

ELLIPTIC PROBLEMS WITH UNBALANCED GROWTH, UNBOUNDED COEFFICIENT AND INTRINSIC OPERATOR

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ABSTRACT. We investigate a variable exponent double phase problem involving an unbounded coefficient in the principal part and a multi-valued convection term defined via an intrinsic operator. Under suitable growth and coercivity conditions, truncation techniques and a surjectivity result for multi-valued pseudomonotone operators yield bounded weak solutions with bounded boundary traces. Furthermore, we examine special cases of the double phase problem by selecting distinct intrinsic operators, including convolution-type operators. The analysis is motivated by [Math. Z. 308 (2024), no. 2, Paper No. 38, 13 pp.] and incorporates mixed boundary conditions and multivalued source terms.

1. INTRODUCTION

Our starting point is the variable exponent double phase inclusion below, whose principal coefficient may be unbounded and whose convection term involves an intrinsic operator

$$\begin{aligned} -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u)] &\in f(x, u, Tu, \nabla u, \nabla(Tu)) && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, Tu) && \text{on } \Gamma_2, \end{aligned} \quad (\text{P1})$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded Lipschitz domain, and

$$\mathcal{K}(\nabla u) = |\nabla u|^{\vartheta(x)-2} \nabla u + \mu(x) |\nabla u|^{\kappa(x)-2} \nabla u, \quad \mathcal{K}(0) = 0.$$

The sets Γ_1 and Γ_2 form a partition of $\partial\Omega$ satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. Here, $\vartheta, \kappa, k \in C(\overline{\Omega})$ fulfill $1 < \vartheta(x) < N$, $\vartheta(x) \leq \kappa(x)$ and $1 < k(x) < \vartheta^*(x) = \frac{N\vartheta(x)}{N-\vartheta(x)}$ for all $x \in \overline{\Omega}$ while $0 \leq \mu(\cdot) \in L^\infty(\Omega)$. Moreover, the intrinsic operator $T: V \rightarrow W^{1,\vartheta}(\Omega)$ (see Section 2 for $W^{1,\Phi}(\Omega)$) is continuous, the coefficient $h: \mathbb{R} \times [0, \infty) \times [0, \infty) \rightarrow [a_0, +\infty)$ with $a_0 > 0$ is continuous and possibly unbounded and $f: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R} \setminus \{\emptyset\}}$ is a multivalued function defined in the domain Ω and $f_{\Gamma_2}: \Gamma_2 \times \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R} \setminus \{\emptyset\}}$ is a multivalued function defined on the boundary Γ_2 . Furthermore,

$$\frac{\partial u}{\partial n} := h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u) \cdot \nu,$$

where ν represents the outward unit normal at Γ_2 .

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The operator governing the variable exponent mixed boundary problem (P1) has a double phase structure. Applications that have motivated research on this class of problems include image processing, transonic flow models, nonlinear versions of Derrick's problem, and the nonlinear theory of composite materials. We refer, for instance, to the works by Bahrouni–Rădulescu–Repovš [4], Benci–D'Avenia–Fortunato–Pisani [8], Charkaoui–Ben-Loghfry–Zeng [12], Cherfils–Il'yasov [13] and Harjulehto–Hästö [24]. In its classical form, this operator reads

$$\operatorname{div} (|\nabla w|^{\vartheta-2} \nabla w + \mu(x) |\nabla w|^{\kappa-2} \nabla w),$$

with the associated growth functional

$$w \mapsto \int_{\Omega} (|\nabla w|^{\vartheta} + \mu(x) |\nabla w|^{\kappa}) \, dx. \quad (1.1)$$

For a weight function $0 \leq \mu(\cdot) \in L^{\infty}(\Omega)$, the integrand of (1.1) given by $\Phi(x, t) = t^{\vartheta} + \mu(x)t^{\kappa}$ exhibits unbalanced growth, that is

$$t^{\vartheta} \leq \Phi(x, t) \leq k_1 (1 + t^{\kappa}) \quad \text{for all } (x, t) \in \Omega \times [0, \infty),$$

with some constant $k_1 > 0$.

Functionals of the type (1.1) were first studied by Zhikov [47] and Marcellini [28, 29] in connection with nonlinear elasticity of composite materials. Such functionals also arise in duality theory and in the study of the Lavrentiev phenomenon; see Zhikov [48]. The qualitative behavior of (1.1) is strongly influenced by the modulating coefficient $\mu(\cdot)$. More precisely, on the set

$$\Omega_{>0} := \{x \in \Omega : \mu(x) > 0\},$$

the functional (1.1) exhibits κ -growth and κ -ellipticity in the gradient, while it reduces to ϑ -growth and ϑ -ellipticity on the complement where μ vanishes. The term “double phase” refers to these two elliptic regimes. The regularity of local minimizers of (1.1) was studied by Baroni–Colombo–Mingione [5, 6] and Colombo–Mingione [14, 15]. Other studies of regularity under unbalanced growth include those of Beck–Mingione [7], Byun–Ok–Song [9], Fuchs–Mingione [20], and Prasad–Tewary [41]. Boundedness of weak solutions for fractional double phase problems with logarithmic perturbation is established by Zeng–Lu–Rădulescu–Winkert [44]. The relevant Musielak–Orlicz Sobolev spaces and properties of double phase operators are treated by Arora–Crespo-Blanco–Winkert [2, 3], Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], Lu–Vetro–Zeng [27], and Vetro–Zeng [42]. For different choices of the right-hand side, existence has been studied by Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], Gasiński–Papageorgiou [22], Gasiński–Winkert [23], Liu–Dai [26], Papageorgiou–Rădulescu–Repovš [37, 38], Papageorgiou–Vetro–Vetro [39], Perera–Squassina [40], Zeng–Bai–Gasiński–Winkert [43] and Zeng–Rădulescu–Winkert [45, 46], see also the references therein.

The second feature of problem (P1) is the occurrence of an unbounded nonlocal coefficient

$$h : \mathbb{R} \times [0, \infty) \times [0, \infty) \rightarrow [a_0, +\infty), \quad a_0 > 0,$$

which is continuous but may be unbounded from above. Importantly, the function h depends not only on the unknown function u , but also on its norms in the spaces $L^{k(\cdot)}(\Omega)$ with $1 < k(x) < \vartheta^*(x)$ for all $x \in \bar{\Omega}$, and in $W^{1,\Phi}(\Omega)$. The study of Dirichlet problems for the p -Laplacian with unbounded coefficients was initiated by

Motreanu [32]. Later, Motreanu–Tornatore [36] investigated a Dirichlet problem driven by a degenerate p -Laplacian with an unbounded weight, where the reaction term involves convection. Further contributions to the theory of Dirichlet problems with unbounded coefficients can be found in Diblík–Galewski–Kossowski–Motreanu [17], Galewski–Motreanu [21], and Motreanu–Tornatore [35]. These works commonly rely on truncation methods as a key tool to address the challenges posed by unbounded weights.

Problem (P1) also involves the intrinsic operator

$$T: V \rightarrow W^{1,\vartheta-}(\Omega).$$

In (P1), the multivalued term f describes convection in Ω , whereas f_{Γ_2} acts on the boundary. Intrinsic operators are of particular relevance due to their wide range of applications. Prominent examples include convolution, truncation, and the inverse of the p -Laplacian, see the paper by Motreanu–Tornatore [35]. Further studies of intrinsic operators in p, q -Laplacian equations include those by Medeiros–Motreanu [30] and Motreanu–Motreanu [33, 34].

We study the variable exponent double phase inclusion (P1) with an unbounded coefficient, a multivalued convection term and an intrinsic operator, together with its constant-exponent counterpart. It is worth noting that problem (P1) covers a wide range of special cases. For example, by choosing $1 < \vartheta(x) \equiv \vartheta$ and $1 < \kappa(x) \equiv \kappa$ for all $x \in \bar{\Omega}$, the corresponding constant exponent inclusion reads

$$\begin{cases} -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_{\Phi}}) \mathcal{L}(\nabla u)] \in f(x, u, Tu, \nabla u, \nabla(Tu)) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} \in f_{\Gamma_2}(x, u, Tu) & \text{on } \Gamma_2, \end{cases}$$

where $\mathcal{L}(\nabla u) = |\nabla u|^{\vartheta-2} \nabla u + \mu(x) |\nabla u|^{\kappa-2} \nabla u$. When $|\Gamma_2| = 0$, (P1) becomes a variable exponent double phase inclusion subject to a Dirichlet condition

$$\begin{aligned} -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_{\Phi}}) \mathcal{K}(\nabla u)] &\in f(x, u, Tu, \nabla u, \nabla(Tu)) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Moreover, if $\mu(x) = 0$ for a.e. $x \in \Omega$, the principal part of problem (P1) is an unbounded coefficient $\vartheta(\cdot)$ -Laplacian, and the inclusion reads

$$\begin{aligned} -\operatorname{div} \left(h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_{\Phi}}) |\nabla u|^{\vartheta(x)-2} \nabla u \right) &\in f(x, u, Tu, \nabla u, \nabla(Tu)) && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, Tu) && \text{on } \Gamma_2. \end{aligned}$$

Several choices are possible for the intrinsic operator T . For further details, see Section 4 and the work by Motreanu–Tornatore [35].

Establishing the existence of weak solutions to (P1) poses two main challenges. First, the unbounded coefficient h , which may be unbounded from above, precludes a direct treatment of (P1) by standard operator theory. Second, the convection term $f(x, u, Tu, \nabla u, \nabla(Tu))$, whose arguments include the gradient of u , generally prevents a direct variational treatment. Our strategy to overcome these obstacles follows the approach by Motreanu–Tornatore [35]. Section 3 introduces an auxiliary problem obtained by truncation. A surjectivity result for pseudomonotone multivalued operators then provides a solution. Uniform bounds for solutions of the truncated problem allow us to recover solutions of the original problem (P1). The

analysis is carried out in two distinct cases, reflecting the different requirements for variable versus constant exponents:

- (i) In the variable exponent case, the coercivity estimates use growth at the minimum exponent ϑ_- .
- (ii) In the constant exponent case, a broader range of subcritical growth is admitted.

In both cases, the composition condition in (H5)(ii) is used for uniform boundedness. Detailed proofs are presented in Subsection 3.1 (variable exponents) and Subsection 3.2 (constant exponents).

Section 2 collects the Musielak-Orlicz space framework for the variable exponent double phase operator and the required definitions for multivalued operators. In Section 3, a surjectivity result yields solutions of an auxiliary truncated problem. Their uniform bounds then give weak solutions of (P1). Finally, in Section 4, we explore applications of problem (P1) by considering various choices of the intrinsic operator T .

2. PRELIMINARIES

The Musielak-Orlicz space framework used below is based primarily on the works of Crespo-Blanco-Gasiński-Harjulehto-Winkert [16] and Lu-Vetro-Zeng [27], with emphasis on variable exponent double phase operators. Related multivalued double phase inclusions are studied by Cen-Costea [11]. We work on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, whose boundary $\partial\Omega$ is Lipschitz.

Definition 2.1.

- (i) A continuous, convex mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ is an N -function when $\varphi(t) = 0$ only for $t = 0$ and the following limits hold:

$$\lim_{t \rightarrow 0^+} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} = +\infty.$$

- (ii) For a mapping $\varphi: \Omega \times [0, \infty) \rightarrow [0, \infty)$, membership in the generalized N -function class, written $\varphi \in N(\Omega)$, requires $\varphi(\cdot, t)$ to be measurable at each $t \geq 0$ and $\varphi(x, \cdot)$ to be an N -function for a.e. $x \in \Omega$.

Definition 2.2. For a function φ , the doubling condition $\varphi \in \Delta_2$ means that a constant $c_\varphi \geq 1$ can be chosen so that

$$\varphi(x, 2t) \leq c_\varphi \varphi(x, t)$$

for a.e. $x \in \Omega$ and for all $t > 0$. Moreover, if $\varphi \in \Delta_2$, then

$$\varphi(x, t + s) \leq c_\varphi \varphi(x, t) + c_\varphi \varphi(x, s)$$

for a.e. $x \in \Omega$ and for all $t, s \geq 0$. For $\varphi, \psi \in N(\Omega)$, the relation $\varphi \prec \psi$ means that φ is weaker than ψ : there are positive c_1, c_2 and a nonnegative $h \in L^1(\Omega)$ for which

$$\varphi(x, t) \leq c_1 \psi(x, c_2 t) + h(x)$$

for a.e. $x \in \Omega$ and for all $t > 0$.

For $\varphi \in N(\Omega)$, its Fenchel conjugate is defined by

$$\varphi^*(x, t) = \sup_{s \geq 0} (st - \varphi(x, s)) \quad \text{for a.e. } x \in \Omega \text{ and for all } t > 0,$$

which satisfies Young's inequality

$$ts \leq \varphi(x, t) + \varphi^*(x, s) \quad \text{for a.e. } x \in \Omega \text{ and for all } s, t > 0.$$

Moreover, for $f \in L^\varphi(\Omega)$ and $g \in L^{\varphi^*}(\Omega)$, we have

$$\int_{\Omega} |f||g| \, dx \leq 2\|f\|_{\varphi}\|g\|_{\varphi^*}.$$

If $\varphi \in \Delta_2$ is convex with derivative φ' , then for some constants $c, \tilde{c} > 1$,

$$c^{-1}\varphi(x, t) \leq \varphi'(x, t)t \leq c\varphi(x, t) \text{ and } \varphi^*(x, \varphi'(x, t)) \leq \tilde{c}\varphi(x, t)$$

for a.e. $x \in \Omega$ and for all $t \geq 0$. To define the Musielak-Orlicz space $L^\varphi(\Omega)$ for $\varphi \in N(\Omega)$, we first introduce its modular

$$\rho_\varphi(u) = \int_{\Omega} \varphi(x, |u|) \, dx,$$

and the set

$$L^\varphi(\Omega) := \{u \in M(\Omega) : \text{there exists } \lambda > 0 \text{ such that } \rho_\varphi(\lambda u) < +\infty\}$$

whose norm is the Luxemburg norm

$$\|u\|_{\varphi, \Omega} := \inf \left\{ \lambda > 0 : \rho_\varphi\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

We suppress the domain in the norm notation, writing $\|u\|_\varphi$ for $\|u\|_{\varphi, \Omega}$. The associated Musielak-Orlicz Sobolev space consists of the functions

$$W^{1, \varphi}(\Omega) := \{u \in L^\varphi(\Omega) : |\nabla u| \in L^\varphi(\Omega)\}$$

with norm

$$\|u\|_{1, \varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi$$

where $\|\nabla u\|_\varphi = \||\nabla u||\|_\varphi$ and $\rho_\varphi(\nabla u) = \rho_\varphi(|\nabla u|)$. When $\varphi \in N(\Omega)$ is locally integrable (i.e. $\varphi(\cdot, t) \in L^1(\Omega)$ for all $t > 0$), completing $C_0^\infty(\Omega)$ in $W^{1, \varphi}(\Omega)$ yields $W_0^{1, \varphi}(\Omega)$. Thus $W_0^{1, \varphi}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1, \varphi}(\Omega)}$. Musielak states the next two results in his monograph [31, Theorems 7.7 and 8.5].

Proposition 2.3.

- (i) For $\varphi \in N(\Omega)$, the norm $\|\cdot\|_\varphi$ makes the Musielak-Orlicz space complete. Equivalently, $(L^\varphi(\Omega), \|\cdot\|_\varphi)$ is a Banach space.
- (ii) Local integrability of $\varphi, \psi \in N(\Omega)$ together with $\varphi \prec \psi$ implies the continuous embedding $L^\psi(\Omega) \hookrightarrow L^\varphi(\Omega)$.

The unit ball properties below are recorded in Musielak [31, Theorem 8.13] and Diening–Harjulehto–Hästö–Růžička [18, Lemma 2.1.14].

Proposition 2.4. *The following assertions are valid:*

- (i) Let $\varphi \in N(\Omega) \cap \Delta_2$. Then

$$L^\varphi(\Omega) = \{u \in M(\Omega) : \rho_\varphi(u) < +\infty\}.$$

- (ii) Let $\varphi \in N(\Omega) \cap \Delta_2$ and $u \in L^\varphi(\Omega)$. Then $\rho_\varphi(u) < 1$ (resp. $= 1; > 1$) if and only if $\|u\|_\varphi < 1$ (resp. $= 1; > 1$).

For completeness and reflexivity of the Sobolev spaces, we use the following sufficient conditions from Musielak [31, Theorem 10.2] and Fan [19, Proposition 1.7 and 1.8].

Theorem 2.5. *Assume that $\varphi \in N(\Omega)$ is locally integrable and satisfies*

$$\inf_{x \in \Omega} \varphi(x, 1) > 0.$$

The spaces $W^{1,\varphi}(\Omega)$ and $W_0^{1,\varphi}(\Omega)$ are then Banach spaces. If φ also satisfies the Δ_2 condition, these spaces are separable. Reflexivity of $L^\varphi(\Omega)$ also entails reflexivity of $W^{1,\varphi}(\Omega)$ and $W_0^{1,\varphi}(\Omega)$.

For the particular choice $\varphi(x, t) = t^{\vartheta(x)}$, the conjugate takes the form

$$\varphi^*(x, t) = \vartheta(x)^{-\vartheta'(x)} (\vartheta(x) - 1) t^{\vartheta'(x)}$$

where $1 < \vartheta(\cdot) \in C(\overline{\Omega})$ and the relation $\frac{1}{\vartheta(x)} + \frac{1}{\vartheta'(x)} = 1$ holds for all $x \in \overline{\Omega}$. This choice of φ yields the variable exponent Lebesgue space $L^{\vartheta(\cdot)}(\Omega)$.

For $\vartheta \in C(\overline{\Omega})$, we write its minimum and maximum as

$$\vartheta_- := \min_{x \in \overline{\Omega}} \vartheta(x) \quad \text{and} \quad \vartheta_+ := \max_{x \in \overline{\Omega}} \vartheta(x).$$

Analogous notation is used for κ . Suppose $1 < \vartheta(x) < N$ for all $x \in \overline{\Omega}$. The critical Sobolev exponents for Ω and $\partial\Omega$, respectively, are

$$\vartheta^*(x) := \frac{N\vartheta(x)}{N - \vartheta(x)} \quad \text{and} \quad \vartheta_*(x) := \frac{(N - 1)\vartheta(x)}{N - \vartheta(x)} \quad \text{for all } x \in \overline{\Omega}.$$

We impose the following hypothesis:

(H0) Let $\vartheta, \kappa \in C(\overline{\Omega})$ such that $1 < \vartheta(x) < N$ and $\vartheta(x) \leq \kappa(x) < \vartheta^*(x)$ for all $x \in \overline{\Omega}$. Let $0 \leq \mu(\cdot) \in L^\infty(\Omega)$. The case $\mu \equiv 0$ is allowed.

The critical exponents corresponding to ϑ_- are denoted by

$$(\vartheta_-)^* := \frac{N\vartheta_-}{N - \vartheta_-} \quad \text{and} \quad (\vartheta_-)_* := \frac{(N - 1)\vartheta_-}{N - \vartheta_-},$$

respectively. The double phase structure is described by the function $\Phi: \Omega \times [0, \infty) \rightarrow [0, \infty)$, namely

$$\Phi(x, t) := t^{\vartheta(x)} + \mu(x)t^{\kappa(x)} \quad \text{for all } (x, t) \in \Omega \times [0, \infty). \quad (2.1)$$

The energy density associated with $\mathcal{K}$ is given by

$$\Psi(x, t) := \frac{t^{\vartheta(x)}}{\vartheta(x)} + \mu(x) \frac{t^{\kappa(x)}}{\kappa(x)}.$$

For $\xi \neq 0$, $\mathcal{K}(\xi) = \partial_t \Psi(x, |\xi|)\xi/|\xi|$, and $\mathcal{K}(0) = 0$. Under (H0), the function Φ in (2.1) is a locally integrable N-function. The associated modular for Φ reads

$$\rho_\Phi(u) = \int_\Omega \Phi(x, |u|) dx.$$

The resulting Musielak-Orlicz space takes the form

$$L^\Phi(\Omega) = \{u \in M(\Omega) : \rho_\Phi(u) < +\infty\},$$

and carries the Luxemburg norm

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 : \rho_\Phi\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Moreover, the Sobolev space $W^{1,\Phi}(\Omega)$ is defined as

$$W^{1,\Phi}(\Omega) := \{u \in L^\Phi(\Omega) : |\nabla u| \in L^\Phi(\Omega)\},$$

while $W_0^{1,\Phi}(\Omega)$ is obtained by completing $C_0^\infty(\Omega)$ in the norm of $W^{1,\Phi}(\Omega)$. On both $W_0^{1,\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ we use the norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \|\nabla u\|_\Phi.$$

Under hypothesis (H0), the Luxemburg norm $\|\cdot\|_\Phi$ makes $L^\Phi(\Omega)$ a Banach space that is both separable and reflexive. These properties also hold for $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$.

An equivalent norm on $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{\hat{\rho}_\Phi} := \inf \left\{ \lambda > 0 : \hat{\rho}_\Phi \left(\frac{u}{\lambda} \right) \leq 1 \right\},$$

with $\hat{\rho}_\Phi$ denoting the modular

$$\hat{\rho}_\Phi(u) = \int_\Omega \left(|\nabla u|^{\vartheta(x)} + \mu(x) |\nabla u|^{\kappa(x)} \right) dx + \int_\Omega |u|^{\vartheta(x)} dx,$$

for $u \in W^{1,\Phi}(\Omega)$, see Amoroso–Crespo-Blanco–Pucci–Winkert [1, Proposition 3.1].

The norm-modular estimates below are given by Amoroso–Crespo-Blanco–Pucci–Winkert [1, Proposition 3.2].

Proposition 2.6. *Let hypothesis (H0) be satisfied and $u \in W^{1,\Phi}(\Omega)$.*

- (i) *If $\|u\|_{\hat{\rho}_\Phi} < 1$, then $\|u\|_{\hat{\rho}_\Phi}^{\kappa_+} \leq \hat{\rho}_\Phi(u) \leq \|u\|_{\hat{\rho}_\Phi}^{\vartheta_-}$.*
- (ii) *If $\|u\|_{\hat{\rho}_\Phi} > 1$, then $\|u\|_{\hat{\rho}_\Phi}^{\vartheta_-} \leq \hat{\rho}_\Phi(u) \leq \|u\|_{\hat{\rho}_\Phi}^{\kappa_+}$.*

We write $X \hookrightarrow Y$ when the embedding of X into Y is continuous, and $X \hookrightarrow\hookrightarrow Y$ when the embedding of X into Y is compact. A function $h: \overline{\Omega} \rightarrow \mathbb{R}$ in $C^{0, \frac{1}{|\log t|}}(\overline{\Omega})$ is log-Hölder continuous if there is a constant $C > 0$ satisfying

$$|h(x) - h(y)| \leq \frac{C}{|\log |x - y||} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } 0 < |x - y| < \frac{1}{2}.$$

For a bounded domain $\Omega \subset \mathbb{R}^N$ and $\gamma > N$, it holds

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0, \frac{1}{|\log t|}}(\overline{\Omega}).$$

The embeddings listed below are established in Proposition 2.16 of Crespo-Blanco–Gasiński–Harjulehto–Winkert [16].

Proposition 2.7. *Let hypothesis (H0) be satisfied. Then the following hold:*

- (i) *$L^\Phi(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ for all $r \in C(\overline{\Omega})$ with $1 \leq r(x) \leq \vartheta(x)$ for all $x \in \Omega$.*
- (ii) *Let $\vartheta \in C(\overline{\Omega}) \cap C^{0, \frac{1}{|\log t|}}(\overline{\Omega})$, then $W^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ and $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ for all $r \in C(\overline{\Omega})$ with $1 \leq r(x) \leq \vartheta^*(x)$ for all $x \in \overline{\Omega}$. Under (H0), if $1 \leq r(x) < \vartheta^*(x)$ for all $x \in \overline{\Omega}$, both embeddings are compact without the additional log-Hölder assumption.*
- (iii) *Let $\vartheta \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, then $W^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ and $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ for all $r \in C(\overline{\Omega})$ with $1 \leq r(x) \leq \vartheta_*(x)$ for all $x \in \overline{\Omega}$.*
- (iv) *$W^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\Omega)$ for all $r \in C(\overline{\Omega})$ with $1 \leq r(x) < \vartheta_*(x)$ for all $x \in \overline{\Omega}$.*

Proposition 2.8. *Let hypothesis (H0) be satisfied. Then the following hold:*

- (i) *$W^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^\Phi(\Omega)$.*

(ii) A constant $C > 0$, independent of u , can be chosen such that

$$\|u\|_{\Phi} \leq C \|\nabla u\|_{\Phi}. \quad (2.2)$$

for all $u \in W_0^{1,\Phi}(\Omega)$.

Remark 2.9. Following the norm-equivalence approach of Cen–Costea [11], we show that $\|\nabla u\|_{\Phi}$ is an equivalent norm on the closed mixed-boundary space

$$V := \{u \in W^{1,\Phi}(\Omega) : \operatorname{Tr} u = 0 \text{ a.e. on } \Gamma_1\}.$$

Take $P(u) := \int_{\Gamma_1} |\operatorname{Tr} u| d\sigma$. The trace estimate implies $P(u) \leq C \|u\|_{1,\Phi}$, and $P(c) = |c| |\Gamma_1|$ for constants c . It remains to prove that, for some $C > 0$,

$$\|u\|_{1,\Phi} \leq C (\|\nabla u\|_{\Phi} + P(u)) \quad \text{for all } u \in W^{1,\Phi}(\Omega).$$

If this estimate failed, homogeneity would provide a sequence $u_n \in W^{1,\Phi}(\Omega)$ with

$$\|u_n\|_{1,\Phi} = 1, \quad \|\nabla u_n\|_{\Phi} + P(u_n) \rightarrow 0.$$

The reflexivity of $W^{1,\Phi}(\Omega)$ and its compact embedding into $L^{\Phi}(\Omega)$ in Proposition 2.8(i) yield, after passage to a subsequence, $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Omega)$ and $u_n \rightarrow u$ in $L^{\Phi}(\Omega)$. Since $\nabla u_n \rightarrow 0$ in L^{Φ} , we have $\nabla u = 0$, so u is constant on the domain Ω . It follows that $u_n \rightarrow u$ in $W^{1,\Phi}(\Omega)$, and continuity of P gives $P(u) = 0$. Thus $u = 0$ because $|\Gamma_1| > 0$, contradicting $\|u_n\|_{1,\Phi} = 1$. Consequently, $\|\nabla u\|_{\Phi} + P(u)$ is equivalent to $\|u\|_{1,\Phi}$ on $W^{1,\Phi}(\Omega)$. On V , $P(u) = 0$, so (2.2) holds and $\|\nabla u\|_{\Phi}$ is an equivalent norm. We retain $\|u\| := \|u\|_{\hat{\rho}_{\Phi}}$ on V , including in the arguments of h . Choose $D \geq 1$ such that $\|u\| \leq D \|\nabla u\|_{\Phi}$ on V and set $a_* := a_0 D^{-\vartheta_-}$. For $\|u\| \geq D$, we have $\|\nabla u\|_{\Phi} \geq \|u\|/D \geq 1$, and the norm-modular inequalities give

$$\int_{\Omega} \Phi(x, |\nabla u|) dx \geq \|\nabla u\|_{\Phi}^{\vartheta_-} \geq D^{-\vartheta_-} \|u\|^{\vartheta_-}.$$

Multiplying by a_0 yields

$$a_0 \int_{\Omega} \Phi(x, |\nabla u|) dx \geq a_* \|u\|^{\vartheta_-}.$$

Write C_r, C_{r,Γ_2} for the embedding constants of V (with norm $\|\cdot\|$) into $L^r(\Omega), L^r(\Gamma_2)$ whenever the relevant embedding holds, and choose G_{ϑ_-} so that $\|\nabla u\|_{\vartheta_-} \leq G_{\vartheta_-} \|u\|$. The output Tu need not have zero trace on Γ_1 .

Next, we recall standard definitions for multivalued operators.

Definition 2.10. Consider a real reflexive Banach space X , its dual X^* , and the duality pairing $\langle \cdot, \cdot \rangle$. For a multivalued operator $B: X \rightarrow 2^{X^*}$, we use the following terminology:

- (i) **pseudomonotone** if and only if
 - (a) For each $v \in X$, the value $B(v)$ is a nonempty, closed, convex and bounded set.
 - (b) On every finite dimensional subspace of X , the restriction of B is upper semicontinuous with respect to the weak topology of X^* .
 - (c) For any sequence $\{v_n\}_{n \in \mathbb{N}} \subset X$ with $v_n \rightharpoonup v$ in X and $v_n^* \in B(v_n)$ with $\limsup \langle v_n^*, v_n - v \rangle \leq 0$, there exists $v^*(w) \in B(v)$ satisfying

$$\liminf \langle v_n^*, v_n - w \rangle \geq \langle v^*(w), v - w \rangle.$$

for all $w \in X$.

- (ii) **generalized pseudomonotone** if and only if $\{v_n\}_{n \in \mathbb{N}} \subset X$, $\{v_n^*\}_{n \in \mathbb{N}} \subset X^*$ with $v_n^* \in B(v_n)$ such that $v_n \rightharpoonup v$ in X , $v_n^* \rightharpoonup v^*$ in X^* and

$$\limsup \langle v_n^*, v_n - v \rangle \leq 0$$

imply $v^* \in B(v)$ and

$$\langle v_n^*, v_n \rangle \rightarrow \langle v^*, v \rangle.$$

- (iii) **coercive** if and only if for $v \in D(B)$ such that $\|v\|_X \rightarrow \infty$, it holds that

$$\frac{\inf\{\langle v^*, v \rangle : v^* \in B(v)\}}{\|v\|_X} \rightarrow +\infty.$$

The following surjective theorem is essential for our existence results, see Carl–Le [10].

Theorem 2.11. *Suppose X is a real reflexive Banach space and $T: X \rightarrow 2^{X^*}$ is bounded and pseudomonotone. Assume the following coercivity condition: a real-valued function $c: \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\lim_{r \rightarrow +\infty} c(r) = +\infty$ and a point $v_0 \in X$ can be chosen so that*

$$\langle v^*, v - v_0 \rangle \geq c(\|v\|_X) \|v\|_X \quad \text{for all } (v, v^*) \in \text{Gr}(T).$$

Its range is then the whole space, that is, $\text{range}(T) = X^*$, so T is surjective.

3. EXISTENCE OF UNIFORMLY BOUNDED WEAK SOLUTIONS

To obtain uniformly bounded weak solutions of (P1), we truncate the unbounded coefficient h to define an auxiliary problem, and then establish its solvability via a surjectivity theorem for multivalued pseudomonotone operators. Uniform estimates independent of the truncation parameters allow us to choose those parameters so that an auxiliary solution also satisfies the original problem.

First, we focus on the problem involving variable exponents and present the corresponding assumptions for the function h , the multivalued functions $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ and $f_{\Gamma_2}: \Gamma_2 \times \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ as well as the intrinsic operator $T: V \rightarrow W^{1, \vartheta^-}(\Omega)$.

- (H1) The function $h: \mathbb{R} \times [0, \infty) \times [0, \infty) \rightarrow [a_0, \infty)$ is continuous, where a_0 is a positive constant.
- (H2) The functions $f: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ and $f_{\Gamma_2}: \Gamma_2 \times \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ are graph measurable with $f(x, \cdot, \cdot, \cdot, \cdot): \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ being upper semicontinuous for a.e. $x \in \Omega$ and $f_{\Gamma_2}(x, \cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ being upper semicontinuous for a.e. $x \in \Gamma_2$. Both maps have nonempty closed convex values.
- (H3) The growth of the interior term is controlled by constants $a_i \geq 0$ for $i = 1, 2, 3, 4$ and a function $\sigma_1 \in L^{\frac{\vartheta_-}{\vartheta_- - 1}}(\Omega)$ through the estimate

$$\begin{aligned} & \sup \{|\eta| : \eta \in f(x, s_1, s_2, \xi_1, \xi_2)\} \\ & \leq \sigma_1(x) + a_1 |s_1|^{\vartheta_- - 1} + a_2 |s_2|^{\vartheta_- - 1} + a_3 |\xi_1|^{\vartheta_- - 1} + a_4 |\xi_2|^{\vartheta_- - 1}, \end{aligned}$$

for a.e. $x \in \Omega$, for all $s_1, s_2 \in \mathbb{R}$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

- (H4) For the boundary term, there are constants $b_i \geq 0$ for $i = 1, 2$ and a function $\sigma_2 \in L^{\frac{\vartheta_-}{\vartheta_- - 1}}(\Gamma_2)$ for which

$$\sup \{|\xi| : \xi \in f_{\Gamma_2}(x, s_1, s_2)\} \leq \sigma_2(x) + b_1 |s_1|^{\vartheta_- - 1} + b_2 |s_2|^{\vartheta_- - 1},$$

for a.e. $x \in \Gamma_2$ and for all $s_1, s_2 \in \mathbb{R}$.

(H5) (i) There are constants $0 \leq \Lambda < a_*$ and $K \geq 0$ such that

$$\int_{\Omega} \eta u \, dx - \int_{\Gamma_2} \zeta u \, d\sigma \leq \Lambda \|u\|^{\vartheta_-} + K \|u\| \quad (3.1)$$

for every $u \in V$ and all measurable selections $\eta(x) \in f(x, u, Tu, \nabla u, \nabla(Tu))$ and $\zeta(x) \in f_{\Gamma_2}(x, u, Tu)$ almost everywhere. The norm and a_* are fixed in Remark 2.9.

(ii) There exist exponents

$$\begin{aligned} \vartheta_- &\leq d < (\vartheta_-)^*, & \vartheta_- &\leq e < (\vartheta_-)_*, \\ \frac{N}{\vartheta_-} &< \gamma_1 < \infty, & \frac{N-1}{\vartheta_- - 1} &< \gamma_2 < \infty. \end{aligned}$$

such that, for every $M > 0$, there exists $L_M > 0$ with the following property. For each $u \in V$ with $\|u\| \leq M$, there are nonnegative functions $\omega_{u,1} \in L^{\gamma_1}(\Omega)$ and $\omega_{u,2} \in L^{\gamma_2}(\Gamma_2)$ satisfying

$$\|\omega_{u,1}\|_{\gamma_1} + \|\omega_{u,2}\|_{\gamma_2, \Gamma_2} \leq L_M$$

and

$$\begin{aligned} &\sup_{\eta \in f(x, u, Tu, \nabla u, \nabla(Tu))} |\eta| \\ &\leq \omega_{u,1}(x) + L_M (|u|^{d-1} + |\nabla u|^{\vartheta_-(d-1)/d}) \quad \text{a.e. in } \Omega, \end{aligned}$$

as well as

$$\sup_{\zeta \in f_{\Gamma_2}(x, u, Tu)} |\zeta| \leq \omega_{u,2}(x) + L_M |u|^{e-1} \quad \text{a.e. on } \Gamma_2.$$

Hypothesis (H5)(i) provides the energy estimate. Hypothesis (H5)(ii) is used in Lemma 3.7 to obtain uniform bounds independent of the truncation parameters.

Remark 3.1. Under (H3)–(H4), a sufficient condition for (H5)(i) is that T satisfies the following estimates with constants $c_i \geq 0$ for $i = 1, 2, 3, 4$:

$$\begin{aligned} \|Tu\|_{\vartheta_-}^{\vartheta_- - 1} &\leq c_1 \|u\|_{\hat{\rho}_{\Phi}}^{\vartheta_- - 1} + c_3, \\ \|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- - 1} &\leq c_2 \|u\|_{\hat{\rho}_{\Phi}}^{\vartheta_- - 1} + c_3, \\ \|Tu\|_{\vartheta_-, \Gamma_2}^{\vartheta_- - 1} &\leq c_4 \|u\|_{\hat{\rho}_{\Phi}}^{\vartheta_- - 1} + c_3, \end{aligned}$$

for all $u \in V$, together with

$$\begin{aligned} \Lambda &:= a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 c_1 C_{\vartheta_-} + a_3 G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-} + a_4 c_2 C_{\vartheta_-} \\ &\quad + b_1 C_{\vartheta_-, \Gamma_2}^{\vartheta_-} + b_2 c_4 C_{\vartheta_-, \Gamma_2} < a_*, \end{aligned} \quad (3.2)$$

Indeed, Hölder's inequality and these estimates give

$$\begin{aligned} \int_{\Omega} \eta u \, dx - \int_{\Gamma_2} \zeta u \, d\sigma &\leq \int_{\Omega} |\eta| |u| \, dx + \int_{\Gamma_2} |\zeta| |u| \, d\sigma \\ &\leq (\|\sigma_1\|_{\vartheta'_-} + a_2 \|Tu\|_{\vartheta_-}^{\vartheta_- - 1} + a_4 \|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- - 1}) \|u\|_{\vartheta_-} \\ &\quad + a_1 \|u\|_{\vartheta_-}^{\vartheta_-} + a_3 \|\nabla u\|_{\vartheta_-}^{\vartheta_- - 1} \|u\|_{\vartheta_-} \\ &\quad + (\|\sigma_2\|_{\vartheta'_-, \Gamma_2} + b_2 \|Tu\|_{\vartheta_-, \Gamma_2}^{\vartheta_- - 1}) \|u\|_{\vartheta_-, \Gamma_2} + b_1 \|u\|_{\vartheta_-, \Gamma_2}^{\vartheta_-} \end{aligned}$$

$$\leq \Lambda \|u\|^{\vartheta_-} + K \|u\|,$$

where

$$K = C_{\vartheta_-} [\|\sigma_1\|_{\vartheta_-'} + (a_2 + a_4)c_3] + C_{\vartheta_-, \Gamma_2} [\|\sigma_2\|_{\vartheta_-', \Gamma_2} + b_2c_3].$$

Thus (3.1) holds with constants independent of u .

If, in addition, $\vartheta_- > N - 1$, the same norm estimates imply (H5)(ii). Indeed,

$$\vartheta'_- > \frac{N}{\vartheta_-} \quad \text{and} \quad \vartheta'_- > \frac{N-1}{\vartheta_- - 1}.$$

Take $d = e = \vartheta_-$ and $\gamma_1 = \gamma_2 = \vartheta'_-$, and set

$$\begin{aligned} \omega_{u,1} &= \sigma_1 + a_2 |Tu|^{\vartheta_- - 1} + a_4 |\nabla(Tu)|^{\vartheta_- - 1}, \\ \omega_{u,2} &= \sigma_2 + b_2 |Tu|^{\vartheta_- - 1}. \end{aligned}$$

Since $(\vartheta_- - 1)\vartheta'_- = \vartheta_-$, the displayed norm estimates give, for $\|u\| \leq M$,

$$\begin{aligned} \|\omega_{u,1}\|_{\vartheta'_-} + \|\omega_{u,2}\|_{\vartheta'_-, \Gamma_2} &\leq \|\sigma_1\|_{\vartheta'_-} + \|\sigma_2\|_{\vartheta'_-, \Gamma_2} \\ &\quad + (a_2c_1 + a_4c_2 + b_2c_4)M^{\vartheta_- - 1} + (a_2 + a_4 + b_2)c_3. \end{aligned}$$

The remaining terms in (H3)–(H4) have exactly the powers required in (H5)(ii). Choose L_M larger than the displayed norm bound and a_1, a_3, b_1 .

We use the following weak formulation of (P1).

Definition 3.2. We say that a function $u \in V$ is a weak solution of problem (P1), if there exist fixed measurable selections $\eta \in f(x, u, Tu, \nabla u, \nabla(Tu))$ a.e. in Ω and $\zeta \in f_{\Gamma_2}(x, u, Tu)$ a.e. on Γ_2 , such that, for every $v \in V$, $h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})\mathcal{K}(\nabla u) \cdot \nabla v \in L^1(\Omega)$, $\eta v \in L^1(\Omega)$, $\zeta v \in L^1(\Gamma_2)$, and the following identity holds:

$$\begin{aligned} &\int_{\Omega} h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\vartheta(x)-2} \nabla u + \mu(x) |\nabla u|^{\kappa(x)-2} \nabla u \right) \cdot \nabla v \, dx \\ &= \int_{\Omega} \eta v \, dx - \int_{\Gamma_2} \zeta v \, d\sigma. \end{aligned} \tag{3.3}$$

The main existence statements for weak solutions of (P1) are given next.

Theorem 3.3. Let hypotheses (H0)–(H5) be satisfied. Then problem (P1) possesses a uniformly bounded weak solution $u \in V$. Its trace on Γ_2 is also bounded.

Next, we consider the particular case when the exponents are constants, that is, $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \bar{\Omega}$. In this scenario, the growth conditions for the multivalued functions $f: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ and $f_{\Gamma_2}: \Gamma_2 \times \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ are weaker as seen below:

(H'3) There exist constants $\tilde{a}_i \geq 0$ for $i = 1, 2, 3, 4$, $\tilde{r}_1 \in (1, \tilde{\vartheta}^*)$ and $\tilde{\sigma}_1 \in L^{\tilde{r}'_1}(\Omega)$ satisfying

$$\begin{aligned} &\sup \{|\eta|: \eta \in f(x, s_1, s_2, \xi_1, \xi_2)\} \\ &\leq \tilde{\sigma}_1(x) + \tilde{a}_1 |s_1|^{\tilde{\vartheta}-1} + \tilde{a}_2 |s_2|^{\tilde{r}_1-1} + \tilde{a}_3 |\xi_1|^{\tilde{\vartheta}-1} + \tilde{a}_4 |\xi_2|^{\frac{\tilde{\vartheta}}{\tilde{r}'_1}}, \end{aligned}$$

for a.e. $x \in \Omega$, for all $s_1, s_2 \in \mathbb{R}$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

(H'4) For the boundary term, choose constants $\tilde{b}_i \geq 0$ for $i = 1, 2$, $\tilde{r}_2 \in (1, \tilde{\vartheta}_*)$ and a function $\tilde{\sigma}_2 \in L^{\tilde{r}'_2}(\Gamma_2)$ so that

$$\sup \{|\xi| : \xi \in f_{\Gamma_2}(x, s_1, s_2)\} \leq \tilde{\sigma}_2(x) + \tilde{b}_1 |s_1|^{\tilde{\vartheta}-1} + \tilde{b}_2 |s_2|^{\tilde{r}_2-1},$$

for a.e. $x \in \Gamma_2$ and for all $s_1, s_2 \in \mathbb{R}$.

(H'5) (i) There are constants $0 \leq \tilde{\Lambda} < a_*$ and $\tilde{K} \geq 0$ such that

$$\int_{\Omega} \eta u \, dx - \int_{\Gamma_2} \zeta u \, d\sigma \leq \tilde{\Lambda} \|u\|^{\tilde{\vartheta}} + \tilde{K} \|u\| \quad (3.4)$$

for every $u \in V$ and all measurable selections specified in (H5)(i).

(ii) Hypothesis (H5)(ii) holds with $\vartheta_- = \tilde{\vartheta}$.

Remark 3.4. Under (H'3)–(H'4), a sufficient condition for (H'5)(i) is that T satisfies the following estimates with constants $\tilde{c}_i \geq 0$ for $i = 1, 2, 3, 4$:

$$\begin{aligned} \|Tu\|_{\tilde{\vartheta}_*}^{\tilde{r}_1-1} &\leq \tilde{c}_1 \|u\|_{\tilde{\rho}_{\Phi}}^{\tilde{\vartheta}-1} + \tilde{c}_3, \\ \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{r}'_1} &\leq \tilde{c}_2 \|u\|_{\tilde{\rho}_{\Phi}}^{\tilde{\vartheta}-1} + \tilde{c}_3, \\ \|Tu\|_{\tilde{\vartheta}_*, \Gamma_2}^{\tilde{r}_2-1} &\leq \tilde{c}_4 \|u\|_{\tilde{\rho}_{\Phi}}^{\tilde{\vartheta}-1} + \tilde{c}_3, \end{aligned}$$

for all $u \in V$, together with

$$\begin{aligned} \tilde{\Lambda} &:= \tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 \tilde{c}_1 C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_1 + 1}} + \tilde{a}_3 G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}} + \tilde{a}_4 \tilde{c}_2 C_{\tilde{r}_1} \\ &\quad + \tilde{b}_1 C_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 \tilde{c}_4 C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1}, \Gamma_2} < a_*. \end{aligned} \quad (3.5)$$

The terms involving T are estimated by Hölder's inequality as follows:

$$\begin{aligned} \int_{\Omega} |Tu|^{\tilde{r}_1-1} |u| \, dx &\leq \|Tu\|_{\tilde{\vartheta}_*}^{\tilde{r}_1-1} \|u\|_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_1 + 1}}, \\ \int_{\Omega} |\nabla(Tu)|^{\tilde{\vartheta}/\tilde{r}'_1} |u| \, dx &\leq \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}/\tilde{r}'_1} \|u\|_{\tilde{r}_1}, \\ \int_{\Gamma_2} |Tu|^{\tilde{r}_2-1} |u| \, d\sigma &\leq \|Tu\|_{\tilde{\vartheta}_*, \Gamma_2}^{\tilde{r}_2-1} \|u\|_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1}, \Gamma_2}. \end{aligned}$$

Here the first and third pairs of Hölder exponents follow from

$$\frac{\tilde{r}_1 - 1}{\tilde{\vartheta}_*} + \frac{\tilde{\vartheta}_* - \tilde{r}_1 + 1}{\tilde{\vartheta}_*} = 1, \quad \frac{\tilde{r}_2 - 1}{\tilde{\vartheta}_*} + \frac{\tilde{\vartheta}_* - \tilde{r}_2 + 1}{\tilde{\vartheta}_*} = 1.$$

By substituting the norm bounds for T and estimating the remaining terms as in Remark 3.1, we obtain (3.4) with the displayed $\tilde{\Lambda}$ and

$$\tilde{K} = \|\tilde{\sigma}_1\|_{\tilde{r}'_1} C_{\tilde{r}_1} + \tilde{c}_3 (\tilde{a}_2 C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_1 + 1}} + \tilde{a}_4 C_{\tilde{r}_1}) + \|\tilde{\sigma}_2\|_{\tilde{r}'_2, \Gamma_2} C_{\tilde{r}_2, \Gamma_2} + \tilde{b}_2 \tilde{c}_3 C_{\frac{\tilde{\vartheta}_*}{\tilde{\vartheta}_* - \tilde{r}_2 + 1}, \Gamma_2}.$$

When $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta} > N - 1$, these estimates also imply (H'5)(ii). The critical domain and trace norm estimates control the corresponding $L^{\tilde{\vartheta}}$ norms on these finite-measure sets, so the argument at the end of Remark 3.1 applies.

We have the following existence result.

Theorem 3.5. Suppose $\vartheta(x) \equiv \tilde{\vartheta} > 1$, $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and assume (H0)–(H2) together with (H'3)–(H'5). Problem (P1) then has a weak solution $u \in V$ that is uniformly bounded in Ω , with bounded trace on Γ_2 .

3.1. Variable exponents. We now give the proof of Theorem 3.3.

Lemma 3.6. *Assume (H0)–(H5). Every weak solution $u \in V$ of (P1), or of its auxiliary problem with h_R , obeys $\|u\|_{\hat{\rho}_\Phi} \leq M_1$ for a constant $M_1 \geq 1$ depending on the problem data and the lower bound a_0 of h .*

Proof. Let u be a solution of (P1). Choosing $v = u$ as a test function in (3.3) yields

$$\begin{aligned} & \int_{\Omega} h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\vartheta(x)} + \mu(x) |\nabla u|^{\kappa(x)} \right) dx \\ &= \int_{\Omega} \eta u dx - \int_{\Gamma_2} \zeta u d\sigma. \end{aligned} \quad (3.6)$$

Assume $\|u\|_{\hat{\rho}_\Phi} \geq D$. By (H1), Proposition 2.6 and Remark 2.9, we obtain

$$\begin{aligned} & \int_{\Omega} h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\vartheta(x)} + \mu(x) |\nabla u|^{\kappa(x)} \right) dx \\ & \geq a_0 \int_{\Omega} \Phi(x, |\nabla u|) dx \geq a_* \|u\|_{\hat{\rho}_\Phi}^{\vartheta_-}. \end{aligned} \quad (3.7)$$

By (H5)(i), (3.6) and (3.7), we obtain

$$(a_* - \Lambda) \|u\|^{\vartheta_-} \leq K \|u\|.$$

Hence, we have

$$\|u\|_{\hat{\rho}_\Phi} \leq \left(\frac{K}{a_* - \Lambda} \right)^{1/(\vartheta_- - 1)} =: M \quad \text{whenever } \|u\|_{\hat{\rho}_\Phi} \geq D.$$

Taking $M_1 = \max\{D, 1, M\}$ shows the assertion. The estimate uses only the lower bound a_0 , and therefore also holds for h_R , independently of R . $\square$

Lemma 3.7. *Under (H0), (H1) and (H5)(ii), let $u \in V$ satisfy the weak formulation of the original or an auxiliary problem, with a principal coefficient bounded below by a_0 , and assume $\|u\|_{\hat{\rho}_\Phi} \leq M$. A constant $M_2(M) > 0$, independent of the upper bound of that coefficient, satisfies $\|u\|_\infty + \|u\|_{\infty, \Gamma_2} \leq M_2(M)$.*

Proof. We use a Moser iteration, as in Motreanu–Tornatore [35]. Put $u^+ = \max\{u, 0\}$ and $u_\ell = \min\{u^+, \ell\}$ for $\ell > 1$. For $\alpha \geq 0$, take

$$v = u^+ u_\ell^{\alpha \vartheta_-}, \quad z = u^+ u_\ell^\alpha.$$

Both functions belong to V , since the corresponding scalar truncation maps are Lipschitz and vanish at zero. Their gradients satisfy

$$\begin{aligned} \nabla v &= (1 + \alpha \vartheta_- \mathbf{1}_{\{0 < u^+ < \ell\}}) u_\ell^{\alpha \vartheta_-} \nabla u^+, \\ \nabla z &= (1 + \alpha \mathbf{1}_{\{0 < u^+ < \ell\}}) u_\ell^\alpha \nabla u^+. \end{aligned}$$

Denote the principal coefficient by $H_u \geq a_0$. Since $|\xi|^{\vartheta_-} \leq 1 + |\xi|^{\vartheta(x)}$, testing the weak formulation with v and discarding the nonnegative extra gradient terms yields

$$\begin{aligned} a_0 \int_{\Omega} u_\ell^{\alpha \vartheta_-} |\nabla u^+|^{\vartheta_-} dx &\leq a_0 \int_{\Omega} u_\ell^{\alpha \vartheta_-} dx + \int_{\Omega} |\eta| u^+ u_\ell^{\alpha \vartheta_-} dx \\ &\quad + \int_{\Gamma_2} |\zeta| u^+ u_\ell^{\alpha \vartheta_-} d\sigma. \end{aligned}$$

Taking (H5)(ii) into account, apply Young's inequality with conjugate exponents $d/(d-1)$ and d to

$$(|\nabla u^+|^{\vartheta_-} u_\ell^{\alpha\vartheta_-})^{(d-1)/d} (u^+ u_\ell^{\alpha\vartheta_-/d}).$$

After choosing the Young parameter in terms of a_0 and L_M , we obtain

$$L_M |\nabla u^+|^{\vartheta_- (d-1)/d} u^+ u_\ell^{\alpha\vartheta_-} \leq \frac{a_0}{2} |\nabla u^+|^{\vartheta_-} u_\ell^{\alpha\vartheta_-} + C_M (u^+)^d u_\ell^{\alpha\vartheta_-}.$$

Moreover, $|\nabla z| \leq (\alpha + 1) u_\ell^\alpha |\nabla u^+|$, $u_\ell^{\alpha\vartheta_-} \leq 1 + z^{\vartheta_-}$ and $u^+ u_\ell^{\alpha\vartheta_-} \leq 1 + z^{\vartheta_-}$. Absorbing the gradient term, we obtain

$$\begin{aligned} \|\nabla z\|_{\vartheta_-}^{\vartheta_-} &\leq C_M (\alpha + 1)^{\vartheta_-} \left[1 + \int_{\Omega} (1 + \omega_{u,1} + |u|^{d-\vartheta_-}) z^{\vartheta_-} dx \right. \\ &\quad \left. + \int_{\Gamma_2} (\omega_{u,2} + |u|^{e-\vartheta_-}) z^{\vartheta_-} d\sigma \right]. \end{aligned} \quad (3.8)$$

Here and below $C_M \geq 1$ may be enlarged. The bounds on $\|\omega_{u,1}\|_{\gamma_1}$ and $\|\omega_{u,2}\|_{\gamma_2, \Gamma_2}$ in (H5)(ii) ensure that C_M is independent of the choice of u with $\|u\| \leq M$, of α and ℓ , and of the upper bound of H_u . Set

$$\begin{aligned} r_\Omega &= \max \left\{ \vartheta_-, \vartheta_- \gamma'_1, \frac{\vartheta_- (\vartheta_-)^*}{(\vartheta_-)^* - d + \vartheta_-} \right\} < (\vartheta_-)^*, \\ r_\Gamma &= \max \left\{ \vartheta_-, \vartheta_- \gamma'_2, \frac{\vartheta_- (\vartheta_-)^*}{(\vartheta_-)^* - e + \vartheta_-} \right\} < (\vartheta_-)^*. \end{aligned}$$

The strict inequalities follow from (H5)(ii). Hölder's inequality yields the interior estimates

$$\begin{aligned} \int_{\Omega} \omega_{u,1} z^{\vartheta_-} dx &\leq \|\omega_{u,1}\|_{\gamma_1} \|z\|_{\vartheta_- \gamma'_1}^{\vartheta_-}, \\ \int_{\Omega} |u|^{d-\vartheta_-} z^{\vartheta_-} dx &\leq \|u\|_{(\vartheta_-)^*}^{d-\vartheta_-} \|z\|_{\frac{\vartheta_- (\vartheta_-)^*}{(\vartheta_-)^* - d + \vartheta_-}}^{\vartheta_-} \quad \text{if } d > \vartheta_-. \end{aligned}$$

The second estimate uses conjugate exponents $(\vartheta_-)^*/(d-\vartheta_-)$ and $(\vartheta_-)^*/((\vartheta_-)^* - d + \vartheta_-)$. If $d = \vartheta_-$, the weight is 1, and the integral is $\|z\|_{\vartheta_-}^{\vartheta_-}$. The same calculations on Γ_2 use γ_2 , e and $(\vartheta_-)^*$. The norm bound on u and the Sobolev and trace embeddings therefore turn (3.8) into

$$\|\nabla z\|_{\vartheta_-}^{\vartheta_-} \leq C_M (\alpha + 1)^{\vartheta_-} \left(1 + \|z\|_{r_\Omega}^{\vartheta_-} + \|z\|_{r_\Gamma, \Gamma_2}^{\vartheta_-} \right).$$

Choose $0 < \theta_\Omega, \theta_\Gamma < 1$ such that

$$\begin{aligned} \frac{1}{r_\Omega} &= \frac{1 - \theta_\Omega}{\vartheta_-} + \frac{\theta_\Omega}{(\vartheta_-)^*}, \\ \frac{1}{r_\Gamma} &= \frac{1 - \theta_\Gamma}{\vartheta_-} + \frac{\theta_\Gamma}{(\vartheta_-)^*}. \end{aligned}$$

Interpolation and the Sobolev and trace inequalities on the partial zero-trace space give

$$\begin{aligned} \|z\|_{r_\Omega}^{\vartheta_-} &\leq C \|z\|_{\vartheta_-}^{\vartheta_- (1-\theta_\Omega)} \|\nabla z\|_{\vartheta_-}^{\vartheta_- \theta_\Omega}, \\ \|z\|_{r_\Gamma, \Gamma_2}^{\vartheta_-} &\leq C \|z\|_{\vartheta_-, \Gamma_2}^{\vartheta_- (1-\theta_\Gamma)} \|\nabla z\|_{\vartheta_-}^{\vartheta_- \theta_\Gamma}. \end{aligned}$$

Young's inequality with exponents $1/\theta_\Omega$, $1/(1-\theta_\Omega)$, and their boundary counterparts implies that, for every $0 < \varepsilon \leq 1$,

$$\|z\|_{r_\Omega}^{\vartheta_-} + \|z\|_{r_\Gamma, \Gamma_2}^{\vartheta_-} \leq \varepsilon \|\nabla z\|_{\vartheta_-}^{\vartheta_-} + C\varepsilon^{-b} \left(\|z\|_{\vartheta_-}^{\vartheta_-} + \|z\|_{\vartheta_-, \Gamma_2}^{\vartheta_-} \right)$$

where $b = \max\{\theta_\Omega/(1-\theta_\Omega), \theta_\Gamma/(1-\theta_\Gamma)\} > 0$. Take $\varepsilon = [2C_M(\alpha+1)^{\vartheta_-}]^{-1}$. The coefficient of the gradient term on the right is then $1/2$, so absorption yields, with $\beta = \vartheta_-(1+b)$,

$$\|\nabla z\|_{\vartheta_-}^{\vartheta_-} \leq C_M(\alpha+1)^\beta \left(1 + \|z\|_{\vartheta_-}^{\vartheta_-} + \|z\|_{\vartheta_-, \Gamma_2}^{\vartheta_-} \right).$$

For $s \geq 1$, write

$$\|w\|_{s, \Omega \cup \Gamma_2} := \left(\int_\Omega |w|^s dx + \int_{\Gamma_2} |w|^s d\sigma \right)^{1/s}.$$

Since $(\vartheta_-)_* < (\vartheta_-)^*$, the same Sobolev and trace inequalities control $\|z\|_{(\vartheta_-)_*, \Omega \cup \Gamma_2}$ by $\|\nabla z\|_{\vartheta_-}$. Letting $\ell \rightarrow \infty$ and using Fatou's lemma therefore yields

$$\begin{aligned} & \max \{1, \|u^+\|_{(\alpha+1)(\vartheta_-)_*, \Omega \cup \Gamma_2}\} \\ & \leq [C_M(\alpha+1)^\beta]^{1/[(\vartheta_-)(\alpha+1)]} \max \{1, \|u^+\|_{\vartheta_-(\alpha+1), \Omega \cup \Gamma_2}\}, \end{aligned} \quad (3.9)$$

whenever the norm on the right is finite. Set $\chi = (\vartheta_-)_*/\vartheta_- > 1$ and take $\alpha_j+1 = \chi^j$ for $j = 0, 1, \dots$. The initial norm is bounded by C_0M , and repeated application of (3.9) bounds all norms with exponents $\vartheta_- \chi^j$ by $P_M \max\{1, C_0M\}$, where

$$\begin{aligned} \log P_M &= \sum_{j=0}^{\infty} \frac{\log C_M + \beta j \log \chi}{\vartheta_- \chi^j} \\ &= \frac{\chi \log C_M}{\vartheta_- (\chi - 1)} + \frac{\beta \chi \log \chi}{\vartheta_- (\chi - 1)^2} < \infty. \end{aligned}$$

Since $dx + d\sigma$ on $\Omega \cup \Gamma_2$ is a finite measure, letting $j \rightarrow \infty$ proves

$$\max\{\|u^+\|_\infty, \|u^+\|_{\infty, \Gamma_2}\} \leq P_M \max\{1, C_0M\}.$$

Testing with $v = -u^- \min\{u^-, \ell\}^{\alpha\vartheta_-}$, where $u^- = \max\{-u, 0\}$, leads to the same estimates for u^- . This proves the asserted bound $M_2(M)$, independently of the truncation parameters. $\square$

Since the coefficient h may be unbounded, the standard operator theory cannot be applied directly to problem (P1). We therefore replace it with the truncated function $h_R: \mathbb{R} \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ defined by

$$h_R(t, s, l) = \begin{cases} h(R_1, R_2, R_3) & \text{if } t > R_1, s > R_2, l > R_3 \\ h(R_1, R_2, l) & \text{if } t > R_1, s > R_2, 0 \leq l \leq R_3 \\ h(R_1, s, R_3) & \text{if } t > R_1, 0 \leq s \leq R_2, l > R_3 \\ h(R_1, s, l) & \text{if } t > R_1, 0 \leq s \leq R_2, 0 \leq l \leq R_3 \\ h(t, R_2, R_3) & \text{if } -R_1 \leq t \leq R_1, s > R_2, l > R_3 \\ h(t, R_2, l) & \text{if } -R_1 \leq t \leq R_1, s > R_2, 0 \leq l \leq R_3 \\ h(t, s, R_3) & \text{if } -R_1 \leq t \leq R_1, 0 \leq s \leq R_2, l > R_3 \\ h(t, s, l) & \text{if } -R_1 \leq t \leq R_1, 0 \leq s \leq R_2, 0 \leq l \leq R_3 \\ h(-R_1, R_2, R_3) & \text{if } t < -R_1, s > R_2, l > R_3 \\ h(-R_1, R_2, l) & \text{if } t < -R_1, s > R_2, 0 \leq l \leq R_3 \\ h(-R_1, s, R_3) & \text{if } t < -R_1, 0 \leq s \leq R_2, l > R_3 \\ h(-R_1, s, l) & \text{if } t < -R_1, 0 \leq s \leq R_2, 0 \leq l \leq R_3, \end{cases} \quad (3.10)$$

for parameters $R_1, R_2, R_3 > 0$. Clearly, h_R is bounded and continuous, with $a_0 \leq h_R \leq M_R$ for some $M_R > 0$. The corresponding auxiliary inclusion reads:

$$\begin{aligned} -\operatorname{div} [h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u)] &\in f(x, u, Tu, \nabla u, \nabla(Tu)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ -h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u) \cdot \nu &\in f_{\Gamma_2}(x, u, Tu) \quad \text{on } \Gamma_2. \end{aligned} \quad (\text{P2})$$

The operator associated with (P2), $\mathcal{J}_R: V \rightarrow V^*$, acts through

$$\begin{aligned} \langle \mathcal{J}_R(u), v \rangle &= \int_{\Omega} h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\vartheta(x)-2} \nabla u + \mu(x) |\nabla u|^{\kappa(x)-2} \nabla u \right) \cdot \nabla v \, dx, \end{aligned}$$

for all $u, v \in V$.

Definition 3.8. We say that a function $\tilde{u} \in V$ is a weak solution of problem (P2), if $h_R(\tilde{u}, \|\tilde{u}\|_{k(\cdot)}, \|\tilde{u}\|_{\hat{\rho}_\Phi}) (|\nabla \tilde{u}|^{\vartheta(x)-2} \nabla \tilde{u} + \mu(x) |\nabla \tilde{u}|^{\kappa(x)-2} \nabla \tilde{u}) \cdot \nabla v \in L^1(\Omega)$ and there exist fixed measurable selections $\eta \in f(x, \tilde{u}, T\tilde{u}, \nabla \tilde{u}, \nabla(T\tilde{u}))$ as well as $\zeta \in f_{\Gamma_2}(x, \tilde{u}, T\tilde{u})$ with $\eta v \in L^1(\Omega)$ and $\zeta v \in L^1(\Gamma_2)$, such that

$$\begin{aligned} &\int_{\Omega} h_R(\tilde{u}, \|\tilde{u}\|_{k(\cdot)}, \|\tilde{u}\|_{\hat{\rho}_\Phi}) \left(|\nabla \tilde{u}|^{\vartheta(x)-2} \nabla \tilde{u} + \mu(x) |\nabla \tilde{u}|^{\kappa(x)-2} \nabla \tilde{u} \right) \cdot \nabla v \, dx \\ &= \int_{\Omega} \eta v \, dx - \int_{\Gamma_2} \zeta v \, d\sigma. \end{aligned}$$

for all $v \in V$. The pointwise memberships hold almost everywhere on the corresponding sets.

The operator $\mathcal{J}_R$ fulfills the following useful properties.

Proposition 3.9. Under (H0) and (H1), $\mathcal{J}_R: V \rightarrow V^*$ is a bounded and continuous operator. It also has the (S_+) -property, i.e.,

$$u_n \rightharpoonup u \quad \text{in } V \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle \mathcal{J}_R(u_n), u_n - u \rangle \leq 0 \quad \Rightarrow \quad u_n \rightarrow u \quad \text{in } V.$$

Proof. We use the basic double phase operator $\mathcal{A}: V \rightarrow V^*$, namely

$$\begin{aligned} \langle \mathcal{A}(u), v \rangle &= \int_{\Omega} \partial_t \Psi(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla v \, dx \\ &= \int_{\Omega} (|\nabla u|^{\vartheta(x)-2} \nabla u + \mu(x) |\nabla u|^{\kappa(x)-2} \nabla u) \cdot \nabla v \, dx, \quad u, v \in V. \end{aligned}$$

The double phase estimates of Crespo-Blanco-Gasiński-Harjulehto-Winkert [16] show that $\mathcal{A}$ is bounded, continuous and of type (S_+) . Their gradient estimates apply on V by the norm equivalence in Remark 2.9, and remain valid when the two exponents coincide or μ vanishes. The strict monotonicity of the first power flux also implies

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle > 0 \quad \text{for } u \neq v.$$

Indeed, equality would imply $\nabla u = \nabla v$ almost everywhere. Their difference is constant, and its zero trace on Γ_1 forces that constant to vanish.

Continuity of h_R provides the finite constant

$$M_R = \max_{(t,s,l) \in [-R_1, R_1] \times [0, R_2] \times [0, R_3]} h(t, s, l).$$

For $H_u = h_R(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi})$, we have $a_0 \leq H_u \leq M_R$. The flux map $\xi \mapsto \mathcal{K}(\xi)$ is continuous from $L^\Phi(\Omega; \mathbb{R}^N)$ to $L^{\Phi^*}(\Omega; \mathbb{R}^N)$ and is bounded on bounded sets. Hölder's inequality gives

$$|\langle \mathcal{J}_R(u), v \rangle| \leq 2M_R \|\mathcal{K}(\nabla u)\|_{\Phi^*} \|\nabla v\|_\Phi.$$

Together with boundedness of $\mathcal{A}$, this estimate bounds $\mathcal{J}_R$.

For continuity, assume $u_n \rightarrow u$ in V , and write $H_n = h_R(u_n, \|u_n\|_{k(\cdot)}, \|u_n\|_{\hat{\rho}_\Phi})$ and $H = H_u$. The subcritical embedding in Proposition 2.7(ii) and convergence in V imply $\|u_n\|_{k(\cdot)} \rightarrow \|u\|_{k(\cdot)}$ and $\|u_n\|_{\hat{\rho}_\Phi} \rightarrow \|u\|_{\hat{\rho}_\Phi}$. Every subsequence has a further subsequence with $u_n(x) \rightarrow u(x)$ almost everywhere, and continuity of h_R implies $H_n(x) \rightarrow H(x)$ there. The power bounds in (H0) imply the Δ_2 condition for Φ^* , and the fixed flux belongs to its Orlicz heart. Since $|H_n - H| \leq 2M_R$, for every $\lambda > 0$ the integrand below tends to zero almost everywhere and is dominated by the integrable function $\Phi^*(x, 2M_R |\mathcal{K}(\nabla u)|/\lambda)$. Thus

$$\rho_{\Phi^*} \left(\frac{(H_n - H)\mathcal{K}(\nabla u)}{\lambda} \right) = \int_\Omega \Phi^* \left(x, \frac{|H_n - H| |\mathcal{K}(\nabla u)|}{\lambda} \right) dx \rightarrow 0.$$

For each $\lambda > 0$ this modular is eventually at most 1, so the Luxemburg norm is eventually at most λ . Since λ is arbitrary, we obtain

$$\|(H_n - H)\mathcal{K}(\nabla u)\|_{\Phi^*} \rightarrow 0. \quad (3.11)$$

The subsequence principle extends this convergence for the whole sequence. For $v \in V$, Hölder's inequality now yields the estimate

$$\begin{aligned} & |\langle \mathcal{J}_R(u_n) - \mathcal{J}_R(u), v \rangle| \\ & \leq \left| \int_\Omega H_n [\mathcal{K}(\nabla u_n) - \mathcal{K}(\nabla u)] \cdot \nabla v \, dx \right| + \left| \int_\Omega (H_n - H) \mathcal{K}(\nabla u) \cdot \nabla v \, dx \right| \\ & \leq 2(M_R \|\mathcal{K}(\nabla u_n) - \mathcal{K}(\nabla u)\|_{\Phi^*} + \|(H_n - H)\mathcal{K}(\nabla u)\|_{\Phi^*}) \|\nabla v\|_\Phi. \end{aligned}$$

Flux continuity and (3.11) show that $\mathcal{J}_R(u_n) \rightarrow \mathcal{J}_R(u)$ in V^* .

To verify the (S_+) -property, assume $u_n \rightharpoonup u$ in V and the stated limsup condition. The compact embedding $V \hookrightarrow L^\Phi(\Omega)$ in Proposition 2.8(i) implies $u_n \rightarrow u$ in $L^\Phi(\Omega)$. From any subsequence we may extract a further subsequence such that

$$u_n(x) \rightarrow u(x) \text{ a.e., } \|u_n\|_{k(\cdot)} \rightarrow s, \quad \|u_n\|_{\hat{\rho}_\Phi} \rightarrow l.$$

Set $\bar{H}(x) = h_R(u(x), s, l)$. As above,

$$\|(H_n - \bar{H})\mathcal{K}(\nabla u)\|_{\Phi^*} \rightarrow 0.$$

Weak convergence of the gradients and Hölder's inequality therefore imply

$$\begin{aligned} I_n &:= \int_\Omega H_n \mathcal{K}(\nabla u) \cdot \nabla(u_n - u) \, dx \\ &= \int_\Omega \bar{H} \mathcal{K}(\nabla u) \cdot \nabla(u_n - u) \, dx + \int_\Omega (H_n - \bar{H}) \mathcal{K}(\nabla u) \cdot \nabla(u_n - u) \, dx \rightarrow 0. \end{aligned}$$

Pointwise monotonicity of the flux, together with $H_n \geq a_0$, yields

$$\begin{aligned} 0 &\leq a_0 \langle \mathcal{A}(u_n) - \mathcal{A}(u), u_n - u \rangle \\ &\leq \int_\Omega H_n [\mathcal{K}(\nabla u_n) - \mathcal{K}(\nabla u)] \cdot \nabla(u_n - u) \, dx \\ &= \langle \mathcal{J}_R(u_n), u_n - u \rangle - I_n. \end{aligned}$$

It follows that $\limsup_n \langle \mathcal{A}(u_n), u_n - u \rangle \leq 0$, since $\langle \mathcal{A}(u), u_n - u \rangle \rightarrow 0$. The (S_+) -property of $\mathcal{A}$ yields strong convergence in V along this further subsequence. Applying the subsequence principle to the arbitrary initial subsequence proves $u_n \rightarrow u$ in V . $\square$

Define the selection sets

$$\mathcal{S}_f(u) := \{\eta \in L^{\vartheta_-'}(\Omega) : \eta(x) \in f(x, u, Tu, \nabla u, \nabla(Tu)) \text{ a.e.}\},$$

$$\mathcal{S}_\Gamma(u) := \{\zeta \in L^{\vartheta_-'}(\Gamma_2) : \zeta(x) \in f_{\Gamma_2}(x, u, Tu) \text{ a.e.}\}.$$

Each selection defines a functional on V by integration against the test function. The corresponding multivalued operators $B_f, B_{f_{\Gamma_2}} : V \rightarrow 2^{V^*}$ are defined by

$$B_f(u) = \left\{ v \mapsto \int_{\Omega} \eta v \, dx : \eta \in \mathcal{S}_f(u) \right\},$$

$$B_{f_{\Gamma_2}}(u) = \left\{ v \mapsto \int_{\Gamma_2} \zeta v \, d\sigma : \zeta \in \mathcal{S}_\Gamma(u) \right\}.$$

We write B_η and B_ζ for the corresponding linear functionals. Hölder's inequality and the embeddings fixed in Remark 2.9 give

$$|B_\eta(v)| \leq C_{\vartheta_-} \|\eta\|_{\vartheta_-'} \|v\|,$$

$$|B_\zeta(v)| \leq C_{\vartheta_-, \Gamma_2} \|\zeta\|_{\vartheta_-', \Gamma_2} \|v\|.$$

Thus these functionals belong to V^* , and $B_f(u)$ and $B_{f_{\Gamma_2}}(u)$ collect all functionals obtained from the selections at u .

By (H2)–(H4), the scalar values are nonempty and compact almost everywhere. Graph measurability and the measurable selection theorem ensure that $\mathcal{S}_f(u)$ and $\mathcal{S}_\Gamma(u)$ are nonempty. Their L^{ϑ_-} -integrability follows from (H3)–(H4) for fixed u .

Convexity of the pointwise values implies convexity of the selection sets. If a sequence of selections converges in L^{ϑ_-} , an almost everywhere convergent subsequence and pointwise closedness show that its limit remains a selection. The sets are therefore norm closed, convex and bounded, hence weakly compact in the reflexive spaces $L^{\vartheta_-}(\Omega)$ and $L^{\vartheta_-}(\Gamma_2)$.

On the ball $\|u\| \leq M$, (H5)(ii) and the continuous embeddings into $L^d(\Omega)$ and $L^e(\Gamma_2)$ imply

$$\|\eta\|_{\min\{\gamma_1, d'\}} + \|\zeta\|_{\min\{\gamma_2, e'\}, \Gamma_2} \leq C_M, \quad \eta \in \mathcal{S}_f(u), \quad \zeta \in \mathcal{S}_\Gamma(u).$$

Indeed, $(d-1)d' = d$ and $\vartheta_-(d-1)d'/d = \vartheta_-$, so

$$\| |u|^{d-1} \|_{d'} = \|u\|_d^{d-1}, \quad \| |\nabla u|^{\vartheta_-(d-1)/d} \|_{d'} = \| \nabla u \|_{\vartheta_-}^{\vartheta_-(d-1)/d}.$$

Similarly, $\| |u|^{e-1} \|_{e', \Gamma_2} = \|u\|_{e, \Gamma_2}^{e-1}$. The growth bounds and the finite-measure embeddings therefore yield the displayed uniform estimate. Since $d, \gamma_1' < (\vartheta_-)^*$ and $e, \gamma_2' < (\vartheta_-)_*$, the associated dual embeddings

$$V \hookrightarrow L^{\max\{d, \gamma_1'\}}(\Omega), \quad V \hookrightarrow L^{\max\{e, \gamma_2'\}}(\Gamma_2)$$

are compact.

Proposition 3.10. *Under (H0)–(H5), $\mathcal{F}_R := \mathcal{J}_R - B_f + B_{f_{\Gamma_2}} : V \rightarrow 2^{V^*}$ is bounded, coercive and pseudomonotone.*

Proof. We first verify the closed-graph property of the selection maps. Suppose $u_n \rightarrow u$ in V , $\eta_n \in \mathcal{S}_f(u_n)$, $\zeta_n \in \mathcal{S}_\Gamma(u_n)$, and

$$\eta_n \rightharpoonup \eta \text{ in } L^{\min\{\gamma_1, d'\}}(\Omega), \quad \zeta_n \rightharpoonup \zeta \text{ in } L^{\min\{\gamma_2, e'\}}(\Gamma_2).$$

Continuity of T implies $Tu_n \rightarrow Tu$ in $W^{1, \vartheta^-}(\Omega)$. After extraction, all interior arguments converge almost everywhere, as do the traces of u_n and Tu_n on Γ_2 . Set $F_u(x) = f(x, u, Tu, \nabla u, \nabla(Tu))$. Upper semicontinuity and compactness of $F_u(x)$ imply

$$\text{dist}(\eta_n(x), F_u(x)) \rightarrow 0 \text{ for a.e. } x \in \Omega.$$

By Mazur's lemma, there are finite convex combinations $\hat{\eta}_n = \sum_{j=n}^{k_n} \alpha_{n,j} \eta_j$, where $\alpha_{n,j} \geq 0$ and $\sum_{j=n}^{k_n} \alpha_{n,j} = 1$, such that $\hat{\eta}_n \rightarrow \eta$ in $L^{\min\{\gamma_1, d'\}}(\Omega)$. A further subsequence converges almost everywhere. Convexity of $F_u(x)$ yields

$$\text{dist}(\hat{\eta}_n(x), F_u(x)) \leq \sup_{j \geq n} \text{dist}(\eta_j(x), F_u(x)) \rightarrow 0.$$

Pointwise closedness therefore implies $\eta(x) \in F_u(x)$ almost everywhere. The growth bound (H3) at the limiting input shows that $\eta \in L^{\vartheta^-}(\Omega)$, so $\eta \in \mathcal{S}_f(u)$. The same argument on Γ_2 , using (H4), proves $\zeta \in \mathcal{S}_\Gamma(u)$.

The preceding selection bounds and the bound for $\mathcal{J}_R$ imply boundedness. Each value is nonempty, convex and weakly compact, hence closed. For $u^* = \mathcal{J}_R(u) - B_\eta + B_\zeta \in \mathcal{F}_R(u)$ and $\|u\| \geq D$, the calculation in Lemma 3.6 yields

$$\langle u^*, u \rangle \geq (a_* - \Lambda) \|u\|^{\vartheta^-} - K \|u\|.$$

Dividing by $\|u\|$ shows that the lower bound tends to $+\infty$ as $\|u\| \rightarrow \infty$. For $\|u\| < D$, boundedness on this ball gives $\langle u^*, u \rangle \geq -C_D \|u\|$. Combining the two estimates verifies the coercivity condition in Theorem 2.11 with $v_0 = 0$.

Let $u_n \rightharpoonup u$, choose $u_n^* = \mathcal{J}_R(u_n) - B_{\eta_n} + B_{\zeta_n} \in \mathcal{F}_R(u_n)$, and suppose $\limsup_n \langle u_n^*, u_n - u \rangle \leq 0$. If $Q = \max\{1, \sup_n \|u_n\|_{\hat{\rho}_\Phi}\} < \infty$, (H5)(ii) and Hölder's inequality give a constant $C_Q > 0$ such that

$$\begin{aligned} \int_{\Omega} |\eta_n| |u_n - u| \, dx &\leq C_Q (\|u_n - u\|_d + \|u_n - u\|_{\gamma_1'}), \\ \int_{\Gamma_2} |\zeta_n| |u_n - u| \, d\sigma &\leq C_Q (\|u_n - u\|_{e, \Gamma_2} + \|u_n - u\|_{\gamma_2', \Gamma_2}). \end{aligned}$$

The compact embeddings just established give

$$\int_{\Omega} \eta_n (u_n - u) \, dx - \int_{\Gamma_2} \zeta_n (u_n - u) \, d\sigma \rightarrow 0.$$

Consequently $\limsup_n \langle \mathcal{J}_R(u_n), u_n - u \rangle \leq 0$, and Proposition 3.9 implies $u_n \rightarrow u$ in V . Fix $w \in V$ and pass to a subsequence attaining $\liminf_n \langle u_n^*, u_n - w \rangle$. Extract selections $\eta_n \rightharpoonup \eta$ in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $\zeta_n \rightharpoonup \zeta$ in $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. The closed-graph property just proved implies $\eta \in \mathcal{S}_f(u)$, $\zeta \in \mathcal{S}_\Gamma(u)$. By continuity of $\mathcal{J}_R$ and strong convergence of the inputs,

$$\liminf_n \langle u_n^*, u_n - w \rangle = \langle \mathcal{J}_R(u) - B_\eta + B_\zeta, u - w \rangle.$$

This verifies condition (c) of pseudomonotonicity. Condition (a) was established above. To verify condition (b), let $E \subset V$ be a finite-dimensional subspace, $u \in$

E , and let $\mathcal{O} \subset V^*$ be weakly open with $\mathcal{F}_R(u) \subset \mathcal{O}$. If $\mathcal{F}_R|_E$ were not upper semicontinuous at u , there would exist sequences satisfying

$$u_n \in E, \quad \|u_n - u\| < \frac{1}{n}, \quad u_n^* \in \mathcal{F}_R(u_n) \setminus \mathcal{O}.$$

The inputs lie in a norm ball, so boundedness of $\mathcal{F}_R$ and reflexivity of V^* imply, after extraction, $u_n^* \rightharpoonup w^*$ in V^* . Write $u_n^* = \mathcal{J}_R(u_n) - B_{\eta_n} + B_{\zeta_n}$. The uniform selection bounds yield a further subsequence with $\eta_n \rightharpoonup \eta$ in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $\zeta_n \rightharpoonup \zeta$ in $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. Since $u_n \rightarrow u$ in V , the closed-graph property implies $\eta \in \mathcal{S}_f(u)$ and $\zeta \in \mathcal{S}_\Gamma(u)$. Continuity of $\mathcal{J}_R$ and the dual embeddings established above show, for every $v \in V$, that

$$\langle w^*, v \rangle = \lim_{n \rightarrow \infty} \langle u_n^*, v \rangle = \langle \mathcal{J}_R(u), v \rangle - \int_{\Omega} \eta v \, dx + \int_{\Gamma_2} \zeta v \, d\sigma.$$

Thus $w^* \in \mathcal{F}_R(u) \subset \mathcal{O}$. Weak convergence and weak openness of $\mathcal{O}$ imply $u_n^* \in \mathcal{O}$ for all sufficiently large n , contradicting the choice of u_n^* . Consequently, $\mathcal{F}_R|_E$ is upper semicontinuous from E with its norm topology to V^* with its weak topology. This verifies condition (b). $\square$

Theorem 3.11. *Under (H0)–(H5), (P2) has a weak solution $u \in V$, and every such solution satisfies $\|u\| \leq M_1$ and $\|u\|_{\infty} + \|u\|_{\infty, \Gamma_2} \leq M_2(M_1)$, independently of R .*

Proof. Proposition 3.10 and Theorem 2.11 yield a solution of $0 \in \mathcal{F}_R(u)$. The two estimates follow from Lemmas 3.6 and 3.7, applied to the auxiliary coefficient $h_R \geq a_0$. $\square$

We can now complete the proof of Theorem 3.3.

Proof of Theorem 3.3. Choose $R_3 \geq M_1$, $R_2 \geq C_k M_1$ and $R_1 \geq M_2(M_1)$, where C_k is the continuous embedding constant into $L^{k(\cdot)}(\Omega)$. Solve (P2) for these parameters. The estimates in Theorem 3.11 imply that all three truncations in (3.10) are inactive at this solution, so $h_R = h$ a.e. in its weak formulation. Thus it solves (P1), with the asserted bounds. $\square$

3.2. Constant exponents. We turn to the proof of Theorem 3.5.

Lemma 3.12. *Suppose $\vartheta(x) \equiv \tilde{\vartheta} > 1$, $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and assume (H0)–(H1) together with (H'3)–(H'5). Every weak solution $u \in V$ of (P1), or of its auxiliary problem with h_R , obeys $\|u\|_{\hat{\rho}_\Phi} \leq \tilde{M}_1$ for a constant $\tilde{M}_1 \geq 1$ depending on the problem data and the lower bound a_0 of h .*

Proof. Take a weak solution $u \in V$ of (P1). By choosing $v = u$ in (3.3) we arrive at

$$\int_{\Omega} h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\tilde{\vartheta}} + \mu(x) |\nabla u|^{\tilde{\kappa}} \right) dx = \int_{\Omega} \eta u \, dx - \int_{\Gamma_2} \zeta u \, d\sigma. \quad (3.12)$$

If $\|u\|_{\hat{\rho}_\Phi} \geq D$, hypotheses (H1), Proposition 2.6 and Remark 2.9 give

$$\int_{\Omega} h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \left(|\nabla u|^{\tilde{\vartheta}} + \mu(x) |\nabla u|^{\tilde{\kappa}} \right) dx \geq a_* \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}}. \quad (3.13)$$

By (H'5)(i), (3.12) and (3.13), we obtain

$$(a_* - \tilde{\Lambda}) \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}} \leq \tilde{K} \|u\|.$$

Hence,

$$\tilde{M} = \left(\frac{\tilde{K}}{a_* - \tilde{\Lambda}} \right)^{1/(\tilde{\vartheta}-1)}.$$

From this we conclude that $\|u\|_{\hat{\rho}_\Phi} \leq \tilde{M}$ whenever $\|u\|_{\hat{\rho}_\Phi} \geq D$. Setting $\tilde{M}_1 = \max\{D, 1, \tilde{M}\}$ completes the proof. As in Lemma 3.6, this estimate is also valid for the auxiliary coefficient h_R , independently of its truncation parameters. $\square$

Lemma 3.13. *Under the assumptions of Lemma 3.12, every such solution satisfies $\|u\|_\infty + \|u\|_{\infty, \Gamma_2} \leq M_2(\tilde{M}_1)$, independently of the truncation parameters.*

Proof. The assertion follows from Lemma 3.7 with $M = \tilde{M}_1$, since (H'5)(ii) supplies (H5)(ii) and its proof uses the $W^{1, \tilde{\vartheta}}$ embeddings and the lower bound a_0 . $\square$

Proposition 3.14. *Under (H0)–(H2) and (H'3)–(H'5), $\mathcal{F}_R: V \rightarrow 2^{V^*}$ is bounded, coercive and pseudomonotone.*

Proof. The selection sets are defined as above, now in $L^{d'_0}(\Omega)$ and $L^{e'_0}(\Gamma_2)$, where $d_0 = \max\{\tilde{\vartheta}, \tilde{r}_1\}$ and $e_0 = \max\{\tilde{\vartheta}, \tilde{r}_2\}$. For fixed u , the growth estimates imply bounded selections in these spaces: the u and ∇u terms have $L^{\tilde{\vartheta}'}$ bounds. The Tu and $\nabla(Tu)$ terms have $L^{\tilde{r}'_1}$ bounds, and the boundary terms have $L^{\tilde{\vartheta}'}$ or $L^{\tilde{r}'_2}$ bounds. On finite-measure sets these bounds imply the asserted $L^{d'_0}, L^{e'_0}$ bounds. On each norm ball, (H'5)(ii) also bounds the selections uniformly in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. The corresponding dual embeddings and boundedness of $\mathcal{J}_R$ therefore imply boundedness of $\mathcal{F}_R$. As above, (H2) and these growth bounds ensure nonempty, convex and weakly compact values, and the Mazur argument in Proposition 3.10 establishes their closed graph for strong convergence of inputs in V and weak convergence of selections in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. The estimate in Lemma 3.12 implies $\langle u^*, u \rangle \geq (a_* - \tilde{\Lambda})\|u\|^{\tilde{\vartheta}} - \tilde{K}\|u\|$ for $u^* \in \mathcal{F}_R(u)$ and $\|u\| \geq D$, proving coercivity. For a weakly convergent input sequence in the pseudomonotonicity condition, (H'5)(ii) yields uniform bounds in $L^{\min\{\gamma_1, d'\}}(\Omega)$ and $L^{\min\{\gamma_2, e'\}}(\Gamma_2)$. The compact embeddings into $L^{\max\{d, \gamma'_1\}}(\Omega)$ and $L^{\max\{e, \gamma'_2\}}(\Gamma_2)$ imply that both source pairings tend to zero, exactly as in Proposition 3.10. Proposition 3.9 then implies strong convergence of the inputs in V . Passing to weak limits of the selections verifies condition (c) as in Proposition 3.10. For condition (b), apply the weak-open-set contradiction from that proposition to a norm-convergent sequence in a finite-dimensional input space. The same uniform bounds and closed-graph argument identify the weak limit with an element of the value at the limiting input. $\square$

Proof of Theorem 3.5. Proposition 3.14 and Theorem 2.11 give an auxiliary solution for every R . Lemmas 3.12 and 3.13 provide bounds independent of R . Choose $R_3 \geq \tilde{M}_1$, $R_2 \geq C_k \tilde{M}_1$ and $R_1 \geq M_2(\tilde{M}_1)$. All truncations are then inactive, and the auxiliary solution solves the original problem. This completes the constant-exponent argument. $\square$

4. APPLICATIONS

Following Motreanu–Tornatore [35], we examine several choices of the intrinsic operator $T: V \rightarrow W^{1, \vartheta}(\Omega)$ and their applications. We verify the sufficient norm

conditions in Remarks 3.1 and 3.4 for each choice. We also verify (H5)(ii) from the concrete assumptions in each application. In the sufficient conditions below, γ_1, γ_2 satisfy the inequalities in (H5)(ii). All domain and trace embedding constants are those fixed in Remark 2.9, unless the full space $W^{1,\vartheta-}(\Omega)$ is specified.

Choosing the convolution operator $Tu = (\rho * Eu)|_\Omega$, where $\rho \in L^1(\mathbb{R}^N)$ and $E: W^{1,\vartheta-}(\Omega) \rightarrow W^{1,\vartheta-}(\mathbb{R}^N)$ is a bounded linear extension operator, gives the inclusion

$$\begin{aligned} -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u)] &\in f(x, u, \rho * Eu, \nabla u, \nabla(\rho * Eu)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, \rho * Eu) \quad \text{on } \Gamma_2. \end{aligned} \quad (4.1)$$

Fix $E_0 > 0$ so that $\|Eu\|_{W^{1,\vartheta-}(\mathbb{R}^N)} \leq E_0 \|u\|_{\hat{\rho}_\Phi}$, and put $L = E_0 \|\rho\|_{L^1(\mathbb{R}^N)}$. For the output of this operator, let S_r, S_{r,Γ_2} denote the embedding constants of the full space $W^{1,\vartheta-}(\Omega)$ into $L^r(\Omega)$ and $L^r(\Gamma_2)$.

Theorem 4.1. *Assume (H0)–(H4) and either $\vartheta_- > N - 1$, or $\rho \in L^{\vartheta'_-}(\mathbb{R}^N)$, $\sigma_1 \in L^{\gamma_1}(\Omega)$ and $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$. Suppose also that*

$$\begin{aligned} a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 L^{\vartheta_- - 1} C_{\vartheta_-} + a_3 G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-} + a_4 L^{\vartheta_- - 1} C_{\vartheta_-} \\ + b_1 C_{\vartheta_-, \Gamma_2}^{\vartheta_-} + b_2 (S_{\vartheta_-, \Gamma_2} L)^{\vartheta_- - 1} C_{\vartheta_-, \Gamma_2} < a_*. \end{aligned}$$

Under these conditions, problem (4.1) has a weak solution $u \in V$ that is uniformly bounded. Its trace on Γ_2 is also bounded.

Proof. To apply Theorem 3.3, we verify the sufficient conditions in Remark 3.1. The extension estimate reads

$$\|Eu\|_{W^{1,\vartheta-}(\mathbb{R}^N)} \leq E_0 \|u\|_{\hat{\rho}_\Phi} \quad \text{for all } u \in V.$$

Young's convolution inequality, including its derivative version, gives

$$\begin{aligned} \|Tu\|_{\vartheta_-, \Omega} &\leq \|\rho * Eu\|_{\vartheta_-, \mathbb{R}^N} \leq \|\rho\|_{1, \mathbb{R}^N} \|Eu\|_{\vartheta_-, \mathbb{R}^N} \leq L \|u\|_{\hat{\rho}_\Phi}, \\ \|\nabla(Tu)\|_{\vartheta_-, \Omega} &\leq \|\rho * \nabla(Eu)\|_{\vartheta_-, \mathbb{R}^N} \leq \|\rho\|_{1, \mathbb{R}^N} \|\nabla(Eu)\|_{\vartheta_-, \mathbb{R}^N} \leq L \|u\|_{\hat{\rho}_\Phi}. \end{aligned} \quad (4.2)$$

Using the full $W^{1,\vartheta-}$ norm as the sum of the function and gradient norms, the same argument also yields

$$\|Tu\|_{W^{1,\vartheta-}(\Omega)} \leq \|\rho\|_{1, \mathbb{R}^N} \|Eu\|_{W^{1,\vartheta-}(\mathbb{R}^N)} \leq L \|u\|_{\hat{\rho}_\Phi}.$$

For the boundary estimate, the trace embedding of the full space implies

$$\|Tu\|_{\vartheta_-, \Gamma_2} \leq S_{\vartheta_-, \Gamma_2} \|Tu\|_{W^{1,\vartheta-}(\Omega)} \leq S_{\vartheta_-, \Gamma_2} L \|u\|_{\hat{\rho}_\Phi}.$$

Thus the norm estimates in Remark 3.1 hold with

$$c_1 = L^{\vartheta_- - 1}, \quad c_2 = L^{\vartheta_- - 1}, \quad c_3 = 0, \quad c_4 = (S_{\vartheta_-, \Gamma_2} L)^{\vartheta_- - 1}.$$

The extension and convolution maps are continuous, so $T: V \rightarrow W^{1,\vartheta-}(\Omega)$ is continuous. The displayed smallness inequality is exactly (3.2) with these constants, so Remark 3.1 implies (H5)(i).

If $\vartheta_- > N - 1$, Remark 3.1 also verifies (H5)(ii). In the other case, the hypotheses on ρ, σ_1, σ_2 apply. Hölder's inequality and the derivative identity in (4.2) yield

$$\|Tu\|_{\infty, \Omega} + \|\nabla(Tu)\|_{\infty, \Omega} \leq \|\rho\|_{\vartheta'_-, \mathbb{R}^N} \|Eu\|_{W^{1,\vartheta-}(\mathbb{R}^N)} \leq E_0 \|\rho\|_{\vartheta'_-, \mathbb{R}^N} \|u\|_{\hat{\rho}_\Phi}.$$

The convolution has bounded continuous representatives, and the same bound applies to its trace. Hence (H3)–(H4) imply (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i + C_M$ for $i = 1, 2$ on every norm ball. Theorem 3.3 therefore applies. $\square$

In particular, a smooth compactly supported kernel and bounded source functions σ_1, σ_2 satisfy the second set of sufficient conditions.

Theorem 4.2. *Let $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and set $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Assume (H0)–(H2), (H'3)–(H'4) and either $\tilde{\vartheta} > N - 1$, or $\rho \in L^{\tilde{\vartheta}'}(\mathbb{R}^N)$, $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$ and $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$. Suppose also that*

$$\begin{aligned} & \tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 (S_{\tilde{\vartheta}*} L)^{\tilde{\vartheta}-1} C_{\frac{\tilde{\vartheta}*}{\tilde{\vartheta}* - \tilde{r}_1 + 1}}^{\tilde{\vartheta}-1} + \tilde{a}_3 G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} + \tilde{a}_4 L^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} \\ & + \tilde{b}_1 C_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 (S_{\tilde{\vartheta}, \Gamma_2} L)^{\tilde{\vartheta}-1} C_{\frac{\tilde{\vartheta}*}{\tilde{\vartheta}* - \tilde{r}_2 + 1}, \Gamma_2}^{\tilde{\vartheta}-1} < a_*. \end{aligned}$$

Under these conditions, problem (4.1) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. Theorem 3.5 reduces the proof to checking the sufficient conditions in Remark 3.4. For constant exponents, (4.2) yields

$$\|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}/\tilde{r}_1'} = \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} \leq L^{\tilde{\vartheta}-1} \|u\|_{\tilde{\rho}_\Phi}^{\tilde{\vartheta}-1}.$$

The critical domain embedding of the full space yields

$$\|Tu\|_{\tilde{\vartheta}*}^{\tilde{r}_1-1} = \|Tu\|_{\tilde{\vartheta}*}^{\tilde{\vartheta}-1} \leq S_{\tilde{\vartheta}*}^{\tilde{\vartheta}-1} \|Tu\|_{W^{1,\tilde{\vartheta}}(\Omega)}^{\tilde{\vartheta}-1} \leq (S_{\tilde{\vartheta}*} L)^{\tilde{\vartheta}-1} \|u\|_{\tilde{\rho}_\Phi}^{\tilde{\vartheta}-1}.$$

For the boundary estimate, the critical trace embedding implies

$$\|Tu\|_{\tilde{\vartheta}, \Gamma_2}^{\tilde{r}_2-1} = \|Tu\|_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}-1} \leq S_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}-1} \|Tu\|_{W^{1,\tilde{\vartheta}}(\Omega)}^{\tilde{\vartheta}-1} \leq (S_{\tilde{\vartheta}, \Gamma_2} L)^{\tilde{\vartheta}-1} \|u\|_{\tilde{\rho}_\Phi}^{\tilde{\vartheta}-1}.$$

Thus the norm estimates in Remark 3.4 hold with $\tilde{c}_1 = (S_{\tilde{\vartheta}*} L)^{\tilde{\vartheta}-1}$, $\tilde{c}_2 = L^{\tilde{\vartheta}-1}$, $\tilde{c}_3 = 0$ and $\tilde{c}_4 = (S_{\tilde{\vartheta}, \Gamma_2} L)^{\tilde{\vartheta}-1}$. The stated inequality (3.5) verifies (H'5)(i) by Remark 3.4. If $\tilde{\vartheta} > N - 1$, the same remark verifies (H'5)(ii). In the other case, the preceding convolution bound and $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$ verify (H'5)(ii) with $d = e = \tilde{\vartheta}$ and $\omega_{u,i} = \tilde{\sigma}_i + C_M$. Thus Theorem 3.5 applies. $\square$

Next, we consider the continuous map $Tu = u^\pm$ with $u^+ = \max\{0, u\}$ and $u^- = -\min\{0, u\}$ in problem (P1). For this choice, problem (P1) becomes

$$\begin{aligned} -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\tilde{\rho}_\Phi}) \mathcal{K}(\nabla u)] & \in f(x, u, u^\pm, \nabla u, \nabla(u^\pm)) \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} & \in f_{\Gamma_2}(x, u, u^\pm) \quad \text{on } \Gamma_2. \end{aligned} \tag{4.3}$$

The next existence statements apply to (4.3). We give the proof for $Tu = u^-$. The argument for $Tu = u^+$ is analogous.

Theorem 4.3. *Assume (H0)–(H4), $\sigma_1 \in L^{\gamma_1}(\Omega)$, $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$, and the inequality*

$$(a_1 + a_2) C_{\vartheta_-}^{\vartheta_-} + (a_3 + a_4) G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-}^{\vartheta_-} + (b_1 + b_2) C_{\vartheta_-, \Gamma_2}^{\vartheta_-} < a_*.$$

Under these conditions, problem (4.3) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. We verify the norm estimates in Remark 3.1. For all $u \in V$, we get

$$\begin{aligned} \|Tu\|_{\vartheta_-} &= \|u^-\|_{\vartheta_-} \leq \|u\|_{\vartheta_-} \leq C_{\vartheta_-} \|u\|_{\hat{\rho}_\Phi}, \\ \|\nabla(Tu)\|_{\vartheta_-} &= \|\nabla u^-\|_{\vartheta_-} \leq \|\nabla u\|_{\vartheta_-} \leq G_{\vartheta_-} \|u\|_{\hat{\rho}_\Phi}, \\ \|Tu\|_{\vartheta_-, \Gamma_2} &= \|u^-\|_{\vartheta_-, \Gamma_2} \leq \|u\|_{\vartheta_-, \Gamma_2} \leq C_{\vartheta_-, \Gamma_2} \|u\|_{\hat{\rho}_\Phi}. \end{aligned}$$

Therefore, the norm estimates in Remark 3.1 hold with

$$c_1 = C_{\vartheta_-}^{\vartheta_- - 1}, \quad c_2 = G_{\vartheta_-}^{\vartheta_- - 1}, \quad c_3 = 0, \quad c_4 = C_{\vartheta_-, \Gamma_2}^{\vartheta_- - 1}.$$

The truncation map is continuous in $W^{1, \vartheta_-}(\Omega)$ and preserves the zero trace on Γ_1 .

To check (H5)(ii), the pointwise bounds $|u^-| \leq |u|$ and $|\nabla u^-| \leq |\nabla u|$ give

$$\begin{aligned} \sup_{\eta \in f(x, u, u^-, \nabla u, \nabla u^-)} |\eta| &\leq \sigma_1(x) + (a_1 + a_2)|u|^{\vartheta_- - 1} + (a_3 + a_4)|\nabla u|^{\vartheta_- - 1}, \\ \sup_{\zeta \in f_{\Gamma_2}(x, u, u^-)} |\zeta| &\leq \sigma_2(x) + (b_1 + b_2)|u|^{\vartheta_- - 1}. \end{aligned}$$

These are (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i$ for $i = 1, 2$. The smallness condition is precisely (3.2) for the displayed constants. Remark 3.1 verifies (H5)(i), and Theorem 3.3 proves the assertion. $\square$

Theorem 4.4. *Let $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \overline{\Omega}$, and set $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Assume (H0)–(H2), (H'3)–(H'4), $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$, $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$, and the inequality*

$$\begin{aligned} &\tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 C_{\tilde{\vartheta}^*}^{\tilde{\vartheta} - 1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + (\tilde{a}_3 + \tilde{a}_4) G_{\tilde{\vartheta}}^{\tilde{\vartheta} - 1} C_{\tilde{\vartheta}} \\ &+ \tilde{b}_1 C_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 C_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta} - 1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_2 + 1}, \Gamma_2} < a_*. \end{aligned}$$

Under these conditions, problem (4.3) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. Again, the proof reduces to checking the sufficient conditions in Remark 3.4. For all $u \in V$, one can estimate that

$$\begin{aligned} \|Tu\|_{\tilde{\vartheta}^*}^{\tilde{r}_1 - 1} &= \|u^-\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta} - 1} \leq C_{\tilde{\vartheta}^*}^{\tilde{\vartheta} - 1} \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta} - 1}, \\ \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}/\tilde{r}_1'} &= \|\nabla u^-\|_{\tilde{\vartheta}}^{\tilde{\vartheta} - 1} \leq G_{\tilde{\vartheta}}^{\tilde{\vartheta} - 1} \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta} - 1}, \\ \|Tu\|_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{r}_2 - 1} &= \|u^-\|_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta} - 1} \leq C_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta} - 1} \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta} - 1}. \end{aligned}$$

Thus, we only need to choose $\tilde{c}_1 = C_{\tilde{\vartheta}^*}^{\tilde{\vartheta} - 1}$, $\tilde{c}_2 = G_{\tilde{\vartheta}}^{\tilde{\vartheta} - 1}$, $\tilde{c}_3 = 0$ and $\tilde{c}_4 = C_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta} - 1}$. The same pointwise estimates verify (H'5)(ii), because $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Remark 3.4 and (3.5) give (H'5)(i), and Theorem 3.5 applies. $\square$

Furthermore, let $Tu = (-\Delta_{\vartheta_-})^{-1} \ell(u)$ with $\ell: \mathbb{R} \rightarrow \mathbb{R}$ continuous and satisfying

$$|\ell(t)| \leq d_1 |t|^{\vartheta_- - 1} + d_2, \quad d_1, d_2 > 0.$$

Here Tu is the unique solution in $W_0^{1, \vartheta_-}(\Omega)$ of $-\Delta_{\vartheta_-}(Tu) = \ell(u)$ with zero Dirichlet data on the whole boundary. This specifies the inverse operator, whose values are

only required to belong to $W^{1,\vartheta_-}(\Omega)$. We consider the problem

$$\begin{aligned} & -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_{\Phi}}) \mathcal{K}(\nabla u)] \\ & \quad \in f(x, u, (-\Delta_{\vartheta_-})^{-1} \ell(u), \nabla u, \nabla((- \Delta_{\vartheta_-})^{-1} \ell(u))) \quad \text{in } \Omega, \\ & \quad u = 0 \quad \text{on } \Gamma_1, \\ & \quad -\frac{\partial u}{\partial n} \in f_{\Gamma_2}(x, u, (-\Delta_{\vartheta_-})^{-1} \ell(u)) \quad \text{on } \Gamma_2. \end{aligned} \quad (4.4)$$

The existence statements for (4.4) are as follows. Let $P_{\vartheta_-}, P_{(\vartheta_-)^*}$ denote the Dirichlet Poincaré and Sobolev constants, and put

$$A_1 = P_{\vartheta_-} d_1 C_{\vartheta_-}^{\vartheta_- - 1}, \quad A_0 = P_{\vartheta_-} d_2 |\Omega|^{1/\vartheta_-'}.$$

Theorem 4.5. *Assume (H0)–(H4) and either $\vartheta_- > N - 1$, or Ω has C^2 boundary, ℓ is bounded, $\sigma_1 \in L^{\gamma_1}(\Omega)$ and $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$. Suppose also that*

$$a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 P_{\vartheta_-}^{\vartheta_- - 1} A_1 C_{\vartheta_-} + a_3 G_{\vartheta_-}^{\vartheta_- - 1} C_{\vartheta_-} + a_4 A_1 C_{\vartheta_-} + b_1 C_{\vartheta_-, \Gamma_2}^{\vartheta_-} < a_*.$$

Under these conditions, problem (4.4) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. Since for a given $u \in V$, Tu solves $-\Delta_{\vartheta_-}(Tu) = \ell(u)$ in Ω , testing with Tu we get

$$\begin{aligned} \|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_-} &= \int_{\Omega} |\nabla(Tu)|^{\vartheta_-} dx = \int_{\Omega} \ell(u) Tu dx \\ &\leq (d_1 \|u\|_{\vartheta_-}^{\vartheta_- - 1} + d_2 |\Omega|^{1/\vartheta_-'}) \|Tu\|_{\vartheta_-} \\ &\leq (d_1 C_{\vartheta_-}^{\vartheta_- - 1} \|u\|_{\hat{\rho}_{\Phi}}^{\vartheta_- - 1} + d_2 |\Omega|^{1/\vartheta_-'}) P_{\vartheta_-} \|\nabla(Tu)\|_{\vartheta_-}. \end{aligned}$$

If the gradient norm is nonzero, divide by it. The resulting inequality is also true when it is zero. Thus

$$\|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- - 1} \leq A_1 \|u\|_{\hat{\rho}_{\Phi}}^{\vartheta_- - 1} + A_0.$$

The Dirichlet Poincaré inequality then implies

$$\|Tu\|_{\vartheta_-}^{\vartheta_- - 1} \leq P_{\vartheta_-}^{\vartheta_- - 1} \|\nabla(Tu)\|_{\vartheta_-}^{\vartheta_- - 1} \leq P_{\vartheta_-}^{\vartheta_- - 1} A_1 \|u\|_{\hat{\rho}_{\Phi}}^{\vartheta_- - 1} + P_{\vartheta_-}^{\vartheta_- - 1} A_0.$$

The trace of Tu vanishes on the whole boundary. Consequently the norm estimates in Remark 3.1 hold with

$$c_1 = P_{\vartheta_-}^{\vartheta_- - 1} A_1, \quad c_2 = A_1, \quad c_3 = \max\{1, P_{\vartheta_-}^{\vartheta_- - 1}\} A_0, \quad c_4 = 0.$$

The continuity and growth assumptions on ℓ imply continuity of $u \mapsto \ell(u)$ from L^{ϑ_-} to $L^{\vartheta_-'}$. To verify continuity of the Dirichlet inverse, let $F_n \rightarrow F$ in $L^{\vartheta_-'}$ and let $w_n, w \in W_0^{1,\vartheta_-}(\Omega)$ solve $-\Delta_{\vartheta_-} w_n = F_n$ and $-\Delta_{\vartheta_-} w = F$. Testing with w_n and using the Dirichlet Poincaré inequality gives

$$\|\nabla w_n\|_{\vartheta_-}^{\vartheta_- - 1} \leq P_{\vartheta_-} \|F_n\|_{\vartheta_-'},$$

where the estimate is also valid if the gradient vanishes. Hence a subsequence converges weakly to some w_0 in $W_0^{1,\vartheta_-}(\Omega)$. Writing A_{ϑ_-} for the Dirichlet ϑ_- -Laplacian, we have

$$\langle A_{\vartheta_-}(w_n), w_n - w_0 \rangle = \int_{\Omega} (F_n - F)(w_n - w_0) dx + \int_{\Omega} F(w_n - w_0) dx \rightarrow 0.$$

The (S_+) property of A_{ϑ_-} implies $w_n \rightarrow w_0$ strongly in $W_0^{1,\vartheta_-}(\Omega)$. Passing to the equation gives $-\Delta_{\vartheta_-} w_0 = F$, and uniqueness implies $w_0 = w$. The subsequence principle yields $w_n \rightarrow w$ in $W_0^{1,\vartheta_-}(\Omega)$ for the full sequence, proving continuity of the inverse. Hence $T: V \rightarrow W^{1,\vartheta_-}(\Omega)$ is continuous. The smallness inequality is (3.2) for these constants, so Remark 3.1 implies (H5)(i).

If $\vartheta_- > N - 1$, the verified norm estimates imply (H5)(ii) by Remark 3.1. In the other case, Ω has C^2 boundary and ℓ is bounded. Put $B = \|\ell\|_\infty$ and choose a ball $B_R(x_0)$ containing $\bar{\Omega}$. Its ϑ_- -Laplacian torsion function is

$$\tau(x) = \frac{R^{\vartheta'_-} - |x - x_0|^{\vartheta'_-}}{\vartheta'_- N^{1/(\vartheta_- - 1)}}, \quad -\Delta_{\vartheta_-} \tau = 1 \quad \text{in } B_R(x_0).$$

Since $|\ell(u)| \leq B$, weak comparison with $\pm B^{1/(\vartheta_- - 1)}\tau$ gives

$$|Tu| \leq B^{1/(\vartheta_- - 1)}\tau \quad \text{in } \Omega, \quad \|Tu\|_\infty \leq C(\Omega, \vartheta_-)B^{1/(\vartheta_- - 1)}.$$

If $B = 0$, uniqueness gives $Tu = 0$. Combining this solution bound with the global Dirichlet boundary regularity estimate for bounded forcing yields

$$\|Tu\|_{W^{1,\infty}(\Omega)} \leq C(\Omega, \vartheta_-, \|\ell\|_\infty),$$

see Lieberman [25]. Substituting these bounds in (H3)–(H4) proves (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i + C_M$ for $i = 1, 2$. Theorem 3.3 therefore applies. $\square$

For example, $\ell(t) = \tanh t$ and bounded σ_1, σ_2 satisfy the bounded-forcing assumptions on a C^2 domain.

Note that if $\vartheta(x) = \tilde{\vartheta}$ for all $x \in \bar{\Omega}$, then $\vartheta_- = \tilde{\vartheta}$.

Theorem 4.6. *Let $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \bar{\Omega}$, and set $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Assume (H0)–(H2), (H'3)–(H'4) and either $\tilde{\vartheta} > N - 1$, or Ω has C^2 boundary, ℓ is bounded, $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$ and $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$. Suppose also that*

$$\tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_1 C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + \tilde{a}_3 G_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} C_{\tilde{\vartheta}} + \tilde{a}_4 A_1 C_{\tilde{\vartheta}} + \tilde{b}_1 C_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}} < a_*.$$

Under these conditions, problem (4.4) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. The argument used for Theorem 4.5 yields the following estimates for every $u \in V$:

$$\begin{aligned} \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} &\leq A_1 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + A_0, \\ \|Tu\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} &\leq P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta}-1} \leq P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_1 \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta}-1} + P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_0, \\ \|Tu\|_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta}-1} &= 0. \end{aligned}$$

Therefore the norm estimates in Remark 3.4 hold with $\tilde{c}_1 = P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1} A_1$, $\tilde{c}_2 = A_1$, $\tilde{c}_3 = \max\{1, P_{\tilde{\vartheta}^*}^{\tilde{\vartheta}-1}\} A_0$ and $\tilde{c}_4 = 0$. The assumed inequality (3.5) verifies (H'5)(i) by Remark 3.4. If $\tilde{\vartheta} > N - 1$, that remark also verifies (H'5)(ii). In the other case, the bounded-forcing regularity argument just proved, together with $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$, verifies (H'5)(ii) with $d = e = \tilde{\vartheta}$ and $\omega_{u,i} = \tilde{\sigma}_i + C_M$. Theorem 3.5 applies in both cases. $\square$

Finally, for $Tu = gu$ with $g \in W^{1,N}(\Omega) \cap L^\infty(\Omega)$ we consider the problem

$$\begin{aligned} -\operatorname{div} [h(u, \|u\|_{k(\cdot)}, \|u\|_{\hat{\rho}_\Phi}) \mathcal{K}(\nabla u)] &\in f(x, u, gu) && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial n} &\in f_{\Gamma_2}(x, u, gu) && \text{on } \Gamma_2. \end{aligned} \quad (4.5)$$

The source $f(x, u, gu)$ is independent of the gradient arguments. Thus $a_3 = a_4 = 0$ and $\tilde{a}_3 = \tilde{a}_4 = 0$ in the corresponding growth bounds. Write $G = \|g\|_\infty$, $G_\Gamma = \|g\|_{\infty, \Gamma_2}$ and $G_1 = \|\nabla g\|_N$. Since $1/\vartheta_- = 1/(\vartheta_-)^* + 1/N$, Hölder's inequality controls the product $u\nabla g$ in L^{ϑ_-} . The trace of g is bounded by its essential supremum in Ω .

Theorem 4.7. *Assume (H0)–(H4), $\sigma_1 \in L^{\gamma_1}(\Omega)$, $\sigma_2 \in L^{\gamma_2}(\Gamma_2)$, and the inequality*

$$a_1 C_{\vartheta_-}^{\vartheta_-} + a_2 (GC_{\vartheta_-})^{\vartheta_- - 1} C_{\vartheta_-} + b_1 C_{\vartheta_-, \Gamma_2}^{\vartheta_-} + b_2 (G_\Gamma C_{\vartheta_-, \Gamma_2})^{\vartheta_- - 1} C_{\vartheta_-, \Gamma_2} < a_*,$$

Under these conditions, problem (4.5) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. For $u \in V$, we calculate that

$$\begin{aligned} \|Tu\|_{\vartheta_-} &= \|gu\|_{\vartheta_-} \leq \|g\|_\infty \|u\|_{\vartheta_-} \leq GC_{\vartheta_-} \|u\|_{\hat{\rho}_\Phi}, \\ \|\nabla(Tu)\|_{\vartheta_-} &= \|g\nabla u + u\nabla g\|_{\vartheta_-} \leq G\|\nabla u\|_{\vartheta_-} + G_1\|u\|_{(\vartheta_-)^*} \\ &\leq (GG_{\vartheta_-} + G_1 C_{(\vartheta_-)^*}) \|u\|_{\hat{\rho}_\Phi}, \\ \|Tu\|_{\vartheta_-, \Gamma_2} &= \|(g)(u)\|_{\vartheta_-, \Gamma_2} \leq G_\Gamma \|u\|_{\vartheta_-, \Gamma_2} \leq G_\Gamma C_{\vartheta_-, \Gamma_2} \|u\|_{\hat{\rho}_\Phi}. \end{aligned}$$

Then, the norm estimates in Remark 3.1 hold with

$$\begin{aligned} c_1 &= (GC_{\vartheta_-})^{\vartheta_- - 1}, & c_2 &= (GG_{\vartheta_-} + G_1 C_{(\vartheta_-)^*})^{\vartheta_- - 1}, \\ c_3 &= 0, & c_4 &= (G_\Gamma C_{\vartheta_-, \Gamma_2})^{\vartheta_- - 1}. \end{aligned}$$

The product rule also proves continuity of the multiplication map into $W^{1, \vartheta_-}(\Omega)$. Finally, the pointwise relations $|gu| \leq G|u|$ and the boundary relation $|gu| \leq G_\Gamma|u|$ give (H5)(ii) with $d = e = \vartheta_-$ and $\omega_{u,i} = \sigma_i$ for $i = 1, 2$. Remark 3.1 and (3.2) give (H5)(i), so Theorem 3.3 applies. $\square$

Theorem 4.8. *Let $\vartheta(x) \equiv \tilde{\vartheta} > 1$ and $\kappa(x) \equiv \tilde{\kappa} > 1$ for all $x \in \bar{\Omega}$, and set $\tilde{r}_1 = \tilde{r}_2 = \tilde{\vartheta}$. Assume (H0)–(H2), (H'3)–(H'4), $\tilde{\sigma}_1 \in L^{\gamma_1}(\Omega)$, $\tilde{\sigma}_2 \in L^{\gamma_2}(\Gamma_2)$, and the inequality*

$$\tilde{a}_1 C_{\tilde{\vartheta}}^{\tilde{\vartheta}} + \tilde{a}_2 (GC_{\tilde{\vartheta}^*})^{\tilde{\vartheta} - 1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_1 + 1}} + \tilde{b}_1 C_{\tilde{\vartheta}, \Gamma_2}^{\tilde{\vartheta}} + \tilde{b}_2 (G_\Gamma C_{\tilde{\vartheta}^*, \Gamma_2})^{\tilde{\vartheta} - 1} C_{\frac{\tilde{\vartheta}^*}{\tilde{\vartheta}^* - \tilde{r}_2 + 1}, \Gamma_2} < a_*.$$

Under these conditions, problem (4.5) has a weak solution $u \in V$ that is uniformly bounded. Its trace is also bounded.

Proof. For $u \in V$, we calculate that

$$\begin{aligned} \|Tu\|_{\tilde{\vartheta}^*}^{\tilde{r}_1 - 1} &= \|gu\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta} - 1} \leq G^{\tilde{\vartheta} - 1} \|u\|_{\tilde{\vartheta}^*}^{\tilde{\vartheta} - 1} \leq (GC_{\tilde{\vartheta}^*})^{\tilde{\vartheta} - 1} \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta} - 1}, \\ \|\nabla(Tu)\|_{\tilde{\vartheta}}^{\tilde{r}_1' - 1} &= \|\nabla(gu)\|_{\tilde{\vartheta}}^{\tilde{\vartheta} - 1} \leq (GG_{\tilde{\vartheta}} + G_1 C_{\tilde{\vartheta}^*})^{\tilde{\vartheta} - 1} \|u\|_{\hat{\rho}_\Phi}^{\tilde{\vartheta} - 1}, \\ \|Tu\|_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{r}_2 - 1} &= \|(gu)\|_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta} - 1} \\ &\leq G_\Gamma^{\tilde{\vartheta} - 1} \|u\|_{\tilde{\vartheta}^*, \Gamma_2}^{\tilde{\vartheta} - 1} \end{aligned}$$

$$\leq (G_\Gamma C_{\tilde{\vartheta}_*, \Gamma_2})^{\tilde{\vartheta}-1} \|u\|_{\tilde{\rho}_\Phi}^{\tilde{\vartheta}-1}.$$

Then, the norm estimates in Remark 3.4 hold with $\tilde{c}_1 = (GC_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1}$, $\tilde{c}_2 = (GG_{\tilde{\vartheta}} + G_1 C_{\tilde{\vartheta}^*})^{\tilde{\vartheta}-1}$, $\tilde{c}_3 = 0$ and $\tilde{c}_4 = (G_\Gamma C_{\tilde{\vartheta}_*, \Gamma_2})^{\tilde{\vartheta}-1}$. As before, the absence of gradient arguments in f , the pointwise product bounds and the stated integrability of the source functions imply (H'5)(ii) with $d = e = \tilde{\vartheta}$. The inequality (3.5) verifies (H'5)(i) by Remark 3.4. Theorem 3.5 concludes the proof. $\square$

REFERENCES

- [1] E. Amoroso, Á. Crespo-Blanco, P. Pucci, P. Winkert, *Superlinear elliptic equations with unbalanced growth and nonlinear boundary condition*, Bull. Sci. Math. **197** (2024), Paper No. 103534, 44 pp.
- [2] R. Arora, Á. Crespo-Blanco, P. Winkert, *Logarithmic double phase problems with generalized critical growth*, NoDEA Nonlinear Differential Equations Appl. **32** (2025), no. 5, Paper No. 98, 74 pp.
- [3] R. Arora, Á. Crespo-Blanco, P. Winkert, *On logarithmic double phase problems*, J. Differential Equations **433** (2025), Paper No. 113247, 60 pp.
- [4] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves*, Nonlinearity **32** (2019), no. 7, 2481–2495.
- [5] P. Baroni, M. Colombo, G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121** (2015), 206–222.
- [6] P. Baroni, M. Colombo, G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Art. 62, 48 pp.
- [7] L. Beck, G. Mingione, *Lipschitz bounds and nonuniform ellipticity*, Comm. Pure Appl. Math. **73** (2020), no. 5, 944–1034.
- [8] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal. **154** (2000), no. 4, 297–324.
- [9] S.-S. Byun, J. Ok, K. Song, *Hölder regularity for weak solutions to nonlocal double phase problems*, J. Math. Pures Appl. (9) **168** (2022), 110–142.
- [10] S. Carl, V.K. Le, “Multi-Valued Variational Inequalities and Inclusions”, Springer, Cham, 2021.
- [11] J. Cen, N. Costea, *Weak solvability for a class of double phase variable exponents inclusion problems*, Commun. Nonlinear Sci. Numer. Simul. **144** (2025), Paper No. 108664, 21 pp.
- [12] A. Charkaoui, A. Ben-Loghfry, S. Zeng, *Nonlinear parabolic double phase variable exponent systems with applications in image noise removal*, Appl. Math. Model. **132** (2024), 495–530.
- [13] L. Cherfils, Y. Il'yasov, *On the stationary solutions of generalized reaction diffusion equations with p - q -Laplacian*, Commun. Pure Appl. Anal. **4** (2005), no. 1, 9–22.
- [14] M. Colombo, G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218** (2015), no. 1, 219–273.
- [15] M. Colombo, G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.
- [16] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, J. Differential Equations **323** (2022), 182–228.
- [17] J. Diblík, M. Galewski, I. Kossowski, D. Motreanu, *On competing (p, q) -Laplacian Dirichlet problem with unbounded weight*, Differential Integral Equations **38** (2025), no. 1-2, 23–42.
- [18] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, “Lebesgue and Sobolev Spaces with Variable Exponents”, Springer, Heidelberg, 2011.
- [19] X. Fan, *Differential equations of divergence form in Musielak-Sobolev spaces and a sub-supersolution method*, J. Math. Anal. Appl. **386** (2012), no. 2, 593–604.
- [20] M. Fuchs, G. Mingione, *Full $C^{1,\alpha}$ -regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth*, Manuscripta Math. **102** (2000), no. 2, 227–250.

- [21] M. Galewski, D. Motreanu, *On variational approach to fourth-order problems with unbounded weight*, Math. Methods Appl. Sci. **46** (2023), no. 18, 18579–18591.
- [22] L. Gasiński, N.S. Papageorgiou, *Constant sign and nodal solutions for superlinear double phase problems*, Adv. Calc. Var. **14** (2021), no. 4, 613–626.
- [23] L. Gasiński, P. Winkert, *Existence and uniqueness results for double phase problems with convection term*, J. Differential Equations **268** (2020), no. 8, 4183–4193.
- [24] P. Harjulehto, P. Hästö, *Double phase image restoration*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 123832, 12 pp.
- [25] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.
- [26] W. Liu, G. Dai, *Existence and multiplicity results for double phase problem*, J. Differential Equations **265** (2018), no. 9, 4311–4334.
- [27] Y. Lu, C. Vetro, S. Zeng, *A class of double phase variable exponent energy functionals with different power growth and logarithmic perturbation*, Discrete Contin. Dyn. Syst. Ser. S **22** (2026), 85–121.
- [28] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q -growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.
- [29] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284.
- [30] A.H.S. Medeiros, D. Motreanu, *A problem involving competing and intrinsic operators*, São Paulo J. Math. Sci. **18** (2024), no. 1, 300–311.
- [31] J. Musielak, “Orlicz Spaces and Modular Spaces”, Springer-Verlag, Berlin, 1983.
- [32] D. Motreanu, *Nonhomogeneous Dirichlet problems with unbounded coefficient in the principal part*, Axioms **11** (2022), no. 12, Paper No. 739.
- [33] D. Motreanu, V.V. Motreanu, *Nonstandard Dirichlet problems with competing (p, q) -Laplacian, convection, and convolution*, Stud. Univ. Babeş-Bolyai Math. **66** (2021), no. 1, 95–103.
- [34] D. Motreanu, V.V. Motreanu, *(p, q) -Laplacian equations with convection term and a intrinsic operator*, in: Differential and Integral Inequalities, Springer Optim. Appl. **151**, Springer, Cham, 2019, 589–601.
- [35] D. Motreanu, E. Tornatore, *Elliptic equations with unbounded coefficient, convection term and intrinsic operator*, Math. Z. **308** (2024), no. 2, Paper No. 38, 13 pp.
- [36] D. Motreanu, E. Tornatore, *Nonhomogeneous degenerate quasilinear problems with convection*, Nonlinear Anal. Real World Appl. **71** (2023), Paper No. 103800, 14 pp.
- [37] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Double-phase problems and a discontinuity property of the spectrum*, Proc. Amer. Math. Soc. **147** (2019), no. 7, 2899–2910.
- [38] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Ground state and nodal solutions for a class of double phase problems*, Z. Angew. Math. Phys. **71** (2020), no. 1, Paper No. 15, 15 pp.
- [39] N.S. Papageorgiou, C. Vetro, F. Vetro, *Multiple solutions for parametric double phase Dirichlet problems*, Commun. Contemp. Math. **23** (2021), no. 4, Paper No. 2050006, 18 pp.
- [40] K. Perera, M. Squassina, *Existence results for double-phase problems via Morse theory*, Commun. Contemp. Math. **20** (2018), no. 2, 1750023, 14 pp.
- [41] H. Prasad, V. Tewary, *Local boundedness of variational solutions to nonlocal double phase parabolic equations*, J. Differential Equations **351** (2023), 243–276.
- [42] C. Vetro, S. Zeng, *Regularity and Dirichlet problem for double-phase energy functionals of different power growth*, J. Geom. Anal. **34** (2024), no. 4, Paper No. 105, 27 pp.
- [43] S. Zeng, Y. Bai, L. Gasiński, P. Winkert, *Existence results for double phase implicit obstacle problems involving multivalued operators*, Calc. Var. Partial Differential Equations **59** (2020), no. 5, Paper No. 176, 18 pp.
- [44] S. Zeng, Y. Lu, V.D. Rădulescu, P. Winkert, *Anisotropic nonlocal double phase problems with logarithmic perturbation: maximum principle and qualitative analysis of solutions*, Partial Differ. Equ. Appl. **7** (2026), no. 1, Paper No. 11, 46 pp.
- [45] S. Zeng, V.D. Rădulescu, P. Winkert, *Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions*, SIAM J. Math. Anal. **54** (2022), no. 2, 1898–1926.
- [46] S. Zeng, V.D. Rădulescu, P. Winkert, *Nonlocal double phase implicit obstacle problems with multivalued boundary conditions*, SIAM J. Math. Anal. **56** (2024), no. 1, 877–912.

- [47] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.
- [48] V.V. Zhikov, *On Lavrentiev’s phenomenon*, Russian J. Math. Phys. **3** (1995), no. 2, 249–269.

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