EXISTENCE AND UNIQUENESS FOR SINGULAR LOGARITHMIC DOUBLE PHASE PROBLEMS

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Abstract. In this work we study singular elliptic equations driven by the logarithmic double phase operator. We first establish the existence of at least one positive bounded weak solution under general structural conditions. Subsequently, we refine this result by proving uniqueness of a positive solution for a class of problems involving mild singularities at the origin and perturbed by sublinear growth nonlinearities. Our approach relies on the framework of pseudomonotone operators along with a new weak maximum principle specifically developed for the logarithmic double phase setting, which plays a key role in our analysis. To the best of our knowledge, this is the first contribution dealing with logarithmic double phase problems that include singular terms.

1. Introduction and main results

Recently, Arora–Crespo-Blanco–Winkert [2] introduced the so-called logarithmic double phase operator, defined as

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right), \tag{1.1}$$

where the unknown function u belongs to the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ generated by the nonlinear function

$$\mathcal{H}_{\log}(x,t)=t^p+\mu(x)t^q\log(e+t)\quad \text{for all } (x,t)\in\overline{\Omega}\times[0,\infty).$$

Here $1<p<N$, $p<q$, e denotes Euler's number, and the weight satisfies $0\le\mu(\cdot)\in L^\infty(\Omega)$. The corresponding energy functional associated with (1.1) is given by

$$u\mapsto\int_\Omega\left(\frac{|\nabla u|^p}{p}+\mu(x)\frac{|\nabla u|^q}{q}\log(e+|\nabla u|)\right)\mathrm{d}x, \tag{1.2}$$

which has been analyzed in special cases in recent years. In particular, we mention the contributions of Baroni–Colombo–Mingione [7], Byun–Oh [12], and De Filippis–Mingione [19], who studied local Hölder regularity of the gradient of local minimizers for functionals of type (1.2).

This operator can be regarded as a generalization of the classical double phase operator

$$\mathcal{K}(u)=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right),\quad u\in W_0^{1,\mathcal{H}}(\Omega),$$

associated with the modular function

$$\mathcal{H}(x,t)=t^p+\mu(x)t^q\quad \text{for all } (x,t)\in\overline{\Omega}\times[0,\infty),$$

which was introduced by Marcellini [42, 43] and Zhikov [59, 60, 61] in the framework of elasticity theory to describe the behavior of composites consisting of two different materials. Since then, the double phase operator has found numerous applications in various fields: for instance, Bahrouni–Rădulescu–Repovš [5] investigated models of transonic flows, Benci–D'Avenia–Fortunato–Pisani [10] studied equations arising in electromagnetism, while Harjulehto–Hästö [33] employed it in problems of image restoration and denoising.

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Closely related problems have recently been treated in the works of Cen–Lu–Vetro–Zeng [16] and Lu–Vetro–Zeng [41], where the density function is given by

$$\mathcal{H}_L(x,t) = (t^p + \mu(x)t^q)\log(e+t),$$

and the corresponding operator takes the form

$$\operatorname{div}\left(\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right), \quad u \in W_0^{1,\mathcal{H}_L}(\Omega).$$

One of the motivations for studying functionals with logarithmic perturbations, such as

$$u \mapsto \int_\Omega |\nabla u| \log(1+|\nabla u|)\,\mathrm{d}x,$$

comes from plasticity theory with logarithmic hardening. For further details, we refer the reader to the seminal contributions of Seregin–Frehse [55] and Fuchs–Seregin [24].

The aim of this paper is to prove existence and uniqueness of solutions for the following elliptic problem

$$\begin{cases} -\mathcal{G}(u) = \beta(x)u^{-\eta} + f(x,u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with a C^2-boundary $\partial\Omega$ and $\mathcal{G}$ is the logarithmic double phase operator given in (1.1). We assume the following hypotheses:

(H_0) $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$ and $\mu \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$ with $\mu(x) \geq 0$ for all $x \in \overline{\Omega}$;

(H_β) $\beta \in W_0^{1,\infty}(\Omega)$, $\beta(x) > 0$ for a.a. $x \in \Omega$ and $c^* d \leq \beta$ on the set $\Omega_\varepsilon = \{x \in \Omega \colon d(x) < \varepsilon\}$ with $c^* > 0$, $\varepsilon > 0$ and $d = d(x,\partial\Omega)$ denotes the distance function defined in (2.3);

(H_η) $0 < \eta < 1$;

(H_f) $f \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function such that the following hold:

(i) There exist $c_1 > 0$ and $\tau \leq p$ such that
$$f(x,s) \leq c_1 s^{\tau-1}$$
for a.a. $x \in \Omega$ and for all $s \geq 0$. In addition, if $\tau = p$, it holds $\widehat{c}_0 c_1 < 1$, where $\widehat{c}_0$ is the constant from the estimate (2.5);

(ii) There exist $\rho > 0$ and $\widehat{\xi}_\rho > 0$ such that
$$|f(x,s)| \leq \widehat{\xi}_\rho$$
for a.a. $x \in \Omega$ and for all $0 \leq s \leq \rho$;

(iii) There exists $\delta > 0$ such that
$$f(x,0) = 0 \leq f(x,s)$$
for a.a. $x \in \Omega$ and for all $0 \leq s \leq \delta$.

For a measurable function $u \colon \Omega \to \mathbb{R}$ we write $0 \prec u$ if for all compact sets $K \subset \Omega$, there exists a positive constant $c_K > 0$ such that $0 < c_K \leq u(x)$ for a.a. $x \in K$. Note that if $0 \prec u$ then $0 < u(x)$ for a.a. $x \in \Omega$.

Our existence result reads as follows.

Theorem 1.1. *Let hypotheses* (H_0), (H_β), (H_η), *and* (H_f) (i)–(iii) *be satisfied. Then, there exists at least one solution* $u_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *of problem* (P) *such that* $0 \prec u_0$.

For our uniqueness result, we need the following monotonicity condition.

(H_f) (iv) $t \mapsto \frac{f(x,t)}{t^{\tau-1}}$ is nonincreasing on $(0,+\infty)$ for a.a. $x \in \Omega$, where τ is from (i) above.

Our second result concerning the uniqueness of (P) is the following one.

Theorem 1.2. *Let hypotheses* (H_0), (H_β), (H_η), *and* (H_f) (i)–(iv) *be satisfied. Then, problem* (P) *admits a unique solution* $u_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *with* $0 \prec u_0$.

The proofs of Theorems 1.1 and 1.2 are based, in particular, on a new weak maximum principle for operators of the form (1.1) developed in Section 2. In addition, we make use of a careful analysis of several auxiliary results in order to construct sub-super-solutions for problem (P). Our work can be seen as a generalization of the results obtained by Failla–Gasiński–Papageorgiou–Skupień [21] for the classical double phase operator without logarithm. To the best of our knowledge, this is the first contribution dealing with logarithmic double phase problems that include singular terms.

The study of singular problems has a wide interest also due to the large range of applications. We refer to Callegari–Nachman [13] for boundary layer problems, Carleman [15] for kinetic gas theory, Keller–Cohen [38] for models in heat generation in conductors or Turing [57] for pattern formation and morphogenesis. From a mathematical perspective, the singular term raises significant and interesting difficulties, as discussed in the recent works for double phase problems without logarithm by Arora–Dwivedi [3], Arora–Fiscella–Mukherjee–Winkert [4], Bahrouni–Rădulescu–Repovš [6], Bai–Gasiński–Papageorgiou [8], Bai–Papageorgiou–Zeng [9], Candito–Failla–Livrea [14], Farkas–Winkert [22], Garain–Mukherjee [25], Guarnotta–Winkert [30], Liu–Dai–Papageorgiou–Winkert [39], Liu–Papageorgiou [40], Papageorgiou–Peng [44], Papageorgiou–Rădulescu–Zhang [46], Papageorgiou–Repovš–Vetro [48], Sim–Son [56], see also the references therein. Finally we also mention papers for strongly singular double phase problems without logarithm, see Papageorgiou–Rădulescu–Yuan [45], Papageorgiou–Rădulescu–Zhang [47] and Pimenta–Winkert [52] and also works involving the logarithmic double phase operator given in (1.1), but without singular nonlinearities, see the papers by Arora–Crespo-Blanco–Winkert [1] and Vetro–Winkert [58]. For the sake of completeness we also refer to the seminal paper by Biagi–Esposito–Vecchi [11].

The paper is organized as follows. In Section 2 we first introduce the underlying function space and state the properties of the logarithmic double phase operator. In addition, we prove a new weak maximum principle for such class of logarithmic operators. In Section 3 we give the proofs of Theorems 1.1 and 1.2 based on the weak maximum principle and the theory of pseudomonotone operators along with suitable auxiliary problems in order to construct sub-super-solutions for problem (P).

2. Preliminaries

In this section we recall first the main properties of our underlying function space and the logarithmic double phase operator given in (1.1). We also prove a version of a weak maximum principle for logarithmic double phase problems which will be used in the sequel. To this end, for $1 \leq r \leq \infty$, we denote $L^r(\Omega)$ the Lebesgue space with norm $\|\cdot\|_r$ while $W_0^{1,r}(\Omega)$ denotes the corresponding Sobolev space with zero traces endowed with the equivalent norm $\|\nabla \cdot\|_r$ for $1 < r < \infty$. Next, we are going to introduce the logarithmic Musielak-Orlicz Sobolev spaces under the hypotheses (H_0). Let $M(\Omega)$ be the set of all measurable functions and suppose that (H_0) holds. Most of the results are taken from Arora–Crespo-Blanco–Winkert [2] as well as the monographs by Harjulehto–Hästö [34] and Papageorgiou–Winkert [50], see also the paper by Crespo-Blanco–Gasiński–Harjulehto–Winkert [17] for the main properties of double phase operators without logarithm. We introduce the nonlinear function $\mathcal{H}_{\log}\colon \overline{\Omega} \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^p + \mu(x) t^q \log(e+t),$$

where e is Euler's number. Then we can define the logarithmic Musielak-Orlicz space $L^{\mathcal{H}_{\log}}(\Omega)$ by

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{ u \in M(\Omega)\colon \rho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x < \infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_{\log}} := \inf\left\{\lambda > 0\colon \rho_{\mathcal{H}_{\log}}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

where $\rho_{\mathcal{H}_{\log}}$ is the corresponding modular function. The related logarithmic Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{u \in L^{\mathcal{H}_{\log}}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}}.$$

Moreover, we set

$$W_0^{1,\mathcal{H}_{\log}}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_{\log}}}.$$

Note that the spaces $L^{\mathcal{H}_{\log}}(\Omega)$, $W^{1,\mathcal{H}_{\log}}(\Omega)$ and $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ are separable and reflexive Banach spaces. Moreover, the space $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ can be endowed with the equivalent norm

$$\|u\| := \|\nabla u\|_{\mathcal{H}_{\log}} \quad \text{for all } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega),$$

see Arora–Crespo-Blanco–Winkert [2, Proposition 3.9].

The embedding results below are established in the work of Arora–Crespo-Blanco–Winkert [2, Proposition 3.7].

Proposition 2.1. *Let hypotheses* (H_0) *be satisfied. Then the following hold:*

(i) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ *is continuous;*

(ii) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^r(\Omega)$ *is compact for all* $1 \leq r < p^*$.

Furthermore, the connection between the norm $\|\cdot\|$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and the modular function $\rho_{\mathcal{H}_{\log}}$ is presented in the next proposition, see Arora–Crespo-Blanco–Winkert [2, Proposition 3.6]. In what follows, we denote by κ the constant defined as

$$\kappa = \frac{e}{e+t_0}, \tag{2.1}$$

with t_0 being the positive solution of $t_0 = e\log(e+t_0)$.

Proposition 2.2. *Let hypotheses* (H_0) *be satisfied,* $\lambda > 0$*,* $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$*, and* κ *as in* (2.1)*. Then the following hold:*

(i) $\|u\| = \lambda$ *if and only if* $\rho_{\mathcal{H}_{\log}}\left(\frac{\nabla u}{\lambda}\right) = 1$*;*

(ii) $\|u\| < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u) < 1$ *(resp.* $= 1, > 1$*);*

(iii) $\min\{\|u\|^p, \|u\|^{q+\kappa}\} \leq \rho_{\mathcal{H}_{\log}}(\nabla u) \leq \max\{\|u\|^p, \|u\|^{q+\kappa}\}$*;*

(iv) $\|u_n\| \to 0$ *if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u_n) \to 0$ *as* $n \to \infty$*;*

(v) $\|u_n\| \to \infty$ *if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u_n) \to \infty$ *as* $n \to \infty$.

We now define the nonlinear operator $V\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ defined by

$$\begin{aligned}\langle V(u), v\rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\ &\quad + \int_\Omega \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x\end{aligned} \tag{2.2}$$

for all $u, v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and $W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$. The key properties of V are stated in the next proposition, see Arora–Crespo-Blanco–Winkert [2, Theorem 4.4].

Theorem 2.3. *Let hypotheses* (H_0) *be satisfied and V be given as in* (2.2). *Then V is bounded, continuous, strictly monotone (hence maximal monotone), and satisfies the* (S_+)-*property, that is, any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leq 0$ converges strongly to u in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

Next, we give the definition of pseudomonotonicity.

Definition 2.4. *Let X be a reflexive Banach space. The operator $K\colon X \to X^*$ is said to be pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle K(u_n), u_n - u\rangle \leq 0$ imply $\liminf_{n\to\infty}\langle K(u_n), u_n - w\rangle \geq \langle Ku, u - w\rangle$ for all $w \in X$. If $K\colon X \to X^*$ is bounded, the definition of pseudomonotonicity is equivalent to the following: if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle K(u_n), u_n - u\rangle \leq 0$ we have $K(u_n) \rightharpoonup Ku$ in X^* and $\langle K(u_n), u_n\rangle \to \langle Ku, u\rangle$, see Frisch–Winkert* [23, Proposition 2.3].

We now recall Hardy's inequality, which is fundamental in the study of singular problems, see Papageorgiou–Winkert [50, Theorem 6.8.33].

Lemma 2.5. *Let $p > 1$. Then, there exist $c_H = c_H(N,p) > 0$ and $b_0 = b_0(N,p) > 0$ such that*

$$\left\|\frac{u}{d^{1-b}}\right\|_p \leq c_H \left\||\nabla u| d^b\right\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega) \text{ and for all } 0 \leq b \leq b_0,$$

where $d = d(x, \partial\Omega)$ denotes the distance function, i.e.,

$$d(x) = d(x, \partial\Omega) = \min_{y\in\partial\Omega} |x - y| \quad \text{for all } x \in \overline{\Omega}. \tag{2.3}$$

Let $g \in L^\infty(\Omega)$ and consider the problem

$$-\mathcal{G}(u) = g(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where $\mathcal{G}$ is the logarithmic double phase operator given in (1.1). Similar to the works by Guedda–Véron [31, Proposition 1.3] and Perera–Squassina [51, Proposition 2.4], we can show that problem (2.4) has a unique solution $\hat{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ such that

$$\|\hat{u}\|_\infty \leq \widehat{c}_0 \|g\|_\infty^{\frac{1}{p-1}}. \tag{2.5}$$

for some $\widehat{c}_0 > 0$.

Finally, we are going to prove a weak maximum principle for logarithmic double phase problems which will be useful in Section 3. Such a result is motivated by the paper of Papageorgiou–Vetro–Vetro [49, Proposition 2.4] and the monograph by Hu–Papageorgiou [37, Theorem 5.10].

Lemma 2.6. *Let hypotheses* (H_0) *be satisfied, $\widehat{\xi} > 0$ and $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$ with $u \geq 0$ a.e. in Ω such that*

$$-\mathcal{G}(u) + \widehat{\xi} u^{q-1} \geq 0 \quad \text{in } \Omega,$$

where $\mathcal{G}$ is the logarithmic double phase operator given in (1.1). *Then it holds $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ and $0 \prec u$.*

Proof. We prove the lemma in two steps.

Step 1: First, assume that $u \in C^1(\Omega)$. By contradiction, assume that the assertion is not true. Then there exist $x_1, x_2 \in \Omega$ and an open ball

$$B_{2\rho}(x_2) = \{x \in \Omega\colon |x - x_2| < 2\rho\}$$

such that $\overline{B}_{2\rho}(x_2) \subset \Omega$ and

$$x_1 \in \partial B_{2\rho}(x_2), \quad u(x_1) = 0, \quad u|_{B_{2\rho}(x_2)} > 0. \tag{2.6}$$

Note that fixing x_1 as above and moving x_2 towards x_1, we decrease the radius 2ρ of the ball, preserving the above properties. From (2.6) and $u \geq 0$, we infer that $x_1 \in \Omega$ is a minimizer of $u(\cdot)$. Thus, $\nabla u(x_1) = 0$. Let

$$b = \min\{u(x)\colon x \in \partial B_\rho(x_2)\}.$$

Again, by (2.6), we have $b > 0$. Moreover, observe that, if $x_2 \to x_1$ (x_1 fixed), then $\rho \to 0^+$, and so by L'Hôpital's rule, we get

$$b \to 0^+ \quad \text{and} \quad \frac{b}{\rho} \to 0^+.$$

Consider the annulus

$$D = \{x \in \Omega \colon \rho < |x - x_2| < 2\rho\}.$$

Since $\mu \in C^{0,1}(\overline{\Omega})$, by Rademacher's Theorem, we know that μ is differentiable a.e. in Ω. Put

$$m = \sup_{x \in D} \left\{ \left| \operatorname{div} \left(\mu(x) \left[\log(e + |\nabla u(x)|) + \frac{|\nabla u(x)|}{q(e + |\nabla u(x)|)} \right] \right) \right| \right\},$$

see Arora–Crespo-Blanco–Winkert [2, Lemma 5.4] and recall that $u \in C^1(\Omega)$ with $u \geq 0$ a.e. in Ω. Moreover, let

$$\vartheta = -\log\left(\frac{b}{\rho}\right) + \frac{N-1}{\rho} + 2m.$$

The function

$$v(t) = b\frac{e^{\frac{\vartheta t}{p-1}} - 1}{e^{\frac{\vartheta \rho}{p-1}} - 1}$$

is such that

$$0 < v(t), v'(t) < 1 \quad \text{for all } t \in [0, \rho]$$

with $\rho > 0$ sufficiently small. A straightforward computation shows that

$$v''(t) = \frac{\vartheta}{p-1} v'(t) \quad \text{for all } t \in (0, \rho). \tag{2.7}$$

Take $x_2 = 0$ and set $r = |x|$ as well as $t = 2\rho - r$. For $t \in [0, \rho]$ and $r \in [\rho, 2\rho]$ put

$$y(r) = v(t)$$

that is,

$$y'(r) = -v'(t) \quad \text{and} \quad y''(r) = v''(t).$$

Set $y(x) = y(r)$ for all $x \in \Omega$, $|x| = r$. Recall that, in polar coordinates, we have

$$\operatorname{div}(\nabla f) = \frac{1}{r^{N-1}} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1} f'\right) = \frac{N-1}{r} f' + f''.$$

Then, by applying (2.7), for $\rho > 0$ sufficiently small, we obtain

$$\begin{aligned}
&\operatorname{div}\left(|\nabla y|^{p-2}\nabla y + \mu(x)\left[\log(e + |\nabla y|) + \frac{|\nabla y|}{q(e + |\nabla y|)}\right] |\nabla y|^{q-2}\nabla y \right) \\
&= |\nabla y|^{q-2}\nabla y \operatorname{div}\left(\mu(x)\left[\log(e + |\nabla y|) + \frac{|\nabla y|}{q(e + |\nabla y|)}\right]\right) \\
&\quad + \mu(x)\left[\log(e + |\nabla y|) + \frac{|\nabla y|}{q(e + |\nabla y|)}\right] \Delta_q y + \Delta_p y \\
&\geq -v'(t)^{q-1} \operatorname{div}\left(\mu(x)\left[\log(e + v'(t)) + \frac{v'(t)}{q(e + v'(t))}\right]\right) + \mu(x)\Delta_q y + \Delta_p y \\
&\geq -mv'(t)^{q-1} + \mu(x)\left[(q-1)v'(t)^{q-2}v''(t) - \frac{N-1}{r} v'(t)^{q-1}\right] \\
&\quad + (p-1)v'(t)^{p-2}v''(t) - \frac{N-1}{r} v'(t)^{p-1} \\
&\geq \mu(x)\left[\frac{q-1}{p-1}\vartheta - \frac{N-1}{r}\right] v'(t)^{q-1} + \left[\vartheta - \frac{N-1}{r} - m\right] v'(t)^{p-1}
\end{aligned}$$

$$\begin{aligned}
&\geq \mu(x)\left[\vartheta - \frac{N-1}{r} - m\right] v'(t)^{q-1} + \left[\vartheta - \frac{N-1}{r} - m\right] v'(t)^{p-1} \\
&\geq \left[\vartheta - \frac{N-1}{r} - m\right] (\mu(x)+1) v'(t)^{p-1} \\
&\geq \left(-\log\left(\frac{b}{\rho}\right)\right)(\mu(x)+1)v'(t)^{p-1} \\
&\geq \widehat{\xi}(x) v(t)^{q-1} = \widehat{\xi}(x) y^{q-1}.
\end{aligned}$$

Thus, we arrive at

$$-\operatorname{div}\left(|\nabla y|^{p-2}\nabla y + \mu(x)\left[\log(e+|\nabla y|) + \frac{|\nabla y|}{q(e+|\nabla y|)}\right]|\nabla y|^{q-2}\nabla y\right) + \widehat{\xi}(x) y^{q-1} \leq 0$$

in $W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$. Since $y \leq u$ on D, by the weak comparison principle, see Pucci–Serrin [53, Theorem 3.4.1], we infer that $y(x) \leq u(x)$ for all $x \in D$. Then, we can compute

$$\lim_{s\to 0^+} \frac{u(x_1 + s(x_2 - x_1)) - u(x_1)}{s} \geq \lim_{s \to 0^+} \frac{y(x_1 + s(x_2-x_1)) - y(x_1)}{s} = v'(0) > 0,$$

a contradiction, since $\nabla u(x_1) = 0$. Thus, the assertion holds if $u \in C^1(\Omega)$.

Step 2: Let $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Let

$$\Omega_0 = \left\{x \in \Omega\colon \text{ for some } \varepsilon > 0,\ B_\varepsilon(x) \subset \Omega,\ u|_{B_\varepsilon(x)} = 0 \text{ a.e.}\right\} \tag{2.8}$$

and $\Omega_+ = \Omega \setminus \Omega_0$. For $x_0 \in \Omega_+$, there exists $\varepsilon > 0$ such that $\overline{B}_{2\varepsilon}(x_0) \subseteq \Omega$ and u is not identically zero on $\partial B_{2\varepsilon}(x_0)$. Next, we consider the problem

$$\begin{cases} -\mathcal{G}(y) + \widehat{\xi}(x)|y|^{q-2}y = 0 & \text{in } B_{2\varepsilon}(x_0), \\ y = u & \text{on } \partial B_{2\varepsilon}(x_0). \end{cases} \tag{2.9}$$

From the direct methods of calculus of variations and the strict monotonicity of the operator, we know that (2.9) admits a unique solution y. By Hästö–Ok [32, Theorem 7.4, Corollary 8.4] we get $y \in C^{1,\nu}_{\rm loc}(B_{2\varepsilon}(x_0))$ for some $\nu \in (0,1)$. Indeed, consider the density function

$$\varphi(x,t) = t^p + \mu(x) t^q \log(e+t)$$

associated to (2.9). In particular, $\varphi(\cdot,\cdot)$ has the structure

$$\varphi(x,t) = a(x)\psi(t) + b(x)\xi(t)$$

with $a(x) = 1$, $b(x) = \mu(x)$, $\psi(t) = t^p$ and $\xi(t) = t^q \log(e+t)$. By (H_0), we have that $b(\cdot)$ is a Lipschitz function, that is, its modulus of continuity is $\omega_\mu(r) = kr$ for some $k > 0$. Moreover, ψ' and ξ' satisfy $(\mathrm{Inc})_{p-1}$ and $(\mathrm{Dec})_{q_1-1}$ for any $q_1 > q$ (see Hästö–Ok [32, Definition 4.1]). Thus, in order to apply Theorem 7.4 in Hästö–Ok [32], it remains to prove that φ satisfies (wVA1) of Hästö–Ok [32, Definition 4.1]. But this is satisfied if

$$\lim_{r \to 0^+} \omega_\mu(r) r^{N(1-\varepsilon)} \xi\psi^{-1}(r^{-N(1-\varepsilon)})) = 0$$

for $\varepsilon \in [0,1)$, see Corollary 8.4 of Hästö–Ok [32]. A straightforward computation shows that

$$\begin{aligned}
&\omega_\mu(r) r^{N(1-\varepsilon)} \xi\left(\psi^{-1}\left(r^{-N(1-\varepsilon)}\right)\right) \\
&\leq k r^1 r^{N(1-\varepsilon)} \xi\left(r^{-\frac{N(1-\varepsilon)}{p}}\right) \\
&= k r^1 r^{N(1-\varepsilon)} r^{-\frac{q}{p}N(1-\varepsilon)} \log\left(e + r^{-\frac{N(1-\varepsilon)}{p}}\right) \\
&= k r^{1-(\frac{q}{p}-1)N(1-\varepsilon)} \log\left(e + r^{-\frac{N(1-\varepsilon)}{p}}\right).
\end{aligned}$$

Assumption (H_0) ensures that $1-\left(\frac{q}{p}-1\right)N(1-\varepsilon)>0$. Then by L'Hôpital's rule, we infer that

$$\lim_{r\to 0^+} kr^{1-\left(\frac{q}{p}-1\right)N(1-\varepsilon)}\log\left(e+r^{-\frac{N(1-\varepsilon)}{p}}\right)=\frac{kN(1-\varepsilon)}{q}\lim_{r\to 0^+} r^{1-\left(\frac{q}{p}-1\right)N(1-\varepsilon)}|\log(r)|=0.$$

Thus, by Hästö–Ok [32, Theorem 7.4], we deduce that $y\in C^{1,\nu}_{\mathrm{loc}}(B_{2\varepsilon}(x_0))$ for some $\nu\in(0,1)$.

Since $y=u$ on $\partial B_{2\varepsilon}(x_0)$, again by the weak comparison principle, we obtain $y(x)\geq 0$ for a.a. $x\in B_{2\varepsilon}(x_0)$. Finally, from Step 1, we have $y(x)>0$ for all $x\in B_{2\varepsilon}(x_0)$ and then, $y(x)\geq k_1>0$ for all $x\in \overline{B}_\varepsilon(x_0)$ for some constant $k_1>0$. Then it follows that $u(x)\geq y(x)\geq k_1>0$ for a.a. $x\in\overline{B}_\varepsilon(x_0)$. Furthermore, we will prove that $\Omega_0=\emptyset$. Assume, by contradiction, that $\Omega_0\neq\emptyset$. Since Ω_0 is open, we can choose $x_0\in\Omega_+\cap\partial\Omega_0$, and the above computation shows that $u>0$ for a.a. $x\in B_\varepsilon(x_0)$. Thus, $u>0$ for a.a. $x\in B_\varepsilon(x_0)\cap\Omega_0$, which contradicts (2.8).

Finally, let $K\subset\Omega$ be compact and consider

$$K\subseteq\bigcup_{i=1}^m B_i$$

with balls B_i such that $u\geq\widehat{c}_i>0$ for a.a. $x\in B_i$. Let $c_K=\min\{\widehat{c}_i\colon i\in\{1,\dots,m\}\}$. Clearly, we infer

$$u(x)\geq c_K>0\quad\text{for a.a. } x\in K.$$

□

3. Proofs of the main results

In this Section we give the proofs of Theorems 1.1 and 1.2. First, we consider the purely singular problem

$$\begin{cases} -\mathcal{G}(u)=\lambda\beta(x)u^{-\eta} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}\tag{3.1}$$

We have the following uniqueness result for problem (3.1).

Proposition 3.1. *Let hypotheses (H_0), (H_β), and (H_η) be satisfied. Then problem (3.1) admits a unique solution $\overline{u}_\lambda\in W^{1,\mathcal{H}_{\log}}_0(\Omega)\cap L^\infty(\Omega)$ for each $\lambda>0$. In particular, $0\prec\overline{u}_\lambda$ and $\overline{u}_\lambda\to 0$ as $\lambda\to 0$.*

Proof. First, we prove that (3.1) admits a unique solution u_λ such that $0\prec u_\lambda$. We follow ideas from the work by Gasiński–Papageorgiou [28]. To this end, let $v\in L^q(\Omega)$, $n\in\mathbb{N}$, and consider the following perturbed problem

$$\begin{cases} -\mathcal{G}(u)=\dfrac{\lambda\beta(x)}{\left(|v|+\frac{1}{n}\right)^\eta} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}\tag{3.2}$$

Note that for each $n\in\mathbb{N}$

$$\ell_n(x):=\frac{\lambda\beta(x)}{\left(|v(x)|+\frac{1}{n}\right)^\eta}\in L^\infty(\Omega).$$

By Theorem 2.3, we know that V is maximal monotone and by definition, see (2.2), $\langle V(u),u\rangle\geq\rho_{\mathcal{H}_{\log}}(\nabla u)$. Thus, V is coercive due to Proposition 2.2 (iii). Since a coercive and maximal monotone operator is surjective, there exists $u_n\in W^{1,\mathcal{H}_{\log}}_0(\Omega)$, $u_n\neq 0$, such that

$$V(u_n)=\ell_n\quad\text{in } W^{1,\mathcal{H}_{\log}}_0(\Omega)^*.$$

This solution is unique since V is strictly monotone because of Theorem 2.3. Choosing $v = -u_n^- \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ as test function in (3.2), we infer

$$\rho_{\mathcal{H}_{\log}}(\nabla u_n^-) \leq \int_\Omega \left(|\nabla u_n^-|^p + \mu(x) \left[\log(e + |\nabla u_n^-|) + \frac{|\nabla u_n^-|}{q(e + |\nabla u_n^-|)} \right] |\nabla u_n^-|^q \right) \mathrm{d}x \leq 0.$$

Then, by Proposition 2.2, we obtain that $u_n \geq 0$.

Let us introduce the map $\xi_n \colon L^p(\Omega) \to L^p(\Omega)$ defined by

$$\xi_n(v) = u_n \quad \text{for all } n \in \mathbb{N}.$$

With view to Proposition 2.1, we know that $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and so ξ_n is well defined.

Next, we prove that ξ_n is a continuous map for each $n \in \mathbb{N}$. Let $\{v_k\}_{k\in\mathbb{N}} \subset L^p(\Omega)$ be a sequence such that $v_k \to v$ in $L^p(\Omega)$. By Cruz-Uribe–Fiorenza [18, Theorem 2.66], up to subsequence if necessary, we have $v_k(x) \to v(x)$ for a.a. $x \in \Omega$ as $k \to +\infty$. Then, it follows that

$$\ell_k(x) = \frac{\lambda\beta(x)}{\left(|v_k(x)| + \frac{1}{n}\right)^\eta} \to \frac{\lambda\beta(x)}{\left(|v(x)| + \frac{1}{n}\right)^\eta} = \ell(x) \quad \text{as } k \to +\infty,$$

for a.a. $x \in \Omega$ and for all $n \in \mathbb{N}$. On the other hand, for a.a. $x \in \Omega$ and for all $k \in \mathbb{N}$, one has

$$0 \leq \frac{\lambda\beta(x)}{\left(|v_k(x)| + \frac{1}{n}\right)^\eta} \leq n\lambda\beta(x).$$

Taking Lebesgue's dominated convergence theorem into account yields

$$\|\ell_k - \ell\|_p \to 0 \quad \text{as } k \to +\infty.$$

Therefore, ξ_n is continuous for all $n \in \mathbb{N}$.

For fixed $n \in \mathbb{N}$, one has that

$$\xi_n\left(L^p(\Omega)\right) \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$$

is a bounded set. Indeed, choose the test function $v = u_n$ in the weak formulation and use again Proposition 2.2. Moreover, Proposition 2.1 allows us to consider the compact embedding $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^p(\Omega)$. Thus,

$$\overline{\xi_n\left(L^p(\Omega)\right)}^{\|\cdot\|_p}$$

is compact in $L^p(\Omega)$. According to Schauder-Tychonoff's fixed point Theorem (see Gasiński–Papageorgiou [26, Theorem 7.2.14]), there exists $\overline{u}_n \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\xi_n(\overline{u}_n) = \overline{u}_n \quad \text{for all } n \in \mathbb{N}.$$

Notice that, by Lemma 2.6, we ensure that $0 \prec \overline{u}_n$.

Now, we prove that the sequence $\{\overline{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is nondecreasing. For this purpose, let $n, m \in \mathbb{N}$ with $n < m$. Clearly, one has

$$-\mathcal{G}(\overline{u}_m) = \frac{\lambda\beta(x)}{\left(\overline{u}_m + \frac{1}{m}\right)^\eta} \geq \frac{\lambda\beta(x)}{\left(\overline{u}_m + \frac{1}{n}\right)^\eta} \quad \text{in } \Omega. \tag{3.3}$$

Consider the Carathéodory function

$$\zeta_n(x,s) = \begin{cases} \dfrac{\beta(x)}{\left(s^+ + \frac{1}{n}\right)^\eta} & \text{if } s \leq \overline{u}_m, \\[2ex] \dfrac{\beta(x)}{\left(\overline{u}_m + \frac{1}{n}\right)^\eta} & \text{if } \overline{u}_m < s, \end{cases} \tag{3.4}$$

and the associated elliptic problem is

$$\begin{cases} -\mathcal{G}(u) = \lambda\zeta_n(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

As in the first part of the proof, we can show that (3.5) possesses a unique solution $\widetilde{u}_n \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ due to the strict monotonicity of V and the fact that $\zeta_n(x,\cdot)$ is nonincreasing. Further, choosing $v = (\widetilde{u}_n - \overline{u}_m)^+ \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ as test function in (3.5), and combining (3.3) as well as (3.4), we infer

$$\begin{aligned} \langle V(\widetilde{u}_n), (\widetilde{u}_n - \overline{u}_m)^+\rangle &= \int_\Omega \frac{\lambda\beta(x)}{\left(\overline{u}_m + \frac{1}{n}\right)^\eta}(\widetilde{u}_n - \overline{u}_m)^+ \,\mathrm{d}x \\ &\le \int_\Omega \frac{\lambda\beta(x)}{\left(\overline{u}_m + \frac{1}{m}\right)^\eta}(\widetilde{u}_n - \overline{u}_m)^+ \,\mathrm{d}x \\ &= \langle V(\overline{u}_m), (\widetilde{u}_n - \overline{u}_m)^+\rangle. \end{aligned}$$

Then, by the monotonicity of V (see Theorem 2.3) we get $\widetilde{u}_n \le \overline{u}_m$. Thus, $\widetilde{u}_n \in [0, \overline{u}_m]$ with $\widetilde{u}_n \neq 0$. By the uniqueness of the solution and (3.4), we obtain $\widetilde{u}_n = \overline{u}_n$, that is, $\overline{u}_n \le \overline{u}_m$. This shows that $\{\overline{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is nondecreasing.

For all $n \in \mathbb{N}$, by Hewitt–Stromberg [35, Theorem 13.17, p.196], we deduce that

$$\langle V(\overline{u}_n), \overline{u}_n\rangle = \int_\Omega \frac{\lambda\beta(x)\overline{u}_n}{\left(\overline{u}_n + \frac{1}{n}\right)^\eta}\,\mathrm{d}x \le \lambda\|\beta\|_\infty \int_\Omega \overline{u}_n^{1-\eta}\,\mathrm{d}x \le \lambda c_5\|\overline{u}_n\| \quad \text{for all } n \in \mathbb{N}$$

for some $c_5 > 0$. Then, by Proposition 2.2 and the above computation, it follows that

$$\rho_{\mathcal{H}_{\log}}(\nabla\overline{u}_n) \le \lambda c_6\|\overline{u}_n\|$$

for some $c_6 > 0$. Without loss of any generality, we may assume that $\|\overline{u}_n\| \ge 1$ for all $n \in \mathbb{N}$, otherwise $\{\overline{u}_n\}_{n\in\mathbb{N}}$ is clearly bounded in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. From Proposition 2.2, we infer

$$\|\overline{u}_n\|^p \le \lambda c_7\|\overline{u}_n\| \quad \text{for all } n \in \mathbb{N}$$

for some $c_7 > 0$. Thus, $\{\overline{u}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. So we may assume that

$$\overline{u}_n \rightharpoonup \overline{u}_\lambda \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad \overline{u}_n \to \overline{u}_\lambda \quad \text{in } L^q(\Omega).$$

This gives

$$\langle V(\overline{u}_n), \overline{u}_n - \overline{u}_\lambda\rangle = \int_\Omega \frac{\lambda\beta(x)}{\left(\overline{u}_n + \frac{1}{n}\right)^\eta}(\overline{u}_n - \overline{u}_\lambda)\,\mathrm{d}x \quad \text{for all } n \in \mathbb{N}.$$

Moreover, since $\{\overline{u}_n\}_{n\in\mathbb{N}}$ is nondecreasing , we get

$$\frac{\beta(x)|\overline{u}_n - \overline{u}_\lambda|}{\left(\overline{u}_n + \frac{1}{n}\right)^\eta} \le \frac{\beta(x)|\overline{u}_n - \overline{u}_\lambda|}{\overline{u}_1^\eta}.$$

Let $d(x) = d(x, \partial\Omega)$ for all $x \in \overline{\Omega}$ be the distance function. Lemma 14.16 of Gilbarg–Trudinger [29] ensures that $d \in C_0^1(\overline{\Omega}) \setminus \{0\}$ with $d(x) \ge 0$ for all $x \in \overline{\Omega}$. By the above computation, we know that $0 \prec \overline{u}_1$. Hence, it is possible to find $c_8 > 0$ such that

$$c_8 d(x) \le u_1,$$

see, e.g, Papageorgiou–Peng [44, Proposition 5.2]. Thus, we infer

$$\frac{\beta(x)|\overline{u}_n - \overline{u}_\lambda|}{\overline{u}_1^\eta} = \overline{u}_1^{1-\eta}\frac{\beta(x)|\overline{u}_n - \overline{u}_\lambda|}{\overline{u}_1} \le c_9\frac{|\overline{u}_n - \overline{u}_\lambda|}{d}$$

for some $c_9 > 0$. By Lemma 2.5, one has

$$\frac{|\overline{u}_n - \overline{u}_\lambda|}{d} \in L^p(\Omega)$$

and

$$\left\|\frac{|\overline{u}_n-\overline{u}_\lambda|}{d}\right\|_p \leq c_{10}\|\nabla(\overline{u}_n-\overline{u}_\lambda)\|_p \leq c_{11} \quad \text{for all } n\in\mathbb{N},$$

for some $c_{10}, c_{11} > 0$. This implies, by setting

$$\Upsilon_n = \frac{\beta(x)|\overline{u}_n-\overline{u}_\lambda|}{\left(\overline{u}_n+\frac{1}{n}\right)^\eta} \in L^p(\Omega)$$

that $\{\Upsilon_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega)$. Thus, $\Upsilon_n \rightharpoonup \Upsilon$ in $L^p(\Omega)$ and $\Upsilon_n \rightharpoonup \Upsilon$ in $L^1(\Omega)$. Moreover, $\{\Upsilon_n\}_{n\in\mathbb{N}}$ is uniformly integrable by the Dunford-Pettis Theorem, (see, e.g, Papageorgiou–Winkert [50, Theorem 4.1.18]). Therefore, one has

$$\frac{\beta(x)|(\overline{u}_n-\overline{u}_\lambda)(x)|}{\overline{u}_1(x)^\eta} \to 0 \quad \text{for a.a. } x\in\Omega.$$

Finally, by Vitali's Theorem (see, e.g, Papageorgiou–Winkert [50, Theorem 2.3.44]), we infer

$$\int_\Omega \frac{\beta(x)|\overline{u}_n-\overline{u}_\lambda|}{\overline{u}_1^\eta}\,\mathrm{d}x \to 0 \quad \text{as } n\to+\infty.$$

That is,

$$\lim_{n\to+\infty}\langle V(\overline{u}_n), \overline{u}_n-\overline{u}_\lambda\rangle = 0$$

and by (S_+)-property of V (see Theorem 2.3), we deduce that

$$\overline{u}_n \to \overline{u}_\lambda \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega)$$

with $\overline{u}_1 \leq \overline{u}_\lambda$. We have, for all $h\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$, that

$$\langle V(\overline{u}_n), h\rangle = \int_\Omega \frac{\lambda\beta(x)}{\left(\overline{u}_n+\frac{1}{n}\right)^\eta} h\,\mathrm{d}x \quad \text{for all } n\in\mathbb{N}$$

and

$$\frac{\beta(x)|h|}{\left(\overline{u}_n+\frac{1}{n}\right)^\eta} \leq \frac{\|\beta\|_\infty|h|}{\overline{u}_1^\eta} \leq \frac{\|\beta\|_\infty\overline{u}_n^{1-\eta}|h|}{\overline{u}_1} \leq c_{12}\frac{|h|}{d} \quad \text{for all } n\in\mathbb{N}$$

for some $c_{12} > 0$. By Lemma 2.5, we have that $\frac{|h|}{d} \in L^p(\Omega)$. Then, by Lebesgue's dominate convergence theorem it follows follows that

$$\int_\Omega \frac{\lambda\beta(x)}{\left(\overline{u}_n+\frac{1}{n}\right)^\eta} h\,\mathrm{d}x \to \int_\Omega \frac{\lambda\beta(x)}{\overline{u}_\lambda^\eta} h\,\mathrm{d}x \quad \text{as } n\to+\infty.$$

Then, passing to the limit as $n\to+\infty$, we infer

$$\langle V(\overline{u}_\lambda), h\rangle = \int_\Omega \frac{\lambda\beta(x)}{\overline{u}_\lambda^\eta} h\,\mathrm{d}x \quad \text{for all } h\in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

This shows that $\overline{u}_\lambda$ is a positive solution of (3.1) satisfying $0 \prec \overline{u}_\lambda$.

Next, we prove that $\beta(x)\overline{u}_\lambda^{-\eta} \in L^\infty(\Omega)$. Let $1 \leq s < +\infty$. Applying again Lemma 2.5, one has

$$\int_\Omega \left(\frac{\beta(x)}{\overline{u}_\lambda^\eta}\right)^s \mathrm{d}x = \int_\Omega \left(\overline{u}_\lambda^{1-\eta}\frac{\beta(x)}{\overline{u}_\lambda}\right)^s \mathrm{d}x \leq c_{13}\int_\Omega \left(\frac{\beta(x)}{d}\right)^s \mathrm{d}x \leq c_{14}\|\nabla\beta\|_s^s,$$

for some c_{13}, c_{14}. Using Gasiński–Papageorgiou [27, Problem 1.9] and passing to the limit as $s\to+\infty$ leads to

$$\left\|\frac{\beta}{\overline{u}_\lambda^\eta}\right\|_\infty \leq c_{15}\|\nabla\beta\|_\infty,$$

for some $c_{15} > 0$, i.e., $\beta(x)\overline{u}_\lambda^{-\eta} \in L^\infty(\Omega)$. Thus, by (2.5), we infer

$$\|\overline{u}_\lambda\|_\infty \leq \widehat{c}_0\left(\lambda\|\beta\overline{u}_\lambda^{-\eta}\|_\infty\right)^{\frac{1}{p-1}}$$

that is,

$$\|\overline{u}_\lambda\|_\infty^{1+\frac{\eta}{p-1}} \leq \lambda^{\frac{1}{p-1}} \widehat{c}_0 \|\beta\|_\infty^{\frac{1}{p-1}}.$$

Therefore,

$$\overline{u}_\lambda \to 0 \quad \text{as } \lambda \to 0^+ \text{ in } L^\infty(\Omega).$$

Moreover, starting from the weak formulation,

$$\langle V(\overline{u}_\lambda), h\rangle = \lambda \int_\Omega \beta(x) \overline{u}_\lambda^{-\eta} h \,\mathrm{d}x \quad \text{for all } h \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$$

and testing with $h = \overline{u}_\lambda \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$, by Hewitt–Stromberg [35, Theorem 13.17, p. 196], we deduce

$$\rho_{\mathcal{H}_{\log}}(\nabla \overline{u}_\lambda) \leq \lambda \int_\Omega \beta(x) \overline{u}_\lambda^{1-\eta} \,\mathrm{d}x \leq \lambda c_{16} \|\overline{u}_\lambda\|^{1-\eta}$$

for some $c_{16} > 0$ and so, $\overline{u}_\lambda \to 0$ as $\lambda \to 0^+$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. □

The previous proposition allows us to build a suitable sub-solution. In particular, we can find $\lambda \in (0,1)$ such that

$$\|\overline{u}_\lambda\|_\infty \leq \delta \quad \text{and} \quad \left\| \frac{\beta}{\overline{u}_\lambda^\eta} \right\|_\infty \geq 1 \tag{3.6}$$

with $\delta > 0$ as in (H$_f$)(iii). We choose $\lambda \in (0,1)$ such that (3.6) holds, and set $\overline{u} = \overline{u}_\lambda$. Clearly, we have

$$-\mathcal{G}(u) = \lambda \beta(x) \overline{u}^{-\eta} \leq \beta(x) \overline{u}^{-\eta} \leq \beta(x) \overline{u}^{-\eta} + f(x, \overline{u})$$

where the last inequality holds in view of (H$_f$)(iii).

Next, we define an appropriate super-solution. For this purpose, let us consider the auxiliary problem

$$\begin{cases} -\mathcal{G}(u) = \mu \left\| \frac{\beta}{\overline{u}^\eta} \right\|_\infty & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

with $\mu > 1$.

We have the following uniqueness result for problem (3.7).

Proposition 3.2. *Let hypotheses (H$_0$), (H$_\beta$), and (H$_\eta$) be satisfied. Then problem (3.7) admits a unique solution $\widehat{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$, with $0 \prec \widehat{u}$ such that $\overline{u} \leq \widehat{u}$.*

Proof. Clearly, problem (3.7) can be written as

$$V(u) = \mu \left\| \frac{\beta}{\overline{u}^\eta} \right\|_\infty \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega)^*.$$

As before, since V is maximal monotone and coercive, it is also surjective. So there exists a unique solution $\widehat{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ of (3.7) since V is strictly monotone as well by Theorem 2.3. From Rădulescu–Stapenhorst–Winkert [54, Theorem 3.1], we have that $\widehat{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$. Then, by Lemma 2.6, we infer that $0 \prec \widehat{u}$. Finally, notice that

$$-V(\overline{u}) = \lambda \beta(x) \overline{u}^{-\eta} \leq \mu \beta(x) \overline{u}^{-\eta} \leq \mu \left\| \frac{\beta}{\overline{u}^\eta} \right\|_\infty = -V(\widehat{u}) \quad \text{in } \Omega.$$

Choosing $v = (\overline{u} - \widehat{u})^+ \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ as test function and using the strict monotonicity of V, we have

$$\langle V(\overline{u}) - V(\widehat{u}), (\overline{u} - \widehat{u})^+ \rangle \leq 0,$$

thus $\overline{u} \leq \widehat{u}$. □

Note that, by (2.5), one has

$$\|\widehat{u}\|_\infty \leq \widehat{c}_0 \left(\mu \left\|\frac{\beta}{\overline{u}^\eta}\right\|_\infty\right)^{\frac{1}{p-1}}.$$

Thus, we get

$$\begin{aligned}
-\mathcal{G}(\widehat{u}) - \beta(x)\widehat{u}^{-\eta} - f(x,\widehat{u}) &\geq (\mu-1)\left\|\frac{\beta}{\overline{u}^\eta}\right\|_\infty - c_1\|\widehat{u}\|_\infty^{\tau-1}\\
&\geq (\mu-1)\left\|\frac{\beta}{\overline{u}^\eta}\right\|_\infty - \widehat{c}_0 c_1\left(\mu\left\|\frac{\beta}{\overline{u}^\eta}\right\|_\infty\right)^{\frac{\tau-1}{p-1}}.
\end{aligned}$$

So, for $\mu>1$ large enough, if $\tau<p$ we infer that $\widehat{u}$ is a super-solution for (3.1), while if $\tau=p$ the assumption (H_f)(i) guarantees that $\widehat{u}$ is a super-solution provided $\mu>\frac{1}{1-\widehat{c}_0c_1}$.

Notice that Proposition 3.2 allows us to consider the Carathéodory function $\ell\colon L^{\mathcal{H}_{\log}}(\Omega)\to L^{\mathcal{H}_{\log}}(\Omega)$ defined by

$$\ell(u)(x)=\begin{cases}\overline{u}(x) & \text{if } u(x)<\overline{u}(x),\\ u(x) & \text{if } \overline{u}(x)\leq u(x)\leq\widehat{u}(x),\\ \widehat{u}(x) & \text{if } \widehat{u}(x)<u(x).\end{cases}\tag{3.8}$$

Clearly, ℓ is continuous and its gradient is given by

$$\nabla\ell(u)(x)=\begin{cases}\nabla\overline{u}(x) & \text{if } u(x)<\overline{u}(x),\\ \nabla u(x) & \text{if } \overline{u}(x)\leq u(x)\leq\widehat{u}(x),\\ \nabla\widehat{u}(x) & \text{if } \widehat{u}(x)<u(x),\end{cases}$$

see Papageorgiou–Winkert [50, p. 382]. Moreover, arguing as in Papageorgiou–Peng [44, Proposition 5.1], we have that $\ell\colon W_0^{1,\mathcal{H}_{\log}}(\Omega)\to W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is continuous.

Next, we introduce Nemytskij operator $N\colon W_0^{1,\mathcal{H}_{\log}}(\Omega)\to L^{\tau'}(\Omega)$ associated to the reaction term defined by

$$N(u)(x)=\beta(x)\ell(u)(x)^{-\eta}+f(x,\ell(u)(x))$$

where τ' is the Hölder conjugate exponent to τ, i.e., $\frac{1}{\tau}+\frac{1}{\tau'}=1$. Furthermore, we define the bounded operator $K\colon W_0^{1,\mathcal{H}_{\log}}(\Omega)\to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ by $K=V-N$.

First, we prove that K is pseudomonotone in the sense of Definition 2.4.

Proposition 3.3. *Let hypotheses (H_0), (H_β), (H_η), and (H_f)(i) be satisfied. Then K is a pseudomonotone and coercive operator.*

Proof. First, we show that K is pseudomonotone. Let $\{u_n\}_{n\in\mathbb{N}}\subset W_0^{1,\mathcal{H}_{\log}}(\Omega)$ be a sequence such that $u_n\rightharpoonup u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and $\limsup_{n\to+\infty}\langle K(u_n),u_n-u\rangle\leq 0$. By the compact embedding $W_0^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow L^\tau(\Omega)$ (see Proposition 2.1), we infer that $u_n\to u$ in $L^\tau(\Omega)$. Thus, combining this with (H_f) and the truncation function (3.8), we deduce that

$$\int_\Omega f(x,\ell(u_n))(u_n-u)\,\mathrm{d}x\to 0\quad\text{as } n\to+\infty.$$

On the other hand, we compute

$$\begin{aligned}
&\left|\int_\Omega \beta(x)\ell(u_n)^{-\eta}(u_n-u)\,\mathrm{d}x\right|\\
&\leq \int_{\{u_n<\overline{u}\}}\beta(x)\overline{u}^{-\eta}(u_n-u)\,\mathrm{d}x+\int_{\{\overline{u}\leq u_n\leq\widehat{u}\}}\beta(x)u_n^{-\eta}(u_n-u)\,\mathrm{d}x\\
&\quad+\int_{\{\widehat{u}<u_n\}}\beta(x)\widehat{u}^{-\eta}(u_n-u)\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\le 3\int_\Omega \beta(x)\overline{u}^{-\eta}(u_n-u)\,\mathrm{d}x\\
&\le 3\left\|\frac{\beta(x)}{\overline{u}^{\eta}}\right\|_\infty \|u_n-u\|_1 \to 0 \quad \text{as } n\to+\infty.
\end{aligned}$$

The above computations lead to

$$\limsup_{n\to+\infty}\langle V(u_n),u_n-u\rangle \le 0$$

and by the (S_+)-property of V (see Theorem 2.3), we obtain that $u_n\to u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. By the continuity of V and N, we infer that

$$K(u_n)\to V(u)-N(u) \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega)^*.$$

Thus

$$u^*=V(u)-N(u)=K(u),$$

and

$$\langle K(u_n),u_n\rangle \to \langle K(u),u\rangle.$$

So, K is pseudomonotone.

Next, we prove that K is coercive. Let $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ be such that $\|u\|\ge 1$. Then, by (H_β), (H_η), (H_f), and recalling Proposition 2.2 as well as the truncation (3.8), we have

$$\begin{aligned}
\langle K(u),u\rangle &= \langle V(u),u\rangle - \int_\Omega N(u)u\,\mathrm{d}x\\
&\ge \rho_{\mathcal{H}_{\log}}(\nabla u) - \int_\Omega \big(\beta(x)\ell(u)^{-\eta}+f(x,\ell(u))\big)\,u\,\mathrm{d}x\\
&\ge \|u\|^p - \int_{\{u<\overline{u}\}} \big(\beta(x)\overline{u}^{-\eta}+f(x,\overline{u})\big)\,u\,\mathrm{d}x\\
&\quad - \int_{\{\overline{u}<u<\widehat{u}\}} \big(\beta(x)u^{-\eta}+f(x,u)\big)\,u\,\mathrm{d}x - \int_{\{u>\widehat{u}\}} \big(\beta(x)\widehat{u}^{-\eta}+f(x,\widehat{u})\big)\,u\,\mathrm{d}x\\
&\ge \|u\|^p - 3\int_\Omega \beta(x)\overline{u}^{-\eta}u\,\mathrm{d}x - c_{17}\|u\|\\
&\ge \|u\|^p - 3\left\|\frac{\beta(x)}{\overline{u}^{\eta}}\right\|_\infty \|u\|_1 - c_{17}\|u\|\\
&\ge \|u\|^p - c_{18}\|u\|
\end{aligned}$$

for some $c_{17},c_{18}>0$. Finally, since $p>1$ (see (H_0)), we have

$$\lim_{\|u\|\to\infty}\frac{\langle K(u),u\rangle}{\|u\|}=+\infty,$$

i.e., K is strongly coercive. $\square$

Proof of Theorem 1.1. By Proposition 3.3, we have that K is a bounded, strongly coercive and pseudomonotone operator. Then, by Papageorgiou–Winkert [50, Theorem 6.1.57.], K is surjective and so, there exists $u_0\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that $K(u_0)=0$, i.e.,

$$V(u_0)=N(u_0) \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega)^*. \tag{3.9}$$

Choosing $v=(\overline{u}-u_0)^+$ and $v=(u_0-\widehat{u})^+$ in (3.9) lead to

$$\langle V(u_0),(\overline{u}-u_0)^+\rangle = \int_\Omega \big(\beta(x)\overline{u}^{-\eta}+f(x,\overline{u})\big)\,(\overline{u}-u_0)^+\,\mathrm{d}x \ge \langle V(\overline{u}),(\overline{u}-u_0)^+\rangle$$

and

$$\langle V(u_0), (u_0 - \widehat{u})^+\rangle = \int_\Omega \left(\beta(x)\widehat{u}^{-\eta} + f(x, \widehat{u})\right)(u_0 - \widehat{u})^+ \,\mathrm{d}x \le \langle V(\widehat{u}), (u_0 - \widehat{u})^+\rangle.$$

By the monotonicity of V (see Theorem 2.3), we infer that $\overline{u} \le u_0 \le \widehat{u}$ with $u_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$. ☐

Now we can prove the uniqueness result to our original problem (P) under the additional condition (H_f) (iv).

Proof of Theorem 1.2. We introduce the functional $j\colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_\zeta(\nabla u^{\frac{1}{\tau}}) + \frac{1}{p}\|\nabla u^{\frac{1}{\tau}}\|_p^p & \text{if } u \ge 0 \text{ and } u^{\frac{1}{\tau}} \in W_0^{1,\mathcal{H}_{\log}}(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where $\zeta(x,t) = \mu(x)t^q \log(e+t)$. By Arora–Crespo-Blanco–Winkert [2, Lemma 3.3] (see also Harjulehto–Hästö [34, eq. (2.1.1), p. 14]) and Díaz–Saá [20, Lemma 1], we infer that j is convex. Let

$$\mathrm{dom}(j) = \left\{u \in L^1(\Omega)\colon j(u) < +\infty\right\}$$

be the effective domain of j. Further, let $v_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ with $0 \prec v_0$ be another solution of (P). For $\varepsilon \in (0,1)$, we set $u_0^\varepsilon = u_0 + \varepsilon$ and $v_0^\varepsilon = v_0 + \varepsilon$. Clearly, $u_0^\varepsilon, v_0^\varepsilon \in \operatorname{int} L^\infty(\Omega)$. By Hu–Papageorgiou [37, Proposition 2.86, p. 90], we infer that

$$\frac{u_0^\varepsilon}{v_0^\varepsilon}, \frac{v_0^\varepsilon}{u_0^\varepsilon} \in L^\infty(\Omega).$$

Put $h = (u_0^\varepsilon)^\tau - (v_0^\varepsilon)^\tau \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ and note that, for $t \in (0,1)$, we get

$$(u_0^\varepsilon)^\tau + th \in \mathrm{dom}(j) \quad \text{and} \quad (v_0^\varepsilon)^\tau + th \in \mathrm{dom}(j).$$

Then, by the convexity of j, Green's identity (see Hu–Papageorgiou [36, Theorem 4.106, p.218]) and Lemma 2 by Díaz–Saá [20], we obtain

$$\begin{aligned} j'(u_0^\varepsilon)(h) &= \frac{1}{\tau}\int_\Omega \frac{-\Delta_\zeta^\mu u_0 - \Delta_p u_0}{(u_0^\varepsilon)^{\tau-1}} h \,\mathrm{d}x,\\ j'(v_0^\varepsilon)(h) &= \frac{1}{\tau}\int_\Omega \frac{-\Delta_\zeta^\mu v_0 - \Delta_p v_0}{(v_0^\varepsilon)^{\tau-1}} h \,\mathrm{d}x. \end{aligned}$$

Furthermore, one has

$$\begin{aligned} 0 &\le \int_\Omega \left(\frac{-\Delta_\zeta^\mu u_0 - \Delta_p u_0}{(u_0^\varepsilon)^{\tau-1}} + \frac{\Delta_\zeta^\mu v_0 + \Delta_p v_0}{(v_0^\varepsilon)^{\tau-1}}\right) h \,\mathrm{d}x\\ &= \int_\Omega \beta(x)\left(\frac{1}{u_0^\eta (u_0^\varepsilon)^{\tau-1}} - \frac{1}{v_0^\eta (v_0^\varepsilon)^{\tau-1}}\right) h \,\mathrm{d}x + \int_\Omega \left(\frac{f(x,u_0)}{(u_0^\varepsilon)^{\tau-1}} - \frac{f(x,v_0)}{(v_0^\varepsilon)^{\tau-1}}\right) h \,\mathrm{d}x, \end{aligned}$$

where

$$\Delta_\zeta^\mu u = \operatorname{div}\left(\mu(x)|\nabla u|^{q-2}\nabla u \log(e + |\nabla u|)\right) \quad \text{for all } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

Passing to the limit as $\varepsilon \to 0^+$ and using Lebesgue's dominate convergence theorem, we deduce that

$$\begin{aligned} 0 \le &\int_\Omega \beta(x)\left(\frac{1}{u_0^{\tau+\eta-1}} - \frac{1}{v_0^{\tau+\eta-1}}\right)(u_0^\tau - v_0^\tau)\,\mathrm{d}x\\ &+ \int_\Omega \left(\frac{f(z,u_0)}{u_0^{\tau-1}} - \frac{f(x,v_0)}{v_0^{\tau-1}}\right)(u_0^\tau - v_0^\tau)\,\mathrm{d}x. \end{aligned}$$

Since $t \mapsto \frac{1}{t^{\tau+\eta-1}}$ is strictly decreasing and due to (H_f) (iv), it follows that $u_0 = v_0$. Thus, u_0 is the unique solution of (P). $\square$

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References

[1] R. Arora, Á. Crespo-Blanco, P. Winkert, *Logarithmic double phase problems with generalized critical growth*, NoDEA Nonlinear Differential Equations Appl. **32** (2025), no. 5, Paper No. 98, 74 pp.

[2] R. Arora, Á. Crespo-Blanco, P. Winkert, *On logarithmic double phase problems*, J. Differential Equations **433** (2025), Paper No. 113247, 60 pp.

[3] A. Arora, G. Dwivedi, *Existence results for singular double phase problem with variable exponents*, Mediterr. J. Math. **20** (2023), no. 3, Paper No. 155, 21 pp.

[4] R. Arora, A. Fiscella, T. Mukherjee, P. Winkert, *On double phase Kirchhoff problems with singular nonlinearity*, Adv. Nonlinear Anal. **12** (2023), no. 1, Paper No. 20220312, 24 pp.

[5] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves*, Nonlinearity **32** (2019), no. 7, 2481–2495.

[6] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, *Nonvariational and singular double phase problems for the Baouendi-Grushin operator*, J. Differential Equations **303** (2021), 645–666.

[7] P. Baroni, M. Colombo, G. Mingione, *Non-autonomous functionals, borderline cases and related function classes*, St. Petersburg Math. J. **27** (2016), 347–379.

[8] Y. Bai, L. Gasiński, N.S. Papageorgiou, *Singular double phase equations with a sign changing reaction*, Commun. Nonlinear Sci. Numer. Simul. **142** (2025), Paper No. 108566, 12 pp.

[9] Y. Bai, N.S. Papageorgiou, S. Zeng, *Parametric singular double phase Dirichlet problems*, Adv. Nonlinear Anal. **12** (2023), no. 1, Paper No. 20230122, 20 pp.

[10] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal. **154** (2000), no. 4, 297–324.

[11] S. Biagi, F. Esposito, E. Vecchi, *Symmetry and monotonicity of singular solutions of double phase problems*, J. Differential Equations **280** (2021), 435–463.

[12] S.-S. Byun, J. Oh, *Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains*, J. Differential Equations **263** (2017), no. 2, 1643–1693.

[13] A. Callegari, A. Nachman, *Some singular, nonlinear differential equations arising in boundary layer theory*, J. Math. Anal. Appl. **64** (1978), no. 1, 96–105.

[14] P. Candito, G. Failla, R. Livrea, *Positive solutions for a p-Laplacian equation with sub-critical singular parametric reaction term*, Z. Anal. Anwend. **44** (2025), no. 1-2, 145–164.

[15] T. Carleman, "Problèmes mathématiques dans la théorie cinétique des gaz", Almqvist & Wiksells Boktryckeri AB, Uppsala, 1957.

[16] J. Cen, Y. Lu, C. Vetro, S. Zeng, *Existence results of variable exponent double-phase multivalued elliptic inequalities with logarithmic perturbation and convections*, Adv. Nonlinear Anal. **14** (2025), no. 1, Paper No. 20240066, 21 pp.

[17] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, J. Differential Equations **323** (2022), 182–228.

[18] D.V. Cruz-Uribe, A. Fiorenza, "Variable Lebesgue Spaces", Birkhäuser/Springer, Heidelberg, 2013.

[19] C. De Filippis, G. Mingione, *Regularity for double phase problems at nearly linear growth*, Arch. Ration. Mech. Anal. **247** (2023), no. 5, Paper No. 85, 50 pp.

[20] S.I. Díaz, J.E. Saá, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 12, 521–524.

[21] G. Failla, L. Gasiński, N.S. Papageorgiou, M. Skupień, *Existence and uniqueness of solutions for singular double phase equations*, Appl. Anal. **105** (2026), no. 1, 79–94.

[22] C. Farkas, P. Winkert, *An existence result for singular Finsler double phase problems*, J. Differential Equations **286** (2021), 455–473.

[23] M.M. Frisch, P. Winkert, *Boundedness, existence and uniqueness results for coupled gradient dependent elliptic systems with nonlinear boundary condition*, Adv. Nonlinear Anal. **13** (2024), no. 1, Paper No. 20240009, 22 pp.

[24] M. Fuchs, G. Seregin, "Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids", Springer-Verlag, Berlin, 2000.

[25] P. Garain, T. Mukherjee, *On an anisotropic double phase problem with singular and sign changing nonlinearity*, Nonlinear Anal. Real World Appl. **70** (2023), Paper No. 103790, 28 pp.

[26] L. Gasiński, N.S. Papageorgiou, "Nonlinear Analysis", Chapman & Hall/CRC, Boca Raton, FL, 2006.

[27] L. Gasiński, N.S. Papageorgiou, "Exercises in Analysis. Part 2. Nonlinear Analysis", Springer, Cham, 2016.

[28] L. Gasiński, N.S. Papageorgiou, *Singular equations with variable exponents and concave-convex nonlinearities*, Discrete Contin. Dyn. Syst. Ser. S **16** (2023), no. 6, 1414–1434.

[29] D. Gilbarg, N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order", Springer-Verlag, Berlin, 2001.

[30] U. Guarnotta, P. Winkert, *Degenerate singular Kirchhoff problems in Musielak-Orlicz spaces*, Nonlinear Anal. **264** (2026), Paper No. 113986, 17 pp.

[31] M. Guedda, L. Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879–902.

[32] P. Hästö, J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 4, 1285–1334.

[33] P. Harjulehto, P. Hästö, *Double phase image restoration*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 123832, 12 pp.

[34] P. Harjulehto, P. Hästö, "Orlicz Spaces and Generalized Orlicz Spaces", Springer, Cham, 2019.

[35] E. Hewitt, K. Stromberg, "Real and Abstract Analysis", Springer-Verlag, New York-Heidelberg, 1975.

[36] S. Hu, N.S. Papageorgiou, "Research Topics in Analysis, Vol. I. Grounding Theory", Birkhäuser/Springer, Cham, 2022.

[37] S. Hu, N.S. Papageorgiou, "Research Topics in Analysis, Vol. II. Applications", Birkhäuser/Springer, Cham, 2024.

[38] H.B. Keller, D.S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.

[39] W. Liu, G. Dai, N.S. Papageorgiou, P. Winkert, *Existence of solutions for singular double phase problems via the Nehari manifold method*, Anal. Math. Phys. **12** (2022), no. 3, Paper No. 75, 25 pp.

[40] Z. Liu, N.S. Papageorgiou, *Singular double phase equations*, Acta Math. Sci. Ser. B (Engl. Ed.) **43** (2023), no. 3, 1031–1044.

[41] Y. Lu, C. Vetro, S. Zeng, *A class of double phase variable exponent energy functionals with different power growth and logarithmic perturbation*, Discrete Contin. Dyn. Syst. Ser. S **22** (2026), 85–121.

[42] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.

[43] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284.

[44] N.S. Papageorgiou, Z. Peng, *Singular double phase problems with convection*, Nonlinear Anal. Real World Appl. **81** (2025), Paper No. 104213, 10 pp.

[45] N.S. Papageorgiou, V.D. Rădulescu, S. Yuan, *Nonautonomous double-phase equations with strong singularity and concave perturbation*, Bull. Lond. Math. Soc. **56** (2024), no. 4, 1245–1262.

[46] N.S. Papageorgiou, V.D. Rădulescu, Y. Zhang, *Anisotropic singular double phase Dirichlet problems*, Discrete Contin. Dyn. Syst. Ser. S **14** (2021), no. 12, 4465–4502.

[47] N.S. Papageorgiou, V.D. Rădulescu, Y. Zhang, *Strongly singular double phase problems*, Mediterr. J. Math. **19** (2022), no. 2, Paper No. 82, 21 pp.

[48] N.S. Papageorgiou, D.D. Repovš, C. Vetro, *Positive solutions for singular double phase problems*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 123896, 13 pp.

[49] N.S. Papageorgiou, C. Vetro, F. Vetro, *Multiple solutions for parametric double phase Dirichlet problems*, Commun. Contemp. Math. **23** (2021), no. 4, Paper No. 2050006, 18 pp.

[50] N.S. Papageorgiou, P. Winkert, "Applied Nonlinear Functional Analysis", Second Revised Edition, De Gruyter, Berlin, 2024.

[51] K. Perera, M. Squassina, *Existence results for double-phase problems via Morse theory*, Commun. Contemp. Math. **20** (2018), no. 2, 1750023, 14 pp.

[52] M.T.O. Pimenta, P. Winkert, *Strongly singular problems with unbalanced growth*, Ann. Mat. Pura Appl. (4) **204** (2025), no. 5, 2129–2145.

[53] P. Pucci, J. Serrin, "The Maximum Principle", Birkhäuser Verlag, Basel, 2007.

[54] V.D. Rădulescu, M.F. Stapenhorst, P. Winkert, *Multiplicity results for logarithmic double phase problems via Morse theory*, Bull. Lond. Math. Soc. **57** (2025), no. 12, 4178–4201.

[55] G.A. Seregin, J. Frehse, *Regularity of solutions to variational problems of the deformation theory of plasticity with logarithmic hardening*, in: "Proceedings of the St. Petersburg Mathematical Society, Vol. V", Amer. Math. Soc., Providence, RI **193**, 1999, 127–152.

[56] I. Sim, B. Son, *On a class of singular double phase problems with nonnegative weights whose sum can be zero*, Nonlinear Anal. **237** (2023), Paper No. 113384, 22 pp.

[57] A.M. Turing, *The chemical basis of morphogenesis*, Philos. Trans. Roy. Soc. London Ser. B **237** (1952), no. 641, 37–72.

[58] F. Vetro, P. Winkert, *Logarithmic double phase problems with convection: existence and uniqueness results*, Commun. Pure Appl. Anal. **23** (2024), no. 9, 1325–1339.

[59] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.

[60] V.V. Zhikov, *On Lavrentiev's phenomenon*, Russian J. Math. Phys. **3** (1995), no. 2, 249–269.
[61] V.V. Zhikov, *On variational problems and nonlinear elliptic equations with nonstandard growth conditions*, J. Math. Sci. (N.Y.) **173** (2011), no. 5, 463–570.

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