathbb{N}}$ *satisfying* $\|u_n\|_{\infty,\Omega} \to 0$ *as* $n \to \infty$*. In addition, if* (F4) *hold, then the weak solutions* u_n *are non-negative.*

Proof First, we introduce the functional $\mathcal{I}\colon W_0^{s,\mathcal{H}}(\Omega) \to \mathbb{R}$ given as

$$\begin{aligned}\mathcal{I}(u) = \int_\Omega\int_\Omega \Bigg(&\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x,y|^{N+sp(x,y)}} \log\left(e + \omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\\ &+ \mu(x,y)\frac{|u(x)-u(y)|^{q(x,y)}}{q(x,y)|x,y|^{N+sp(x,y)}} \log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\Bigg)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

for all $u \in W_0^{s,\mathcal{H}}(\Omega)$. Recalling Proposition 2.14, it is not hard to check that $\mathcal{I} \in C^1(W_0^{s,\mathcal{H}}(\Omega),\mathbb{R})$ and its Gâteaux derivative $\mathcal{A}\colon W_0^{s,\mathcal{H}}(\Omega) \to (W_0^{s,\mathcal{H}}(\Omega))^*$ is given by

$$\begin{aligned}\langle \mathcal{A}(u), v\rangle = \int_\Omega\int_\Omega \Bigg(&\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}} \log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\\ &+ \frac{\omega|u(x)-u(y)|^{p(x,y)-1}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s(p(x,y)+1)}\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)}\\ &+ \mu(x,y)\frac{|u(x)-u(y)|^{q(x,y)}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sq(x,y)}} \log\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)\\ &+ \mu(x,y)\frac{\omega|u(x)-u(y)|^{q(x,y)-1}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s(q(x,y)+1)}\left(e+\omega\frac{|u(x)-u(y)|}{|x-y|^s}\right)}\Bigg)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

for all $u, v \in W_0^{s,\mathcal{H}}(\Omega)$.

In order to apply Lemma 5.1, we first modify the nonlinear function f to $\tilde{f}$. Precisely, one can deduce from (F2) and (F3) that there exits $a_1 \in (0,a)$ such that

$$F(x,t) \geq |t|^{r(x)} \quad \text{for a.a. } x \in \Omega \text{ and for all } |t| < a_1. \tag{5.2}$$

Next, we choose $a_2 \in (0, a_1/2)$ and take $\phi \in C^1(\mathbb{R},\mathbb{R})$ to be an even function satisfying

$$\phi(t) = \begin{cases} 1, & |t| \leq a_2,\\ 0, & |t| \geq 2a_2,\end{cases} \qquad |\phi'(t)| \leq 2/a_2 \quad \text{and} \quad \phi'(t)t \leq 0.$$

Next, we define

$$\tilde{F}(x,t) := \phi(t)F(x,t) + (1-\phi(t))\beta|t|^{p^-},$$

where

$$\beta \in \left(0, \min\left\{\frac{1}{p^-C_{e1}C_{e2}}, \frac{1}{q^+C'^{p^-}C_{e3}^{p^-}}\right\}\right) \tag{5.3}$$

with C' given by (2.1), C_{e1} is the embedding constant from $W_0^{s,p^-}(\Omega)$ to $L^{p^-}(\Omega)$, C_{e2} is the constant such that $\|u\|_{W_0^{s,p^-}} \leq C_{e2}[u]_{s,p^-}$ and C_{e3} is the embedding constant from $W_0^{s,\mathcal{H}}(\Omega)$

to $L^{p^-}(\Omega)$. Then, the modified function $\tilde{f}$ is given by

$$\tilde{f}(x,t) := \frac{\partial}{\partial t}\tilde{F}(x,t).$$

Moreover, we consider the modified energy functional $\tilde{E}\colon W_0^{s,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{E}(u) = \mathcal{I}(u) - \int_\Omega \tilde{F}(x,u)\,\mathrm{d}x, \quad u \in W_0^{s,\mathcal{H}}(\Omega).$$

By the definition of $\tilde{F}$ and $\tilde{f}$, we see that $\tilde{F}$ is even in t and

$$\tilde{f}(x,t) = \phi'(t)F(x,t) + \phi(t)f(x,t) - \phi'(t)\beta|t|^{p^-} + (1-\phi(t))\beta p^-|t|^{p^--2}t. \tag{5.4}$$

Thus,

$$p^-\tilde{F}(x,t) - \tilde{f}(x,t)t = \phi(t)[p^-F(x,t) - f(x,t)t] - \phi'(t)t[F(x,t) - \beta|t|^{p^-}]. \tag{5.5}$$

Recalling the definition of ϕ, by (5.2), (5.4) and (5.5) we get

$$p^-\tilde{F}(x,t) - \tilde{f}(x,t)t \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \tag{5.6}$$

$$p^-\tilde{F}(x,t) - \tilde{f}(x,t)t = 0 \quad \text{if and only if } t = 0 \text{ or } |t| \geq 2a_2. \tag{5.7}$$

Recalling (F1) and the definition of ϕ, $\tilde{F}$ and $\tilde{f}$ we can find $C > 0$ such that

$$\tilde{F}(x,t) \leq C + \beta|t|^{p^-} \quad \text{and} \quad |\tilde{f}(x,t)| \leq C\left(1 + |t|^{p^--1}\right) \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{5.8}$$

Hence, invoking that $\mathcal{I} \in C^1(W_0^{s,\mathcal{H}}(\Omega),\mathbb{R})$ and $W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ one can prove that $\tilde{E} \in C^1(W_0^{s,\mathcal{H}}(\Omega),\mathbb{R})$.

Now, we are ready to show that $\tilde{E}$ fulfills the conditions given by Lemma 5.1. It is not hard to see that $\tilde{E}$ is even and $E(0) = 0$. Utilizing (5.8) and Proposition 2.8, we get

$$\begin{aligned}
\tilde{E}(u) &\geq \frac{1}{q^+}\left([u]_{s,\mathcal{H}}^{p^-} - 1\right) - \beta\|u\|_{L^{p^-}(\Omega)}^{p^-} - C|\Omega| \\
&\geq \frac{1}{q^+}[u]_{s,\mathcal{H}}^{p^-} - \beta C_{e3}^{p^-}\|u\|_{s,\mathcal{H}}^{p^-} - C|\Omega| - \frac{1}{q^+} \\
&\geq \frac{1}{q^+}[u]_{s,\mathcal{H}}^{p^-} - \beta C_{e3}^{p^-}C'^{p^-}[u]_{s,\mathcal{H}}^{p^-} - C|\Omega| - \frac{1}{q^+}.
\end{aligned}$$

Note that the range of β given in (5.3) implies that $\tilde{E}$ is coercive and bounded from below on $W_0^{s,\mathcal{H}}(\Omega)$. Due to (5.8) and the compact embedding $W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow\hookrightarrow L^{p^-}(\Omega)$ we infer that the operator $u \mapsto \int_\Omega \tilde{f}(x,t)\,\mathrm{d}x$ is compact. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,\mathcal{H}}(\Omega)$ be a (PS)-sequence, that is $\tilde{E}(u_n)$ is bounded and $\tilde{E}'(u_n) \to 0$. Then, by the coercivity of $\tilde{E}$, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Since $W_0^{s,\mathcal{H}}(\Omega)$ is reflexive, $\{u_n\}_{n\in\mathbb{N}}$ possesses a subsequence still denoted by $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \rightharpoonup u_* \in W_0^{s,\mathcal{H}}(\Omega)$. Hence, due to the compactness of $u \mapsto \int_\Omega \tilde{f}(x,t)\,\mathrm{d}x$ and applying the (S_+)-property of $\mathcal{A}$, we deduce that $u_n \to u_* \in W_0^{s,\mathcal{H}}(\Omega)$. This shows that $\tilde{E}$ satisfies the (PS)-condition.

Next, we choose a fixed $n \in \mathbb{N}$ and let $\phi_1,\dots,\phi_n$ be linearly independent functions. We set $X_n := \operatorname{span}\{\phi_1,\dots,\phi_n\}$. Since X_n is a finite dimensional space, the norms $\|\cdot\|_{\infty,\Omega}$,

$[\,\cdot\,]_{s,\mathcal{H},\Omega}$ and $\|\cdot\|_{L^{p^-}(\Omega)}$ are equivalent on X_n. Thus one can find $c_1, c_2 > 0$ such that

$$c_1\|u\|_{\infty,\Omega} \le [u]_{s,\mathcal{H},\Omega} \le c_2\|u\|_{L^{p^-}(\Omega)} \quad \text{for all } u \in X_n. \tag{5.9}$$

According to hypotheses (F2) and (F3) we can find $a_3 \in (0, a_2)$ satisfying

$$F(x,t) \ge \frac{2c_2^{p^-}}{p^-}|t|^{p^-} \tag{5.10}$$

for a.a. $x \in \Omega$ and for all $|t| \le a_3$. Next, we take $r_n := \min\{1, a_3c_1\}$, then by (5.9) we see that for any $u \in X_n$ with $[u]_{s,\mathcal{H}}^{p^-} = r_n$ we have $|u|_{s,\mathcal{H}} < 1$ as well as $\|u\|_{\infty,\Omega} \le a_3$. Note that $\tilde{F}(x,u) = F(x,u)$ for $\|u\|_{\infty,\Omega} \le a_3$. Then, Proposition 2.8 and inequality (5.10) yield

$$\begin{aligned}
\tilde{E}(u) &\le \frac{1}{p^-}[u]_{s,\mathcal{H}}^{p^-}\\
&\quad - \frac{2c_2^{p^-}}{p^-}\|u\|_{L^{p^-}}^{p^-} \le \frac{1}{p^-}[u]_{s,\mathcal{H}}^{p^-} - \frac{2}{p^-}[u]_{s,\mathcal{H}}^{p^-} = -\frac{(r_n)^{p^-}}{p^-} \quad \text{for all } u \in X_n \cap S_{r_n},
\end{aligned}$$

which implies

$$\sup_{u \in X_n \cap S_{r_n}} \tilde{E}(u) < 0.$$

Using Lemma 5.1 we infer that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,\mathcal{H}}(\Omega)$ with

$$\tilde{E}'(u_n) = 0 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \tilde{E}(u_n) \to 0 \quad \text{as } n \to \infty.$$

Moreover, recall that $u_n \to u_*$ in $W_0^{s,\mathcal{H}}(\Omega)$, due to $\tilde{E} \in C^1(W_0^{s,\mathcal{H}}(\Omega), \mathbb{R})$, we have $\tilde{E}(u_*) = \langle \tilde{E}'(u_*), u_*\rangle = 0$, which gives $\frac{1}{p^-}\langle \tilde{E}'(u_*), u_*\rangle - \tilde{E}(u_*) = 0$. Taking this and (5.6) into account we arrive at

$$\begin{aligned}
0 &\le \int_\Omega\int_\Omega \Bigg(\left(\frac{1}{p^-} - \frac{1}{p(x,y)}\right) \frac{|u_*(x) - u_*(y)^{p(x,y)}|}{|x-y|^{N+sp(x,y)}} \log\left(e + \omega\frac{|u_*(x) - u_*(y)|}{|x-y|^s}\right)\\
&\quad + \mu(x,y)\left(\frac{1}{p^-} - \frac{1}{q(x,y)}\right) \frac{|u_*(x) - u_*(y)^{q(x,y)}|}{|x-y|^{N+sq(x,y)}} \log\left(e + \omega\frac{|u_*(x) - u_*(y)|}{|x-y|^s}\right)\\
&\quad + \frac{\omega|u_*(x) - u_*(y)|^{p(x,y)+1}}{p^-|x-y|^{N+s(p(x,y)+1)}\left(e + \omega\frac{|u_*(x)-u_*(y)|}{|x-y|^s}\right)}\\
&\quad + \mu(x,y)\frac{\omega|u_*(x) - u_*(y)|^{q(x,y)+1}}{p^-|x-y|^{N+s(q(x,y)+1)}\left(e + \omega\frac{|u_*(x)-u_*(y)|}{|x-y|^s}\right)}\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&= -\int_\Omega \left(\tilde{F}(x,u_*(x)) - \frac{1}{p^-}\tilde{f}(x,u_*(x))u_*(x)\right)\mathrm{d}x \le 0.
\end{aligned}$$

From the above inequalities and (5.7) we see that

$$\begin{aligned}
0 &\le \int_\Omega\int_\Omega \Bigg(\left(\frac{1}{p^-} - \frac{1}{p(x,y)}\right) \frac{|u_*(x) - u_*(y)^{p(x,y)}|}{|x-y|^{N+sp(x,y)}} \log\left(e + \omega\frac{|u_*(x) - u_*(y)|}{|x-y|^s}\right)\\
&\quad + \mu(x,y)\left(\frac{1}{p^-} - \frac{1}{q(x,y)}\right) \frac{|u_*(x) - u_*(y)^{q(x,y)}|}{|x-y|^{N+sq(x,y)}} \log\left(e + \omega\frac{|u_*(x) - u_*(y)|}{|x-y|^s}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\omega|u_*(x)-u_*(y)|^{p(x,y)+1}}{p^-|x-y|^{N+s(p(x,y)+1)}\left(e+\omega\frac{|u_*(x)-u_*(y)|}{|x-y|^s}\right)}\\
&+\mu(x,y)\frac{\omega|u_*(x)-u_*(y)|^{q(x,y)+1}}{p^-|x-y|^{N+s(q(x,y)+1)}\left(e+\omega\frac{|u_*(x)-u_*(y)|}{|x-y|^s}\right)}\Bigg)\,\mathrm{d}x\,\mathrm{d}y\\
&=0
\end{aligned}
$$

and for a.a. $x\in\Omega$,

$$u_*=0,$$

or

$$|u_*(x)|\geq 2a_2 \quad \text{and} \quad u_*=c,$$

where c is constant. Hence, $\tilde{F}(x,u_*(x))=0$ or $\tilde{F}(x,u_*(x))=\beta|u_*|^{p^-}$. Moreover, $p(x,y)=p^-$ for a.a. $x\in\Omega$ satisfying $|u_*(x)|\geq 2a_2$. This implies

$$
\begin{aligned}
0=\tilde{E}&\geq\int_\Omega\int_\Omega\frac{1}{p(x,y)}\frac{|u_*(x)-u_*(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega\tilde{F}(x,u_*(x))\,\mathrm{d}x\\
&=\int_\Omega\int_\Omega\frac{1}{p^-}\frac{|u_*(x)-u_*(y)|^{p^-}}{|x-y|^{N+sp^-}}\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega\tilde{F}(x,u_*(x))\,\mathrm{d}x\\
&\geq\frac{1}{p^-}[u_*]_{s,p^-}^{p^-}-\int_\Omega\beta|u_*|^{p^-}\,\mathrm{d}x\\
&=\frac{1}{p^-}[u_*]_{s,p^-}^{p^-}-\beta\|u_*\|_{L^{p^-}}^{p^-}\\
&\geq\frac{1}{p^-}[u_*]_{s,p^-}^{p^-}-\beta C_{e1}C_{e2}[u_*]_{s,p^-}^{p^-}
\end{aligned}
$$

Due to $\beta<\frac{1}{p^-C_{e1}C_{e2}}$, it holds that $u_*=0$. That means $u_n\to 0$ in $W_0^{s,\mathcal{H}}(\Omega)$, so $\|u_n\|_{\mathcal{B},\Omega}\to 0$ as $n\to\infty$. Note that under the hypotheses (F1), f satisfies hypotheses (H2) (or (H2')). Then we deduce from Theorem 4.2 (or Theorem 4.4) that $\|u_n\|_{\infty,\Omega}\to 0$. Hence, $\|u_n\|_{\infty,\Omega}\leq a_2$ for n large enough, which means that $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of weak solutions to problem (5.1) for n large enough.

Furthermore, if $f(x,t)\geq 0$ for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$, we see that

$$
\begin{cases}
(-\Delta)^s_{\mathcal{H}}u\geq 0, & \text{in } \Omega,\\
u=0 & \text{on } \mathbb{R}^N\setminus\Omega.
\end{cases}
$$

Hence, employing Theorem 3.1 we see that $u(x)\geq 0$ for $x\in\Omega$ and if there exists some point $x_0\in\Omega$ such that $u(x_0)=0$, then $u(x)=0$ for a.a. $x\in\mathbb{R}^N$. This ends the proof. □

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