EXISTENCE AND MULTIPLICITY OF SIGN-CHANGING SOLUTIONS FOR THE KLEIN-GORDON-BORN-INFELD SYSTEM

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Abstract. In this paper, we study the following nonhomogeneous Klein-Gordon equation coupled with the Born-Infeld theory:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \mathbb{R}^3. \end{cases}$$

Here, $\omega > 0$ is a constant frequency. Under suitable assumptions on f and V, we prove the existence of sign-changing solutions for this system by using the method of invariant sets of descending flow. Moreover, when f is odd with respect to u, the system admits a sequence of sign-changing solutions whose energies tend to infinity.

1. Introduction

In this paper, we are concerned with the Klein-Gordon equation coupled with Born-Infeld theory

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\omega > 0$ is a constant, $\Delta_4\phi = \operatorname{div}(|\nabla\phi|^2\nabla\phi)$, $V \in C(\mathbb{R}^3,\mathbb{R})$, and $f \in C(\mathbb{R}^3 \times \mathbb{R},\mathbb{R})$. Here, the functions V and f represent suitable nonlinearities. This system represents a coupled model of the Klein-Gordon equation and the Born-Infeld theory, describing the interaction between a scalar matter field and a non-linear electromagnetic field. Specifically, the Klein-Gordon equation is a classical field equation governing spin-0 scalar particles:

$$\psi_{tt} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0, \tag{1.2}$$

where $\psi = \psi(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, m is a real constant, and $p \in [2, 2^*)$. Indeed, (1.2) is the Euler-Lagrange equation with respect to the Lagrangian density

$$\mathcal{L}_{\text{NLKG}} = \frac{1}{2}\left[\left|\frac{\partial\psi}{\partial t}\right|^2 - |\nabla\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

In general, the electromagnetic field is described by the gauge potential (ϕ, A), where $\phi\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ and $A\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The corresponding electric field is given by

$$E = -\nabla\phi - A_t$$

2020 *Mathematics Subject Classification.* 35A15, 35J20, 35J50, 35J60, 78A30.

Key words and phrases. Born-Infeld theory, existence and multiplicity of solutions, invariant sets of descending flow, Klein-Gordon equation, sign-changing solutions, variational methods.

and the magnetic induction field by

$$B = \nabla \times A.$$

Assume that ψ is a charged field and let e denote the electric charge. By gauge invariance arguments, the interaction between ψ and the electromagnetic field can be described as

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + ie\phi, \quad \nabla \mapsto \nabla - ieA.$$

Then, we obtain the following Lagrangian density:

$$\mathcal{L}_0 = \frac{1}{2}\left[\left|\frac{\partial \psi}{\partial t} + ie\phi\psi\right|^2 - |\nabla\psi - ieA\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

The Born-Infeld theory is a nonlinear electromagnetic theory proposed by Born–Infeld [2] to resolve the divergence of the energy of point charges in classical electrodynamics. Its Lagrangian avoids divergence of the field strength by incorporating a nonlinear term. The Lagrangian density of the Born-Infeld theory can be written as

$$\mathcal{L}_{BI} = \frac{b^2}{4\pi}\left\{1 - \sqrt{1 - \frac{1}{b^2}\left(|E|^2 - |B|^2\right)}\right\},$$

where $b \gg 1$ is the Born-Infeld parameter and, as $b \to \infty$, it reduces to the classical Maxwell theory. As in d'Avenia–Pisani [7] and Fortunato–Orsina–Pisani [9], let $\beta = \frac{1}{2b^2}$ and consider the second order expansion of $\mathcal{L}_{BI}$ as $\beta \to 0^+$. Then the Lagrangian density takes the form

$$\mathcal{L}_{BI,1} = \frac{1}{4\pi}\left[\frac{1}{2}\left(|E|^2 - |B|^2\right) + \frac{1}{4}\beta\left(|E|^2 - |B|^2\right)^2\right].$$

The total action of the nonlinear Klein-Gordon-Born-Infeld system is given by

$$\mathcal{S} = \iint (\mathcal{L}_0 + \mathcal{L}_{BI,1})\,\mathrm{d}x\,\mathrm{d}t.$$

Under the electrostatic solitary wave ansatz

$$\psi(x,t) = u(x)e^{i\omega t}, \quad \phi = \phi(x), \quad A = 0,$$

and taking $e = 1$, where $u, \phi \colon \mathbb{R}^3 \to \mathbb{R}$ and ω is a positive frequency parameter, the total action leads to the system

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

System (1.3) coincides with (1.1) when $V(x) \equiv m^2 - \omega^2$ and $f(x,u) = |u|^{p-2}u$.

In recent years, several authors have focused on the system (1.3). Fortunato–Orsina–Pisani [9] first proposed this model and established the existence of infinitely many radially symmetric solutions for $|\omega| < |m|$ and $p \in (4,6)$. Mugnai [15] extended this result to the range $p \in (2,4]$, under the additional condition $0 < \omega <$

$\sqrt{\frac{1}{2}p-1}\,m$. Furthermore, Yu [21] studied the model

$$\begin{cases} \nabla \cdot \dfrac{\nabla \phi}{\sqrt{1-\dfrac{1}{b^2}|\nabla \phi|^2}} = u^2(\omega+\phi), \\ \Delta u = \left(m^2 - (\omega+\phi)^2\right) u - |u|^{p-2}u, \end{cases} \tag{1.4}$$

establishing the existence of solutions either on a smooth bounded domain or in $\mathbb{R}^3$, and analyzing their asymptotic behavior.

Teng–Zhang [18] investigated the case where the nonlinear terms in both (1.3) and (1.4) are of the form $\frac{1}{p}|u|^p - \frac{1}{2^*}|u|^{2^*}$ with $p \in (4, 2^*)$, and established the existence of a solitary wave solution under suitable parameter conditions. Building on this result, He–Li–Chen–O'Regan [10] employed a variant of the monotonicity trick introduced by Jeanjean [11] and Struwe [16]. They extended the admissible range of p to $(2, 6)$, and showed that the corresponding solutions are ground state solutions. In a different direction, Teng [17] studied problem (1.3) with a general nonlinear term $f(x, u)$ on a smooth bounded domain, proving the existence and multiplicity of solutions. Subsequently, Che–Chen [3] established the existence of infinitely many negative energy solutions for the system (1.1) with sublinear nonlinearity, by exploiting genus properties in critical point theory. Under weaker assumptions, Wen–Tang–Chen [20] proved the existence of infinitely many solutions of the system (1.1) without relying on the Ambrosetti Rabinowitz condition ((AR)-condition for short) by means of the symmetric mountain pass theorem. Moreover, under the general (AR)-condition and additional assumptions on ω, a ground state solution was also obtained. Furthermore, Chen–Li [4] introduced a perturbation term $h(x)$ into the system (1.3) and proved the existence of two radially symmetric solutions for $p \in (2, 6)$ by applying Ekeland's variational principle and the mountain pass theorem. Further results on Klein-Gordon equations coupled with Born-Infeld theory can be found in Chen–Song [5], Fei–Zhang [8], Liao–Li–Chen [12], Wang–Xiong–Zhang [19], and the references therein.

To the best of our knowledge, the only available results concerning the existence and multiplicity of sign-changing solutions for the system (1.3) are due to Zhang–Liu [23], who studied this system under the assumptions that $0 < \omega < \sqrt{\frac{p}{2}-1}|m|$ with $2 < p < 4$, or $0 < \omega < |m|$ with $4 \le p < 6$, and established the existence and multiplicity of sign-changing solutions by employing the method of invariant sets of descending flow.

Motivated by the above works, and inspired by the papers of Bartsch–Liu [1], Liu–Liu–Wang [13], Liu–Wang–Zhang [14], and Zhang [22], in this paper, we investigate the existence and multiplicity of sign-changing solutions to the system (1.1). Recall that a solution (u, ϕ) to (1.1) is called a sign-changing solution if u changes its sign.

Formally, under suitable assumptions on V and f, the system (1.1) admits a variational structure. More precisely, solutions of (1.1) correspond to critical points of the functional $\mathcal{F}\colon H^1(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{F}(u, \phi) = &\frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - (2\omega+\phi)\phi u^2\right) \mathrm{d}x \\ &- \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi|^2 \,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi|^4 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x, u)\,\mathrm{d}x, \end{aligned}$$

where $F(x,t)=\int_0^t f(x,s)\,\mathrm{d}s$, $H^1(\mathbb{R}^3)$ denotes the classical Sobolev space endowed with the norm

$$\|u\|_{H^1}:=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2\right)\,\mathrm{d}x\right)^{\frac{1}{2}},$$

and $\mathcal{D}(\mathbb{R}^3)$ is defined as the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R})$ with respect to the norm

$$\|\phi\|_{\mathcal{D}}=\left(\int_{\mathbb{R}^3}|\nabla\phi|^2\,\mathrm{d}x\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^3}|\nabla\phi|^4\,\mathrm{d}x\right)^{\frac{1}{4}}.$$

However, variational methods cannot be applied directly because the functional is strongly indefinite on $H^1(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)$. This difficulty can be overcome by means of the classical reduction method, which was successfully employed by Chen–Li [4]. The key idea is to establish the relationship between solutions of the associated minimization problem and solutions of the second equation in the system (1.1), with u fixed. According to d'Avenia–Pisani [7], for any fixed $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u\in\mathcal{D}(\mathbb{R}^3)$ solving the second equation of the system (1.1). Consequently, the problem reduces to finding sign-changing critical points of the functional $u\mapsto\mathcal{F}(u,\phi_u)$, see Section 2 for more details.

Furthermore, it is worth emphasizing that compactness fails, since the embedding $H^1(\mathbb{R}^3)\hookrightarrow L^q(\mathbb{R}^3)$ for $q\in[2,6]$, is only continuous and not compact. To overcome this lack of compactness, we restrict our analysis to the space

$$H=\left\{u\in H^1(\mathbb{R}^3)\colon\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)\,\mathrm{d}x<+\infty\right\}.$$

This space is a Hilbert space endowed with the inner product and associated norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+V(x)uv)\,\mathrm{d}x,\quad \|u\|=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)\,\mathrm{d}x\right)^{\frac{1}{2}},$$

respectively. We impose the following assumptions.

(V) $V\in C(\mathbb{R}^3,\mathbb{R})$, $\inf_{x\in\mathbb{R}^3}V(x)=V_0>0$ and there exists a constant $r>0$ such that, for any $M>0$,

$$\lim_{|y|\to\infty}\left|\left\{x\in\mathbb{R}^3\colon|x-y|\le r,V(x)\le M\right\}\right|=0,$$

where $|\cdot|$ denotes the Lebesgue measure.

(F_1) $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ and $\lim\limits_{t\to0}\frac{f(x,t)}{t}=0$ uniformly in $x\in\mathbb{R}^3$.

(F_2) $\limsup\limits_{|t|\to+\infty}\frac{|f(x,t)|}{|t|^{p-1}}<\infty$ uniformly in $x\in\mathbb{R}^3$ for some $p\in(2,6)$.

(F_3) There exists $\mu_1>2$ such that $tf(x,t)\ge\mu_1F(x,t)>0$ for all $t\neq0$, where $F(x,t)=\int_0^t f(x,s)\,\mathrm{d}s$.

(F_4) $f(x,t)=-f(x,-t)$ for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}$.

The following theorem is our first main result.

Theorem 1.1. *Assume that* (V) *and* (F_1)–(F_3) *hold. If* $\mu_1\ge4$ *or* $\mu_1\in(2,4)$ *and* $0<\omega<\left(\frac{\mu_1-2}{4-\mu_1}V_0\right)^{1/2}$, *then the system* (1.1) *possesses at least one sign-changing solution. Moreover, if* (F_4) *also holds, then system* (1.1) *admits infinitely many sign-changing solutions* $\{u_j,\phi_{u_j}\}$ *such that* $\mathcal{F}(u_j,\phi_{u_j})\to\infty$ *as* $j\to\infty$.

In particular, our work can be seen as an improvement of Zhang–Liu [23], in which infinitely many sign-changing solutions for (1.1) were obtained with $V(x) \equiv m^2 - \omega^2$ and $f(x,u) = |u|^{p-2}u$ with $2 < p < 6$. Condition (F_3) is the classical (AR)-condition, which ensures the compactness property. In many works on multiple sign changing solutions, this condition also provides a lower bound for the functional $F(x,u)$ and is therefore crucial, see, for instance, Liu–Wang–Zhang [14] and Zhang [22]. In what follows, we study the existence and multiplicity of sign-changing solutions for the system (1.1) under assumptions weaker than the (AR)-condition:

(F_1') $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_0 > 0$ and $p \in (2,6)$ such that
$$|f(x,t)| \le c_0 \left(|t| + |t|^{p-1}\right),$$
for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

(F_2') $\lim\limits_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = +\infty$ uniformly in $x \in \mathbb{R}^3$.

(F_3') There exists $\mu_2 > 2$ and $\gamma > 0$ such that
$$f(x,t)t - \mu_2 F(x,t) + \gamma t^2 \ge 0,$$
for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ and $f(x,t)t > 0$ for $t \neq 0$.

We now state our second main result.

Theorem 1.2. *Assume that* (V) *and* (F_1')–(F_3') *hold. Then, system* (1.1) *possesses at least one sign-changing solution. Moreover, if* (F_4) *also holds, then system* (1.1) *admits infinitely many sign-changing solutions* $\{u_j, \phi_{u_j}\}$ *such that* $\mathcal{F}(u_j, \phi_{u_j}) \to \infty$ *as* $j \to \infty$.

Remark 1.3. *Note that the conditions* (F_2') *and* (F_3') *are significantly weaker than the* (AR)*-condition. There exist many functions satisfying* (F_1')–(F_3'), *but not the* (AR)*-condition. For example,* $f(x,t) = a|t|^{p-2}t + bt$, *where* $a > 0$, $b > 0$, *and* $2 < p < 5$.

It is worth emphasizing that, after overcoming the strong indefiniteness via the reduction method, the nonlocal term associated with the system (1.1) does not admit an explicit representation and fails to satisfy the decomposability property
$$\phi_u = \phi_{u^+} + \phi_{u^-},$$
which is typical for the Schrödinger-Poisson system, due to the nonlinear nature of the nonlocal term. For the Schrödinger-Poisson system, the existence of sign-changing solutions is usually established by imposing a monotonicity condition on the nonlinearity and then applying the constraint method. Specifically, one first constructs the so-called sign-changing Nehari manifold, then seeks a minimizing sequence on this manifold, and finally proves that such a minimizing sequence is attained, see, for example, Chen–Tang [6] and Zhong–Tang [24]. However, this approach is no longer applicable to the present system (1.1). To overcome this difficulty, inspired by Zhang [22], we employ abstract critical point theorems that do not depend on the explicit analytical form of ϕ_u and do not rely on any decomposition property.

For Theorem 1.1, the compactness condition can be established by standard arguments. We introduce an auxiliary operator A (see Section 3), which is used to construct a pseudo-gradient vector field and thereby obtain the desired invariant sets of the flow. Since A is only continuous, it cannot be used directly to construct invariant sets for the descending flow. To overcome this issue, we further introduce a locally Lipschitz continuous operator B, which preserves the main properties of A, and is suitable for defining the flow. Applying abstract critical point theorems then yields the desired result. For Theorem 1.2, the absence of the (AR)-condition makes the compactness argument more involved, and the properties of the auxiliary operator cannot be obtained directly. To address this difficulty, inspired by Wen–Tang–Chen [20], we employ a contradiction argument and ultimately derive the result by means of an abstract critical point theorem.

This paper is organized as follows. In Section 2, we establish the variational framework for the system (1.1) and present several preliminary results, including abstract critical point theorems. Section 3 is devoted to the construction of suitable auxiliary operators and the proof of Theorem 1.1. Section 4 focuses on the proof of Theorem 1.2, where new auxiliary operators are introduced to handle the weaker assumptions.

2. Variational setting and preliminaries

In this section, we develop the variational framework for the system (1.1) and establish several preliminary results. We then verify that the associated functional satisfies the Palais Smale condition ((PS)-condition for short) under the assumptions of each theorem, and conclude by recalling an abstract critical point theorem.

Throughout this paper, we use $C, C_1, C_2, \cdots, C', \cdots$ to denote various positive constants whose values may change from line to line. We denote by $|\cdot|_q$ the usual norm in $L^q(\mathbb{R}^3)$ for $q \in [2,\infty)$ and by $\|\cdot\|_\infty$ the norm of $L^\infty(\mathbb{R}^3)$. Moreover, $\|\cdot\|_{\mathcal{D}}$ denotes the standard norm in $\mathcal{D}(\mathbb{R}^3)$. Finally, "$\to$" and "$\rightharpoonup$" denote strong and weak convergence, respectively, in the relevant function spaces.

Since the functional $\mathcal{F}$ is strongly indefinite on $H^1(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)$ both from above and below, as explained in the introduction, we introduce a reduced functional depending on a single variable by solving the second equation of the system (1.1).

To this end, we require the following lemma.

Lemma 2.1. *For every $u \in H^1(\mathbb{R}^3)$, the following statements hold:*

(i) *there exists a unique $\phi = \phi_u \in \mathcal{D}(\mathbb{R}^3)$ which solves*

$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2;$$

(ii) $-\omega \le \phi_u \le 0$ *whenever* $u(x) \neq 0$*;*

(iii) $\|\phi_u\|_{\mathcal{D}} \le C|u|^2_{\frac{12}{5}} \le C'\|u\|^2$*;*

(iv) *if $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, then $\{\phi_{u_n}\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}(\mathbb{R}^3)$;*

(v) *if $u_n \to u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \to \phi_u$ in $\mathcal{D}(\mathbb{R}^3)$.*

Proof. Note that (i) was proved in Lemma 3 of d'Avenia–Pisani [7] while (ii) can be found in Lemma 2.3 of Mugnai [15]. We prove (iii), (iv), and (v).

(iii) Multiplying $\Delta\phi_u + \beta\Delta_4\phi_u = 4\pi(\omega+\phi_u)u^2$ by ϕ_u and integrating over $\mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,\mathrm{d}x + \beta\int_{\mathbb{R}^3} |\nabla\phi_u|^4\,\mathrm{d}x = -4\pi\int_{\mathbb{R}^3} (\omega+\phi_u)\phi_u u^2\,\mathrm{d}x. \tag{2.1}$$

Since $-\omega \le \phi_u \le 0$, we have $0 \le -(\omega+\phi_u)\phi_u \le \omega|\phi_u|$. Using this along with Hölder's inequality and Sobolev embedding, it follows from (2.1) that

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,\mathrm{d}x \le C\int_{\mathbb{R}^3} |\phi_u|u^2\,\mathrm{d}x \le C|\phi_u|_6|u|^2_{\frac{12}{5}} \le C_1|\nabla\phi_u|_2|u|^2_{\frac{12}{5}}.$$

Thus,

$$|\nabla\phi_u|_2 \le C|u|^2_{\frac{12}{5}}.$$

Similarly we can show that

$$|\nabla\phi_u|_4 \le C|u|^2_{\frac{12}{5}}.$$

Combining this, we have

$$\|\phi_u\|_{\mathcal{D}} = |\nabla\phi_u|_2 + |\nabla\phi_u|_4 \le C|u|^2_{\frac{12}{5}} \le C'\|u\|^2. \tag{2.2}$$

Hence (iii) follows.

(iv) This follows directly from (2.2).

(v) Let $u_n \to u$ in $H^1(\mathbb{R}^3)$. For any $\psi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla(\phi_{u_n} - \phi_u)\cdot\nabla\psi\,\mathrm{d}x + \beta\int_{\mathbb{R}^3}(|\nabla\phi_{u_n}|^2\nabla\phi_{u_n} - |\nabla\phi_u|^2\nabla\phi_u)\cdot\nabla\psi\,\mathrm{d}x \\
&= -4\pi\int_{\mathbb{R}^3}[\omega(u_n^2 - u^2) + (\phi_{u_n}u_n^2 - \phi_u u^2)]\psi\,\mathrm{d}x \\
&= -4\pi\int_{\mathbb{R}^3}\left[(\omega + \phi_{u_n})(u_n^2 - u^2) + (\phi_{u_n} - \phi_u)u^2\right]\psi\,\mathrm{d}x.
\end{aligned}$$

Taking $\psi = \phi_{u_n} - \phi_u$ and using the standard inequality

$$C_p|x-y|^p \le \left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y) \quad \text{for } x, y \in \mathbb{R}^N,\ p \ge 2,$$

we obtain

$$\begin{aligned}
&|\nabla\phi_{u_n} - \nabla\phi_u|_2^2 + |\nabla\phi_{u_n} - \nabla\phi_u|_4^4 + 4\pi\int_{\mathbb{R}^3}|\phi_{u_n} - \phi_u|^2u^2\,\mathrm{d}x \\
&\le C\int_{\mathbb{R}^3}|\omega + \phi_{u_n}|\,|u_n^2 - u^2|\,|\phi_{u_n} - \phi_u|\,\mathrm{d}x
\end{aligned}$$

Since $-\omega \le \phi_{u_n} \le 0$, we have $|\omega + \phi_{u_n}| \le \omega$, and therefore, by Hölder's inequality and Sobolev embedding,

$$\begin{aligned}
|\nabla\phi_{u_n} - \nabla\phi_u|_2^2 + |\nabla\phi_{u_n} - \nabla\phi_u|_4^4 &\le C\int_{\mathbb{R}^3}|u_n^2 - u^2|\,|\phi_{u_n} - \phi_u|\,\mathrm{d}x \\
&\le C|u_n^2 - u^2|_{\frac{6}{5}}|\phi_{u_n} - \phi_u|_6 \\
&\le C'|u_n^2 - u^2|_{\frac{6}{5}}|\nabla(\phi_{u_n} - \phi_u)|_2.
\end{aligned}$$

Moreover,

$$|u_n^2 - u^2|_{\frac{6}{5}} \le |u_n - u|_{\frac{12}{5}}|u_n + u|_{\frac{12}{5}}$$

Since $u_n \to u$ in $H^1(\mathbb{R}^3)$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, hence also in $L^{12/5}(\mathbb{R}^3)$. Thus,

$$|u_n^2 - u^2|_{\frac{6}{5}} \le C|u_n - u|_{\frac{12}{5}} \to 0,$$

which implies that

$$|\nabla\phi_{u_n} - \nabla\phi_u|_2 \to 0.$$

Going back to the previous estimate, we also obtain

$$|\nabla\phi_{u_n} - \nabla\phi_u|_4 \to 0.$$

Hence, $\phi_{u_n} \to \phi_u$ in $\mathcal{D}(\mathbb{R}^3)$. $\square$

By Lemma 2.1, the system (1.1) can be reduced to the following one variable equation

$$-\Delta u + V(x)u - (2\omega + \phi_u)\phi_u u = f(x,u), \quad x \in \mathbb{R}^3. \tag{2.3}$$

Accordingly, the functional $\mathcal{F}$ can be reduced to a one-variable functional defined on H by

$$\begin{aligned}
J(u) &:= \mathcal{F}(u,\phi_u)\\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - (2\omega+\phi_u)\phi_u u^2\right)\,\mathrm{d}x\\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x.
\end{aligned}$$

Combining this with (2.1), it follows that

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2\right)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} (\omega+\phi_u)\,\phi_u u^2\,\mathrm{d}x\\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2\right)\,\mathrm{d}x + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x\\
&\quad + \frac{\beta}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2\right)\,\mathrm{d}x + \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 + \phi_u^2 u^2\right)\,\mathrm{d}x - \int_{\mathbb{R}^3} (\omega+\phi_u)\,\phi_u u^2\,\mathrm{d}x\\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 + \phi_u^2 u^2\right)\,\mathrm{d}x + \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x\\
&\quad + \frac{\beta}{4\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 + \phi_u^2 u^2\right)\,\mathrm{d}x + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^2\,\mathrm{d}x\\
&\quad + \frac{3\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 \,\mathrm{d}x \\ &\quad + \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_u^2 u^2 \,\mathrm{d}x \\ &\quad + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x + \frac{3\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x. \end{aligned} \tag{2.4}$$

The two representations in (2.4) are equivalent.

Under the assumptions (V) and (F_1') (or (F_1) and (F_2)), by an argument similar to that in Mugnai [15], the functional J belongs to $C^1(H,\mathbb{R})$, and its Fréchet derivative is given by

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^3} \left(\nabla u\cdot\nabla v + V(x)uv - (2\omega+\phi_u)\phi_u uv\right) \mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)v\,\mathrm{d}x, \tag{2.5}$$

for $u, v \in H$.

The following lemma characterizes the relationship between the critical points of J and the weak solutions of (1.1), see Lemma 2.4 in Mugnai [15].

Lemma 2.2. *The following statements are equivalent:*

(i) $(u,\phi) \in H\times\mathcal{D}(\mathbb{R}^3)$ *is a critical point of* $\mathcal{F}$*, that is,* (u,ϕ) *is a weak solution of* (1.1)*;*

(ii) u *is a critical point of* J *and* $\phi = \phi_u$*.*

By Lemma 2.2, for simplicity, we will write $u \in H$ instead of $(u,\phi_u) \in H\times\mathcal{D}(\mathbb{R}^3)$ to denote a weak solution of (1.1).

The following lemma plays a crucial role in establishing the compactness property, see Zou–Schechter [25].

Lemma 2.3. *Suppose that* (V) *holds. Then the embedding* $H \hookrightarrow L^q(\mathbb{R}^3)$ *for* $q \in [2,6)$ *is compact.*

Next, we present the compactness results under different conditions.

Lemma 2.4. *Under the assumptions of Theorem 1.1, the functional* J *satisfies the* (PS)*-condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset H$ be a sequence such that $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J'(u_n) \to 0$ as $n\to\infty$. The proof consists of two steps. First, we show that the (PS)-sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H. By (2.4), (2.5), Lemma 2.1 and (F_3), we have

$$\begin{aligned} &|J(u_n)| + \frac{1}{\mu_1}\|J'(u_n)\|_*\|u_n\| \\ &\geq J(u_n) - \frac{1}{\mu_1}\langle J'(u_n), u_n\rangle \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_n)\,\mathrm{d}x + \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^4\,\mathrm{d}x \\ &\quad - \frac{1}{\mu_1}\|u_n\|^2 + \frac{1}{\mu_1}\int_{\mathbb{R}^3} (2\omega+\phi_{u_n})\phi_{u_n}{u_n}^2\,\mathrm{d}x + \frac{1}{\mu_1}\int_{\mathbb{R}^3} f(x,u_n)u_n\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\mu_1-2}{2\mu_1}\|u_n\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2\,\mathrm{d}x+\frac{1}{\mu_1}\int_{\mathbb{R}^3}(2\omega+\phi_{u_n})\phi_{u_n}{u_n}^2\,\mathrm{d}x\\
&\geq \frac{\mu_1-2}{2\mu_1}\|u_n\|^2+\frac{4-\mu_1}{2\mu_1}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2\,\mathrm{d}x.
\end{aligned}
$$

Case 1: $\mu_1\geq 4$. By Lemma 2.1 (ii),

$$
\frac{4-\mu_1}{2\mu_1}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2\,\mathrm{d}x\geq 0. \tag{2.6}
$$

Case 2: $\mu_1\in(2,4)$ and $0<\omega<\left(\frac{(\mu_1-2)V_0}{4-\mu_1}\right)^{1/2}$. From Lemma 2.1 (ii), we have

$$
\begin{aligned}
&\frac{\mu_1-2}{2\mu_1}\|u_n\|^2+\frac{4-\mu_1}{2\mu_1}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2\,\mathrm{d}x\\
&\geq \frac{\mu_1-2}{2\mu_1}|\nabla u_n|_2^2+\frac{(\mu_1-2)V_0-(4-\mu_1)\omega^2}{2\mu_1}|u_n|_2^2\geq C\|u_n\|^2.
\end{aligned} \tag{2.7}
$$

Hence, in both cases, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded.

Second, we prove that the sequence $\{u_n\}_{n\in\mathbb{N}}$ admits a strongly convergent subsequence. By Lemma 2.3, there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, and a function $u\in H$ such that

$$
\begin{aligned}
u_n&\rightharpoonup u &&\text{in } H,\\
u_n&\to u &&\text{in } L^q(\mathbb{R}^3) \text{ for } q\in[2,6),\\
u_n(x)&\to u(x) &&\text{for a.a. } x\in\mathbb{R}^3.
\end{aligned}
$$

From (2.5), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(|\nabla(u_n-u)|^2+V(x)(u_n-u)^2\right)\,\mathrm{d}x\\
&=\langle J'(u_n)-J'(u),u_n-u\rangle+\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)(u_n-u)\,\mathrm{d}x\\
&\quad+2\omega\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,\mathrm{d}x+\int_{\mathbb{R}^3}(\phi_{u_n}^2u_n-\phi_u^2u)(u_n-u)\,\mathrm{d}x.
\end{aligned} \tag{2.8}
$$

By (F_1) and (F_2), for any small $\delta>0$, there exists $C_\delta>0$ such that

$$
|f(x,t)|\leq\delta|t|+C_\delta|t|^{p-1},\quad t\in\mathbb{R}. \tag{2.9}
$$

Since $H\hookrightarrow L^q(\mathbb{R}^3)$ is compact for $q\in[2,6)$, by Lemma 2.3, we have $u_n\to u$ in $L^2(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$. Using (2.9) and Hölder's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\left(f(x,u_n)-f(x,u)\right)(u_n-u)|\,\mathrm{d}x\\
&\leq\int_{\mathbb{R}^3}\big(\delta(|u_n|+|u|)+C_\delta(|u_n|^{p-1}+|u|^{p-1})\big)|u_n-u|\,\mathrm{d}x\\
&\leq\delta\big(|u_n|_2+|u|_2\big)|u_n-u|_2+C_\delta\big(|u_n|_p^{p-1}+|u|_p^{p-1}\big)|u_n-u|_p\to 0,
\end{aligned}
$$

as $n\to\infty$. Thus,

$$
\int_{\mathbb{R}^3}|\left(f(x,u_n)-f(x,u)\right)(u_n-u)|\,\mathrm{d}x\to 0,\quad\text{as } n\to\infty. \tag{2.10}
$$

By Lemma 2.1, Hölder's inequality and the Sobolev embedding, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)u_n(u_n-u)\,\mathrm{d}x\right| &\le |\phi_{u_n}-\phi_u|_6|u_n-u|_3|u_n|_2\\ &\le C\|\phi_{u_n}-\phi_u\|_{\mathcal{D}}|u_n-u|_3|u_n|_2\to 0,\quad \text{as } n\to\infty,\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^3}\phi_u(u_n-u)^2\,\mathrm{d}x\right|\le|\phi_u|_6|u_n-u|_3|u_n-u|_2\to 0,\quad\text{as } n\to\infty.$$

Thus, we get

$$\begin{aligned}\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,\mathrm{d}x &= \int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)u_n(u_n-u)\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3}\phi_u(u_n-u)^2\,\mathrm{d}x\to 0\end{aligned}\tag{2.11}$$

as $n\to\infty$. Moreover, since $|\phi_{u_n}^2u_n|_{\frac{3}{2}}\le|\phi_{u_n}|_6^2|u_n|_3$, using Hölder's inequality, the sequence $\{\phi_{u_n}^2u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{3}{2}}(\mathbb{R}^3)$. Hence,

$$\left|\int_{\mathbb{R}^3}(\phi_{u_n}^2u_n-\phi_u^2u)(u_n-u)\,\mathrm{d}x\right|\le|\phi_{u_n}^2u_n-\phi_u^2u|_{\frac{3}{2}}|u_n-u|_3\to 0,\tag{2.12}$$

as $n\to\infty$. Passing to the limit in (2.8) and using (2.10), (2.11), and (2.12), we obtain

$$\|u_n-u\|^2\to 0.$$

Hence, $u_n\to u$ in H. □

Lemma 2.5. *Under the assumptions of Theorem 1.2, the functional J satisfies the* (PS)*-condition.*

Proof. The proof follows the same strategy as in Lemma 2.4 and is divided into two parts: boundedness and strong convergence. Since the argument for strong convergence is identical to that in Lemma 2.4, we only present the proof of boundedness.

We argue by contradiction. Assume that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}\subset H$ such that $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded, $J'(u_n)\to 0$, and $\|u_n\|\to\infty$ as $n\to\infty$. Taking (2.4), (2.5), Lemma 2.1 and (F_3') into account, we obtain

$$\begin{aligned}&|J(u_n)|+\frac{1}{\mu_2}\|J'(u_n)\|\|u_n\|\\ &\ge J(u_n)-\frac{1}{\mu_2}\langle J'(u_n),u_n\rangle\\ &\ge\left(\frac12-\frac{1}{\mu_2}\right)\|u_n\|^2+\left(\frac12+\frac{1}{\mu_2}\right)\int_{\mathbb{R}^3}\phi_{u_n}^2u_n^2\,\mathrm{d}x+\frac{2}{\mu_2}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3}\left(\frac{1}{\mu_2}f(x,u_n)u_n-F(x,u_n)\right)\mathrm{d}x\\ &\quad+\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^2\,\mathrm{d}x+\frac{3\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4\,\mathrm{d}x\\ &\ge\left(\frac12-\frac{1}{\mu_2}\right)\|u_n\|^2-\left(\frac{2}{\mu_2}\omega^2+\frac{\gamma}{\mu_2}\right)\int_{\mathbb{R}^3}u_n^2\,\mathrm{d}x.\end{aligned}\tag{2.13}$$

Set $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$. Passing to a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and using Lemma 2.3, there exists $v \in H$ such that

$$\begin{aligned} v_n &\rightharpoonup v && \text{in } H,\\ v_n &\to v && \text{in } L^q(\mathbb{R}^3) \text{ for } q\in[2,6),\\ v_n(x) &\to v(x) && \text{for a.a. } x\in\mathbb{R}^3. \end{aligned}$$

Dividing (2.13) by $\|u_n\|^2$, we have

$$\frac{|J(u_n)|}{\|u_n\|^2} + \frac{1}{\mu_2\|u_n\|}\|J'(u_n)\| \geq \left(\frac{1}{2}-\frac{1}{\mu_2}\right) - \left(\frac{2}{\mu_2}\omega^2+\frac{\gamma}{\mu_2}\right)\int_{\mathbb{R}^3} v_n^2\,\mathrm{d}x.$$

Passing to the limit yields

$$\left(\frac{2}{\mu_2}\omega^2+\frac{\gamma}{\mu_2}\right)|v|_2^2 = \left(\frac{2}{\mu_2}\omega^2+\frac{\gamma}{\mu_2}\right)\lim_{n\to\infty}|v_n|_2^2 \geq \left(\frac{1}{2}-\frac{1}{\mu_2}\right) > 0$$

and thus $v \neq 0$. For a.a. $x \in \{y\in\mathbb{R}^3 : v(y)\neq 0\}$, we have $\lim_{n\to\infty}|u_n(x)| = \infty$. Using (2.4), assumption (F_2'), Lemma 2.1, and Fatou's lemma gives

$$\begin{aligned} 0 &= \lim_{n\to\infty}\frac{J(u_n)}{\|u_n\|^2}\\ &= \frac{1}{2} - \lim_{n\to\infty}\frac{\omega}{\|u_n\|^2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x\\ &\quad - \lim_{n\to\infty}\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^3}\left(\frac{1}{2}\phi_{u_n}^2u_n^2+\frac{1}{8\pi}|\nabla\phi_{u_n}|^2+\frac{\beta}{16\pi}|\nabla\phi_{u_n}|^4\right)\mathrm{d}x\\ &\quad - \lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,\mathrm{d}x\\ &\leq \frac{1}{2} - \varliminf_{n\to\infty}\frac{\omega}{\|u_n\|^2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x - \varliminf_{n\to\infty}\int_{\mathbb{R}^3}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,\mathrm{d}x\\ &\leq \frac{1}{2} + \varlimsup_{n\to\infty}\frac{\omega^2|u_n|_2^2}{\|u_n\|^2} - \int_{\mathbb{R}^3}\varliminf_{n\to\infty}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,\mathrm{d}x = -\infty, \end{aligned}$$

which is a contradiction. Therefore, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H. □

Finally, we introduce some definitions and recall the abstract critical point theory. For further details and proofs, we refer to Liu–Liu–Wang [13].

Let X be a Banach space, $I\in C^1(X,\mathbb{R})$, $P,Q\subset X$ be open sets, $M = P\cap Q$, $\Sigma = \partial P\cap\partial Q$, and $W = P\cup Q$. For $c\in\mathbb{R}$, define

$$K_c = \{x\in X : I(x)=c, I'(x)=0\} \quad\text{and}\quad I^c = \{x\in X : I(x)\leq c\}.$$

Definition 2.6. *The pair $\{P,Q\}$ is called an admissible family of invariant sets with respect to I at level c provided that the following deformation property holds: if $K_c\setminus W=\emptyset$, then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$, there exists $\eta\in C(X,X)$ satisfying*

(i) $\eta(\overline{P})\subset\overline{P}$, $\eta(\overline{Q})\subset\overline{Q}$;
(ii) $\eta\,|_{I^{c-2\varepsilon}} = \mathrm{id}$;
(iii) $\eta(I^{c+\varepsilon}\setminus W)\subset I^{c-\varepsilon}$.

The following result can be found in Liu–Liu–Wang [13, Theorem 2.4].

Theorem 2.7. *Assume that $\{P, Q\}$ is an admissible family of invariant sets with respect to I at any level $c \geq c_* := \inf_{u\in\Sigma} I(u)$, and that there exists a map $\varphi_0 \colon \Delta \to X$ satisfying*

(i) $\varphi_0(\partial_1\Delta) \subset P$ *and* $\varphi_0(\partial_2\Delta) \subset Q$*;*
(ii) $\varphi_0(\partial_0\Delta) \cap M = \emptyset$*;*
(iii) $\sup_{u\in\varphi_0(\partial_0\Delta)} I(u) < c_*$*,*

where

$$\begin{aligned}\Delta &= \{(t_1, t_2) \in \mathbb{R}^2 \colon t_1, t_2 \geq 0, t_1 + t_2 \leq 1\},\\ \partial_1\Delta &= \{0\} \times [0, 1],\\ \partial_2\Delta &= [0, 1] \times \{0\},\\ \partial_0\Delta &= \{(t_1, t_2) \in \mathbb{R}^2 \colon t_1, t_2 \geq 0, t_1 + t_2 = 1\}.\end{aligned}$$

Define

$$c = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\setminus W} I(u),$$

where

$$\Gamma := \{\varphi \in C(\Delta, X) \colon \varphi(\partial_1\Delta) \subset P, \varphi(\partial_2\Delta) \subset Q, \varphi|_{\partial_0\Delta} = \varphi_0|_{\partial_0\Delta}\}.$$

Then $c \geq c_*$ and $K_c \setminus W \neq \emptyset$.

To obtain the second statement of Theorem 1.1, we introduce the following setting. Let $G \colon X \to X$ be an isometric involution, that is,

$$G^2 = \mathrm{id} \quad \text{and} \quad d(Gx, Gy) = d(x, y) \quad \text{for all } x, y \in X.$$

Assume that I is G-invariant, i.e., $I(Gx) = I(x)$ for any $x \in X$, and that $Q = GP$. A subset $F \subset X$ is said to be symmetric if $Gx \in F$ for any $x \in F$. The genus of a closed symmetric subset $F \subset X \setminus \{0\}$ is denoted by $\gamma(F)$.

Definition 2.8. *The set P is called a G-admissible invariant set with respect to I at level c if there exist $\varepsilon_0 > 0$ and a symmetric open neighborhood N of $K_c \setminus W$ with $\gamma(\overline{N}) < \infty$, such that for every $\varepsilon \in (0, \varepsilon_0)$, there exists $\eta \in C(X, X)$ satisfying*

(i) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$*;*
(ii) $\eta \circ G = G \circ \eta$*;*
(iii) $\eta\,|_{I^{c-2\varepsilon}} = \mathrm{id}$*;*
(iv) $\eta(I^{c+\varepsilon} \setminus (N \cup W)) \subset I^{c-\varepsilon}$*.*

Finally, the following result is taken from Liu–Liu–Wang [13, Theorem 2.5].

Theorem 2.9. *Assume that P is a G-admissible invariant set with respect to I at any level $c \geq c_* := \inf_{u\in\Sigma} I(u)$, and that for any $n \in \mathbb{N}$, there exists a continuous map $\varphi_n \colon B_n \to X$, where $B_n := \{x \in \mathbb{R}^n \colon |x| \leq 1\}$, satisfying*

(i) $\varphi_n(0) \in M := P \cap Q$, $\varphi_n(-t) = G\varphi_n(t)$ *for* $t \in B_n$*;*
(ii) $\varphi_n(\partial B_n) \cap M = \emptyset$*;*
(iii) $\sup_{u\in \mathrm{Fix}_G \cup \varphi_n(\partial B_n)} I(u) < c_*$*, where* $\mathrm{Fix}_G := \{u \in X \colon Gu = u\}$*.*

For $j \in \mathbb{N}$, define

$$c_j = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} I(u),$$

where

$$\Gamma_j = \big\{B\colon B=\varphi(B_n\setminus Y) \text{ for some } \varphi\in G_n,\ n\geq j, \text{ and open } Y\subset B_n \\ \text{such that } -Y=Y \text{ and } \gamma(\bar{Y})\leq n-j\big\}$$

and

$$G_n = \big\{\varphi\in C(B_n,X)\colon \varphi(-t)=G\varphi(t) \text{ for } t\in B_n, \varphi(0)\in M \\ \text{and } \varphi|_{\partial B_n}=\varphi_n|_{\partial B_n}\big\}.$$

Then for $j\geq 2$, $c_j\geq c_*$, $K_{c_j}\setminus W\neq\emptyset$ and $c_j\to\infty$ as $j\to\infty$.

3. Proof of Theorem 1.1

In this section, we apply the abstract critical point theory to establish the main results for system (1.1). The approach is based on a minimax scheme combined with invariant sets of a descending flow. To this end, we first construct a descending flow via an auxiliary operator $A\colon H\to H$. More precisely, for each $u\in H$, we define $v=Au\in H$ as the unique solution of

$$-\Delta v+V(x)v-(2\omega+\phi_u)\phi_u v=f(x,u),\quad v\in H. \tag{3.1}$$

Lemma 3.1. *Suppose that* (F_1)–(F_3) *hold. Then:*

(i) *the operator A is well-defined;*
(ii) *the operator A maps bounded sets into bounded sets;*
(iii) *the operator A is continuous;*
(iv) *the operator A is compact;*
(v) *if* (F_4) *holds, then A is odd.*

Proof. (i) For any $u\in H$, define the functional

$$\begin{aligned} K(v) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2+V(x)v^2\right)\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u v^2\,\mathrm{d}x \\ &\quad -\int_{\mathbb{R}^3}f(x,u)v\,\mathrm{d}x,\quad v\in H. \end{aligned} \tag{3.2}$$

Clearly, $K\in C^1(H,\mathbb{R})$. By Lemma 2.1, (2.9), and Hölder's inequality, noting that $-(2\omega+\phi_u)\phi_u\geq 0$, we derive

$$K(v)\geq\frac{1}{2}\|v\|^2-\int_{\mathbb{R}^3}f(x,u)v\,\mathrm{d}x\geq\frac{1}{2}\|v\|^2-\delta\|u\|\|v\|-C_\delta\|u\|^{p-1}\|v\|.$$

Hence $K(v)\to\infty$ as $\|v\|\to\infty$, so K is coercive, bounded from below, strictly convex, and weakly lower semi-continuous. Therefore, K admits a unique minimizer $v=Au\in H$, which is the unique solution of (3.1). Thus, A is well-defined.

(ii) Let $v=Au$. From Lemma 2.1, the definition of A, (2.9), and the fact that $-\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u v^2\,\mathrm{d}x\geq 0$, we have

$$\|v\|^2\leq\|v\|^2-\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u v^2\,\mathrm{d}x=\int_{\mathbb{R}^3}f(u)v\,\mathrm{d}x\leq\delta\|u\|\|v\|+C_\delta\|u\|^{p-1}\|v\|,$$

which means that A maps bounded sets into bounded sets.

(iii) Let $u_n\to u$ in H as $n\to\infty$, and set $v=Au$, $v_n=Au_n$. From (3.1) and (3.2), we have

$$\|v_n-v\|^2\leq\|v_n-v\|^2-\int_{\mathbb{R}^3}(2\omega+\phi_{u_n})\phi_{u_n}(v_n-v)^2\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v)\,\mathrm{d}x + 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) v(v_n - v)\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} (\phi_{u_n}^2 - \phi_u^2) v(v_n - v)\,\mathrm{d}x.$$

As in Lemma 2.4, as $n \to \infty$, we have

$$\int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v) \to 0, \tag{3.3}$$

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) v(v_n - v) \to 0. \tag{3.4}$$

Moreover, by Hölder's inequality and Lemma 2.1,

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n}^2 - \phi_u^2) v(v_n - v)\,\mathrm{d}x \right| \le C \|\phi_{u_n} - \phi_u\|_{\mathcal{D}} \|v\| \|v_n - v\|. \tag{3.5}$$

It follows from (3.3), (3.4), and (3.5) that $\|v_n - v\|^2 \to 0$ as $n \to \infty$, so A is continuous.

(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset H$ be bounded and set $v_n = Au_n$. Then $\{v_n\}_{n\in\mathbb{N}}$ is bounded as well. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in H as $n \to \infty$. For all $\psi \in H$, we have the identity

$$\int_{\mathbb{R}^3} (\nabla v_n \cdot \nabla \psi + V(x) v_n \psi)\,\mathrm{d}x - \int_{\mathbb{R}^3} (2\omega + \phi_{u_n})\phi_{u_n} v_n \psi\,\mathrm{d}x = \int_{\mathbb{R}^3} f(u_n)\psi\,\mathrm{d}x. \tag{3.6}$$

Since $u_n \rightharpoonup u$ in H, Lemma 2.3 gives $u_n \to u$ in $L^p(\mathbb{R}^3)$ for $p \in [2, 6)$. Using (2.9) and Hölder's inequality, for any $\psi \in H$,

$$\begin{aligned}\left| \int_{\mathbb{R}^3} \big(f(u_n) - f(u)\big)\psi\,\mathrm{d}x \right| &\le \int_{\mathbb{R}^3} \big(\delta(|u_n| + |u|) + C_\delta(|u_n|^{p-1} + |u|^{p-1})\big)|\psi|\,\mathrm{d}x \\ &\le \delta(|u_n|_2 + |u|_2)|\psi|_2 + C_\delta(|u_n|_p^{p-1} + |u|_p^{p-1})|\psi|_p \to 0.\end{aligned}$$

Moreover, Lemma 2.1 and Hölder's inequality imply

$$\int_{\mathbb{R}^3} (2\omega + \phi_{u_n})\phi_{u_n} v_n \psi\,\mathrm{d}x \to \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u v \psi\,\mathrm{d}x.$$

Passing to the limit in (3.6) gives $v = Au$. The identity

$$\|v_n - v\|^2 = \int_{\mathbb{R}^3} \left((2\omega + \phi_{u_n})\phi_{u_n} v_n - (2\omega + \phi_u)\phi_u v\right)(v_n - v)\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v)\,\mathrm{d}x$$

and the argument in (iii) lead to $v_n \to v$, hence A is compact.

(v) Since $\phi_u = \phi_{-u}$, by the definition of A and assumption (F_4), it follows immediately that $A(-u) = -A(u)$. Thus, A is odd. □

Remark 3.2. *From Lemma 3.1, the following three statements are equivalent:*

(i) *u is a solution of (2.3);*
(ii) *u is a critical point of the functional J;*
(iii) *u is a fixed point of the operator A.*

The following lemma is crucial for the construction of the descending flow.

Lemma 3.3. *The following statements hold:*

(i) *$\langle J'(u), u - Au\rangle \ge \|u - Au\|^2$ for all $u \in H$;*

(ii) $\|J'(u)\|_* \le \|u - Au\|(1 + C\|u\|^2)$ *for some* $C > 0$ *and for all* $u \in H$;
(iii) *for* $a < b$ *and* $\alpha > 0$, *there exists* $\beta > 0$ *such that* $\|u - Au\| \ge \beta$ *whenever* $u \in H$, $J(u) \in [a, b]$ *and* $\|J'(u)\|_* \ge \alpha$.

Proof. (i) Let $v = Au$. From (2.5) and (3.1), we directly obtain

$$\langle J'(u), u - Au\rangle = \|u - Au\|^2 - \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u(u - Au)^2 \, \mathrm{d}x \ge \|u - Au\|^2.$$

(ii) For any $\psi \in H$, Lemma 2.1, Sobolev's embedding theorem, and Hölder's inequality give

$$\begin{aligned}\langle J'(u), \psi\rangle &= \langle J'(u), \psi\rangle - (Au, \psi) + (Au, \psi)\\ &= (u - Au, \psi) + \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u(u - Au)\psi \, \mathrm{d}x\\ &\le \|u - Au\|\|\psi\| + C\|u\|^2\|u - Au\|\|\psi\|.\end{aligned}$$

Taking the supremum over $\|\psi\| = 1$ yields the result.

(iii) By (F_3), Lemma 2.1, (2.4) and (3.1), we compute

$$\begin{aligned}&J(u) - \frac{1}{\mu_1}(u, u - Au)\\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 \, \mathrm{d}x + \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 \, \mathrm{d}x - \int_{\mathbb{R}^3} F(x, u) \, \mathrm{d}x\\ &\quad - \frac{1}{\mu_1}\|u\|^2 + \frac{1}{\mu_1}\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uAu \, \mathrm{d}x + \frac{1}{\mu_1}\int_{\mathbb{R}^3} f(x, u)u \, \mathrm{d}x\\ &\ge \left(\frac{1}{2} - \frac{1}{\mu_1}\right)\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 \, \mathrm{d}x\\ &\quad + \frac{1}{\mu_1}\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u u(Au - u) \, \mathrm{d}x + \frac{1}{\mu_1}\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u u^2 \, \mathrm{d}x\\ &\ge \left(\frac{1}{2} - \frac{1}{\mu_1}\right)\|u\|^2 + \frac{4 - \mu_1}{2\mu_1}\int_{\mathbb{R}^3} \omega\phi_u u^2 \, \mathrm{d}x + \frac{1}{\mu_1}\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u u(Au - u) \, \mathrm{d}x.\end{aligned}$$

Then, by Lemma 2.1, Hölder's inequality, (2.6) and (2.7), we deduce

$$\begin{aligned}\|u\|^2 &\le C_1\left(|J(u)| + \|u\|\|u - Au\| + \left|\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u u(Au - u) \, \mathrm{d}x\right|\right)\\ &\le C\left(|J(u)| + \|u\|\|u - Au\| + \|u\|^2\|u - Au\|\right).\end{aligned} \tag{3.7}$$

Assume that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H$ such that $J(u_n) \in [a, b]$, $\|J'(u_n)\|_* \ge \alpha$ and $\|u_n - Au_n\| \to 0$ as $n \to \infty$. Then (3.7) gives the boundedness of $\{\|u_n\|\}_{n\in\mathbb{N}}$, but (ii) implies $\|J'(u_n)\|_* \to 0$, which is a contradiction. □

We define the positive and negative cones by

$$P^+ := \{u \in H \colon u \ge 0\} \quad \text{and} \quad P^- := \{u \in H \colon u \le 0\}.$$

For $\varepsilon > 0$, set

$$P^+_\varepsilon := \{u \in H \colon \operatorname{dist}(u, P^+) < \varepsilon\} \quad \text{and} \quad P^-_\varepsilon := \{u \in H \colon \operatorname{dist}(u, P^-) < \varepsilon\},$$

where

$$\operatorname{dist}(u, P^\pm) = \inf_{v\in P^\pm} \|u - v\|.$$

Clearly, $P_\varepsilon^- = -P_\varepsilon^+$. Let $W = P_\varepsilon^+ \cup P_\varepsilon^-$. Then W is an open and symmetric subset of H and $H \setminus W$ consists only of sign-changing functions.

The following lemma shows that, for ε sufficiently small, all sign-changing solutions of (2.3) lie in $H \setminus W$.

Lemma 3.4. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the following statements hold:*

(i) *$A(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ and every nontrivial solution $u \in P_\varepsilon^-$ is negative;*
(ii) *$A(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and every nontrivial solution $u \in P_\varepsilon^+$ is positive.*

Proof. The proof is similar to the proof of Lemma 3.4 in Liu–Wang–Zhang [14]. Since the arguments for (i) and (ii) are analogous, we only prove (i).

For any $u \in H$, recall that $\operatorname{dist}(u, P^\mp) = \|u^\pm\|$. Let $u \in H$ and $v = Au$. By Sobolev's embedding theorem, for each $q \in [2, 6]$, there exists $\beta_q > 0$ such that $|w|_q \le \beta_q \|w\|$ for all $w \in H$. Applying this to $w = u^\pm$ gives

$$|u^\pm|_q = \inf_{w \in P^\mp} |u - w|_q \le \beta_q \inf_{w \in P^\mp} \|u - w\| = \beta_q \operatorname{dist}\left(u, P^\mp\right). \tag{3.8}$$

Since $\operatorname{dist}(v, P^-) \le \|v^+\|$, (3.1) and Lemma 2.1 yield

$$\begin{aligned} \operatorname{dist}(v, P^-)\|v^+\| \le \|v^+\|^2 &= \int_{\mathbb{R}^3} f(x,u) v^+ \,\mathrm{d}x + \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u v v^+ \,\mathrm{d}x \\ &\le \int_{\mathbb{R}^3} f(x,u) v^+ \,\mathrm{d}x. \end{aligned}$$

By (F_3), we have $f(x,t) \le 0$ for all $x \in \mathbb{R}^3$ when $t \le 0$. Thus

$$\int_{\mathbb{R}^3} f(x,u) v^+ \,\mathrm{d}x \le \int_{\mathbb{R}^3} f\left(x, u^+\right) v^+ \,\mathrm{d}x.$$

Combining this with (2.9), we obtain

$$\begin{aligned} \operatorname{dist}(v, P^-)\left\|v^+\right\| &\le \delta \left|u^+\right|_2 \left|v^+\right|_2 + C_\delta \left|u^+\right|_p^{p-1} \left|v^+\right|_p \\ &\le C\left(\delta \operatorname{dist}\left(u, P^-\right) + C_\delta \operatorname{dist}\left(u, P^-\right)^{p-1}\right)\left\|v^+\right\|. \end{aligned}$$

Therefore,

$$\operatorname{dist}(Au, P^-) \le C\left(\delta \operatorname{dist}\left(u, P^-\right) + C_\delta \operatorname{dist}\left(u, P^-\right)^{p-1}\right).$$

Choosing $\delta > 0$ small and then $\varepsilon > 0$ sufficiently small, there exists $\varepsilon_0 > 0$ such that for all $u \in P_\varepsilon^-$ with $\varepsilon \in (0, \varepsilon_0)$,

$$\operatorname{dist}(Au, P^-) \le \frac{1}{2} \operatorname{dist}\left(u, P^-\right),$$

which implies $A\left(\partial P_\varepsilon^-\right) \subset P_\varepsilon^-$. If $u \in P_\varepsilon^-$ satisfies $Au = u \ne 0$, then $u \in P^-$. By the strong maximum principle, it follows that $u < 0$ in $\mathbb{R}^3$. $\square$

Since A is only continuous, it is not sufficient for constructing a descending gradient flow. Therefore, in the following lemma, we construct a locally Lipschitz continuous operator B on $H_0 := H \setminus E$, which preserves the key properties of A, where E denotes the set of fixed points of A.

Lemma 3.5. *There exists a locally Lipschitz continuous operator $B\colon H_0 \to H$ such that the following properties hold:*

(i) *$B(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and $B(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ for $\varepsilon \in (0, \varepsilon_0)$;*

(ii) $\frac{1}{2}\|u - B(u)\| \leq \|u - A(u)\| \leq 2\|u - B(u)\|$ *for all* $u \in H_0$;
(iii) $\langle J'(u), u - B(u)\rangle \geq \frac{1}{2}\|u - A(u)\|^2$ *for all* $u \in H_0$;
(iv) *if* (F_4) *holds, then* B *is odd.*

Proof. The proof is analogous to Lemma 4.1 in Bartsch–Liu [1] and Lemma 3.5 in Zhang [22]. For completeness, we outline the main steps.

Let C be as in Lemma 3.3. For $u \in H_0$, define

$$\Delta_1(u) = \frac{1}{2}\|u - Au\|, \quad \Delta_2(u) = \frac{1}{2}\|u - Au\|(1 + C\|u\|^2)^{-1}. \tag{3.9}$$

Then $\Delta_1, \Delta_2 \in C(H_0, \mathbb{R})$. By the continuity of A, for each $u \in H_0$, there exists $r(u) \in (0,1)$ such that for all $v, w \in N(u) := \{z \in H \colon \|z - u\| \leq r(u)\} \subset H_0$,

$$\|Av - Aw\| \leq \min\{\Delta_1(v), \Delta_1(w), \Delta_2(v), \Delta_2(w)\}. \tag{3.10}$$

Let $\mathcal{V}$ be a locally finite open refinement of $\{N(u)\}_{u \in H_0}$, and define

$$\mathcal{W} = \{V \in \mathcal{V} \colon V \cap \overline{P_\varepsilon^+} \neq \emptyset,\ V \cap \overline{P_\varepsilon^-} \neq \emptyset,\ V \cap \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} = \emptyset\}.$$

Set

$$\mathcal{U} := \bigcup_{V \in \mathcal{V} \setminus \mathcal{W}} V \cup \bigcup_{V \in \mathcal{W}} \{V \setminus \overline{P_\varepsilon^+}, V \setminus \overline{P_\varepsilon^-}\}.$$

Then $\mathcal{U}$ is a locally finite open refinement of $\{N(u)\}_{u \in H_0}$, satisfying

$$\text{if } U \cap \overline{P_\varepsilon^+} \neq \emptyset \text{ and } U \cap \overline{P_\varepsilon^-} \neq \emptyset, \text{ then } U \cap \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \neq \emptyset.$$

Let $\{\pi_U\}_{U \in \mathcal{U}}$ be a partition of unity subordinated to $\mathcal{U}$, defined by

$$\pi_U(u) := \left(\sum_{V \in \mathcal{U}} \rho_V(u)\right)^{-1} \rho_U(u), \quad \rho_U(u) = \operatorname{dist}(u, H_0 \setminus U).$$

For each $U \in \mathcal{U}$, we choose $a_U \in U$ such that:

- if $U \cap \overline{P_\varepsilon^+} \neq \emptyset$ then $a_U \in \overline{P_\varepsilon^+}$;
- if $U \cap \overline{P_\varepsilon^-} \neq \emptyset$ then $a_U \in \overline{P_\varepsilon^-}$.

Note that this is possible due to the construction of $\mathcal{U}$. Define the operator

$$B(u) = \sum_{U \in \mathcal{U}} \pi_U(u) A(a_U).$$

By local finiteness of $\mathcal{U}$, the operator B is locally Lipschitz continuous. Property (i) follows directly from the construction.

For (ii) and (iii), note that, for any $u \in H_0$,

$$\|Bu - Au\| \leq \sum_{U \in \mathcal{U}} \pi_U(u) \|A(a_U) - Au\|.$$

Combining (3.9) and (3.10), we obtain

$$\|Bu - Au\| < \frac{1}{2}\|u - Au\|, \tag{3.11}$$

$$\|Bu - Au\| < \frac{1}{2}\|u - Au\|(1 + C\|u\|^2)^{-1}. \tag{3.12}$$

Inequality (3.11) immediately yields (ii). By Lemma 3.3, (ii) and (3.12), we compute

$$\langle J'(u), u - Bu\rangle \geq \langle J'(u), u - Au\rangle - \|J'(u)\|_* \|Bu - Au\|$$

$$\geq \frac{1}{2}\|u - Au\|^2,$$

which proves (iii).

Finally, if (F_4) holds, then A is odd. Define

$$\widetilde{B}(u) = \frac{1}{2}(B(u) - B(-u)).$$

Then $\widetilde{B}$ is odd, locally Lipschitz continuous, and still satisfies (i)–(iii). We keep the notation B for simplicity. □

Next, we apply Theorems 2.7 and 2.9 to prove Theorem 1.1. Let $X = H$, $P = P_\varepsilon^+$, $Q = P_\varepsilon^-$, $W = P_\varepsilon^+ \cup P_\varepsilon^-$, and $\Sigma = \partial P_\varepsilon^- \cap \partial P_\varepsilon^+$. We first show that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional J at any level $c \in \mathbb{R}$.

Lemma 3.6. *$\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional J at any level $c \in \mathbb{R}$.*

Proof. To verify the claim, we construct a descending flow that yields an invariant set satisfying Definition 2.6, based on the locally Lipschitz continuous operator B from Lemma 3.5. Let $\varepsilon \in (0, \varepsilon_0)$ be fixed.

Since J satisfies the (PS)-condition, see Lemma 2.4, standard deformation arguments, compare with the proof of Lemma 3.6 in Liu–Wang–Zhang [14], imply that there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, there exists a continuous map $\sigma \colon [0,1] \times H \to H$ satisfying:

(i) $\sigma(0, u) = u$ for all $u \in H$;
(ii) $\sigma(t, u) = u$ for all $t \in [0,1]$ and all $u \notin J^{-1}[c - \delta, c + \delta]$;
(iii) $\sigma(1, J^{c+\delta} \setminus W) \subset J^{c-\delta}$;
(iv) $\sigma(t, \overline{P_\varepsilon^+}) \subset \overline{P_\varepsilon^+}$ and $\sigma(t, \overline{P_\varepsilon^-}) \subset \overline{P_\varepsilon^-}$ for all $t \in [0,1]$.

Defining $\eta(u) := \sigma(1, u)$, we conclude that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional J at any level $c \in \mathbb{R}$. □

Corollary 3.7. *If (F_4) holds, then P_ε^+ is a G-admissible invariant set for the functional J at any level c.*

This follows by applying the same argument as in Lemma 3.6.

Lemma 3.8. *For $q \in [2, 6]$, there exists $\alpha_q > 0$, independent of ε, such that*

$$|u|_q \leq \alpha_q \varepsilon \quad \textit{for all } u \in M = P_\varepsilon^+ \cap P_\varepsilon^-.$$

Proof. For $u \in M = P_\varepsilon^+ \cap P_\varepsilon^-$, we have $\operatorname{dist}(u, P^+) < \varepsilon$ and $\operatorname{dist}(u, P^-) < \varepsilon$. Hence,

$$\|u\| = (\|u^+\|^2 + \|u^-\|^2)^{1/2} < \sqrt{2}\varepsilon.$$

By (3.8), for any $q \in [2, 6]$, there exists $\alpha_q > 0$, independent of ε, such that

$$|u|_q \leq \beta_q \|u\| < \alpha_q \varepsilon.$$

□

Lemma 3.9. *Under the assumptions of Theorem 1.1, if $\varepsilon > 0$ is sufficiently small, then*

$$J(u) \geq \frac{\varepsilon^2}{8} \quad \textit{for all } u \in \Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-.$$

In particular, $c_* \geq \frac{\varepsilon^2}{8}$.

Proof. Let $u \in \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$. Then $\|u^\pm\| \geq \operatorname{dist}(u, P^\mp) = \varepsilon$. By ($F_1$) and ($F_2$), for any $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$F(x,t) \leq \frac{1}{4\alpha_2^2}|t|^2 + C_\alpha |t|^p \quad \text{for all } t \in \mathbb{R}.$$

Combining Lemmas 2.1 and 3.8 and taking ε sufficiently small, we obtain

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{4\alpha_2^2}|u|_2^2 - C_\alpha |u|_p^p \geq \frac{\varepsilon^2}{8}.$$

□

Proof of Theorem 1.1. Choose $u_1, u_2 \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ such that

$$\operatorname{supp}(u_1) \cap \operatorname{supp}(u_2) = \emptyset, \quad u_1 \leq 0, \quad u_2 \geq 0.$$

For $(t,s) \in \Delta$, we define $\varphi_0(t,s) := R(tu_1 + su_2)$. Then, for all $t, s \in [0,1]$, we have

$$\varphi_0(0,s) = Rsu_2 \in P_\varepsilon^+ \quad \text{and} \quad \varphi_0(t,0) = Rtu_1 \in P_\varepsilon^-.$$

Hence, (i) of Theorem 2.7 holds.

For every $t \in [0,1]$, set $u_t = \varphi_0(t, 1-t)$. Since the supports of u_1 and u_2 are disjoint, we obtain

$$\|u_t\|^2 = R^2(t^2\|u_1\|^2 + (1-t)^2\|u_2\|^2) \tag{3.13}$$

and for $q \in [2,6]$,

$$|u_t|_q^q = R^q(t^q|u_1|_q^q + (1-t)^q|u_2|_q^q).$$

Hence $|u_t|_q^q \to \infty$ as $R \to \infty$ uniformly in $t \in [0,1]$. By Lemma 3.8, choosing R sufficiently large, yields $\varphi_0(\partial_0\Delta) \cap M = \emptyset$, so condition (ii) in Theorem 2.7 holds.

From (F_3), there exist constants $C_1, C_2 > 0$ such that

$$F(x,t) \geq C_1|t|^{\mu_1} - C_2 \quad \text{for all } t \in \mathbb{R}.$$

Then, by Lemma 2.1, (2.4) and (3.13), we have

$$\begin{aligned} J(u_t) &\leq \frac{1}{2}\|u_t\|^2 + C_4\|u_t\|^2 - \int_{\operatorname{supp}(u_1)\cup\operatorname{supp}(u_2)} F(x,u_t) \\ &\leq C_4\|u_t\|^2 - C_1|u_t|_{\mu_1}^{\mu_1} + C_2. \end{aligned}$$

It is obvious that $J(u_t) \to -\infty$ as $R \to \infty$ uniformly in $t \in [0,1]$. According to Lemma 3.9, for R sufficiently large and $\varepsilon > 0$ sufficiently small, it follows

$$\sup_{u \in \varphi_0(\partial_0\Delta)} J(u) < 0 < c_*.$$

Thus (iii) in Theorem 2.7 is also satisfied. Then, by Theorem 2.7, the functional J possesses at least one critical point $u \in H \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$, which is a sign-changing solution of (1.1).

Now assume that (F_4) holds. Choose functions $\{u_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ such that

$$\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset \quad \text{for } i \neq j.$$

For any $n \in \mathbb{N}$, let $t = (t_1, t_2, \dots, t_n) \in B_n$ and define $\varphi_n \in C(B_n, H)$ by

$$\varphi_n(t) = R_n \sum_{i=1}^n t_i u_i.$$

Arguing as above, for R_n sufficiently large, all assumptions of Theorem 2.9 are satisfied. Hence J admits infinitely many critical points $\{u_j\} \subset H \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$, which correspond to infinitely many sign-changing solutions of (1.1). Moreover, $J(u_j) \to \infty$ as $j \to \infty$. This completes the proof. □

4. Proof of Theorem 1.2

In this section, since the (AR)-condition is no longer assumed, certain properties of the auxiliary operator cannot be obtained directly. To overcome this difficulty, inspired by Wen–Tang–Chen [20], we employ a contradiction argument. We establish Theorem 1.2 by applying an abstract critical point theorem. As in Section 3, we first introduce an auxiliary operator $\tilde{A}\colon H \to H$. For any $u \in H$, we define $v = \tilde{A}u \in H$ as the unique solution of the equation (3.1).

Arguing as in Lemma 3.1, the operator $\tilde{A}$ is well defined, continuous, and compact. The proofs are analogous and therefore omitted.

Lemma 4.1. *The following statements hold:*

(i) $\langle J'(u), u - \tilde{A}u\rangle \geq \|u - \tilde{A}u\|^2$ *for all* $u \in H$*;*

(ii) *there exists a constant* $C > 0$*, independent of* u*, such that* $\|J'(u)\|_* \leq \|u - \tilde{A}u\|(1 + C\|u\|^2)$ *for all* $u \in H$*.*

Lemma 4.2. *For any* $a < b$ *and* $\alpha > 0$*, there exists* $\beta > 0$ *such that* $\|u - \tilde{A}u\| \geq \beta$ *whenever* $u \in H$*,* $J(u) \in [a, b]$ *and* $\|J'(u)\|_* \geq \alpha$*.*

Proof. For $u \in H$, by (2.4) and (3.1), together with Lemma 2.1 and (F$_3$'), it follows that

$$\begin{aligned}
&J(u) - \frac{1}{\mu_2}(u, u - \tilde{A}u) \\
&= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \phi_u^2 u^2 \,\mathrm{d}x + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x + \frac{3\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} F(x, u) \,\mathrm{d}x - \frac{1}{\mu_2}\|u\|^2 + \frac{1}{\mu_2}\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u u\tilde{A}u \,\mathrm{d}x \\
&\quad + \frac{1}{\mu_2}\int_{\mathbb{R}^3} f(x, u)u \,\mathrm{d}x \qquad\qquad (4.1)\\
&\geq \left(\frac{1}{2} - \frac{1}{\mu_2}\right)\|u\|^2 + \frac{2\omega}{\mu_2}\int_{\mathbb{R}^3} \phi_u u\tilde{A}u \,\mathrm{d}x + \frac{1}{\mu_2}\int_{\mathbb{R}^3} \phi_u^2 u\tilde{A}u \,\mathrm{d}x - \frac{\gamma}{\mu_2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu_2}\right)\|u\|^2 + \frac{2\omega}{\mu_2}\int_{\mathbb{R}^3} \phi_u u(\tilde{A}u - u) \,\mathrm{d}x + \frac{1}{\mu_2}\int_{\mathbb{R}^3} \phi_u^2 u(\tilde{A}u - u) \,\mathrm{d}x \\
&\quad - \frac{2\omega^2}{\mu_2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x - \frac{\gamma}{\mu_2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x.
\end{aligned}$$

By Lemma 2.1 and Hölder's inequality, we derive

$$\left|\int_{\mathbb{R}^3} \phi_u u(\tilde{A}u - u)\right| \leq C_1\|u\|\|u - \tilde{A}u\|,$$

$$\left|\int_{\mathbb{R}^3} \phi_u^2 u(\tilde{A}u - u)\right| \leq C_2\|u\|\|u - \tilde{A}u\|.$$

Together with (4.1), it follows that

$$|J(u)| + C_3\|u\|\|u - \tilde{A}u\| \geq \left(\frac{1}{2} - \frac{1}{\mu_2}\right)\|u\|^2 - \left(\frac{2\omega^2}{\mu_2} + \frac{\gamma}{\mu_2}\right)\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x. \qquad (4.2)$$

Assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H$ such that $J(u_n) \in [a,b]$, $\|J'(u_n)\|_* \geq \alpha$, and $\|u_n - \tilde{A}u_n\| \to 0$. Arguing as in Lemma 2.5, we deduce from (4.2) that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H. By Lemma 4.1 (ii), we obtain $\|J'(u_n)\|_* \to 0$ as $n \to \infty$, which is a contradiction. □

Lemma 4.3. *There exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$, the following statements hold:*

(i) *$\tilde{A}(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ and every nontrivial solution $u \in P_\varepsilon^-$ is negative;*
(ii) *$\tilde{A}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and every nontrivial solution $u \in P_\varepsilon^+$ is positive.*

Lemma 4.4. *There exists a locally Lipschitz continuous operator*

$$\tilde{B}\colon H \setminus \tilde{E} \to H, \quad \tilde{E} := \mathrm{Fix}(\tilde{A}),$$

such that the following statements hold:

(i) *$\tilde{B}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and $\tilde{B}(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ for $\varepsilon \in (0, \varepsilon_1)$;*
(ii) *$\frac{1}{2}\|u - \tilde{B}u\| \leq \|u - \tilde{A}u\| \leq 2\|u - \tilde{B}u\|$ for all $u \in H \setminus \tilde{E}$;*
(iii) *$\langle J'(u), u - \tilde{B}u\rangle \geq \frac{1}{2}\|u - \tilde{A}u\|^2$ for all $u \in H \setminus \tilde{E}$;*
(iv) *if* (F$_4$) *holds, then $\tilde{B}$ is odd.*

Lemma 4.5. *Under the assumptions of Theorem 1.2, if $\varepsilon > 0$ is sufficiently small, then $J(u) \geq \frac{\varepsilon^2}{8}$ for all $u \in \Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, that is, $c_* \geq \frac{\varepsilon^2}{8}$.*

Proof. From assumption (F$_1$'), we still have $F(x,t) \leq \frac{1}{4\alpha_2^2}|t|^2 + C_\alpha |t|^p$ for all $t \in \mathbb{R}$ uniformly in $x \in \mathbb{R}^3$. Then, following the same reasoning as in Lemma 3.9, we can prove that $c_* \geq \frac{\varepsilon^2}{8}$. □

We apply Theorem 2.7 to the functional J. We claim that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for J at any level c.

This follows from the same argument as in Section 3, combined with Lemmas 2.5 and 4.1–4.5, which ensure the validity of the deformation properties required in Definition 2.6. Therefore, $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional J at any level c. Consequently, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Choose $u_1, u_2 \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ such that

$$\operatorname{supp}(u_1) \cap \operatorname{supp}(u_2) = \emptyset, \quad u_1 \leq 0, \quad u_2 \geq 0.$$

For $(t,s) \in \Delta$, define

$$\varphi_0(t,s) := R\left(tu_1 + su_2\right).$$

As in the proof of Theorem 1.1, the map φ_0 satisfies conditions (i) and (ii) of Theorem 2.7.

By (F$_3$'), there exists $C_1, C_2 > 0$ such that $F(x,t) \geq -C_1|t|^2 + C_2|t|^{\mu_2}$ for all $t \in \mathbb{R}$, uniformly in $x \in \mathbb{R}^3$. Using Lemma 2.1, (2.4) and (3.13), we obtain

$$\begin{aligned} J(u_t) &\leq \frac{1}{2}\|u_t\|^2 + C_3\|u_t\|^2 - \int_{\operatorname{supp}(u_1)\cup\operatorname{supp}(u_2)} F(x,u_t) \\ &\leq C_4\|u_t\|^2 + C_1|u_t|_2^2 - C_2|u_t|_{\mu_2}^{\mu_2}. \end{aligned}$$

Since $\mu_2 > 2$, it follows that $J(u_t) \to -\infty$ as $R \to \infty$ uniformly in $t \in [0,1]$. According to Lemma 4.5, for R sufficiently large and sufficiently small $\varepsilon > 0$, we have

$$\sup_{u\in\varphi_0(\partial_0\Delta)} J(u) < 0 < c_*.$$

Therefore,

$$c = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\setminus W} J(u)$$

is a critical value of J with $c \geq c_*$, and there exists $u \in H \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$ such that $J(u) = c$ and $J'(u) = 0$. Hence, (u, ϕ_u) is a sign-changing solution of (1.1).

Next, choose $\{u_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ such that $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset$ for $i \neq j$. For $t = (t_1, t_2, \ldots, t_n) \in B_n$, we define $\varphi_n \in C(B_n, H)$ by

$$\varphi_n(t) = R_n \sum_{i=1}^n t_i u_i, \quad R_n > 0.$$

Arguing as above, for sufficiently large R_n, all assumptions of Theorem 2.9 are satisfied. For any $j \geq 2$, it follows that

$$0 < \inf_{u\in\Sigma} J(u) := c_* \leq c_j := \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} J(u) \to \infty, \quad \text{as } j \to \infty$$

and there exists $\{u_j\}_{j\geq 2} \subset H \setminus W$ such that $J(u_j) = c_j$ and $J'(u_j) = 0$. The proof is complete. □

Acknowledgments

This work is supported by the Natural Science Foundation of Chongqing, China (No. CSTB2024NSCQ-MSX0843), the Research Fund of the National Natural Science Foundation of China (No. 12361024) and the Team Building Project for Graduate Tutors in Chongqing (No. yds223010).

References

[1] T. Bartsch, Z. Liu, *On a superlinear elliptic p-Laplacian equation*, J. Differential Equations **198** (2004), no. 1, 149–175.

[2] M. Born, L. Infeld, *Foundations of the new field theory*, Proc. R. Soc. Lond., Ser. A **144** (1934), 425–451.

[3] G. Che, H. Chen, *Infinitely many solutions for the Klein-Gordon equation with sublinear nonlinearity coupled with Born-Infeld theory*, Bull. Iranian Math. Soc. **46** (2020), no. 4, 1083–1100.

[4] S.-J. Chen, L. Li, *Multiple solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory on* $\mathbb{R}^3$, J. Math. Anal. Appl. **400** (2013), no. 2, 517–524.

[5] S.-J. Chen, S.-Z. Song, The existence of multiple solutions for the Klein-Gordon equation with concave and convex nonlinearities coupled with Born-Infeld theory on $\mathbb{R}^3$, Nonlinear Anal. Real World Appl. **38** (2017), 78–95.

[6] S. Chen, X. Tang, *Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in* $\mathbb{R}^3$, Z. Angew. Math. Phys. **67** (2016), no. 4, Art. 102, 18 pp.

[7] P. d'Avenia, L. Pisani, *Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations*, Electron. J. Differential Equations **2002** (2002), No. 26, 13 pp.

[8] J. Fei, Q. Zhang, *On solutions for a class of Klein-Gordon equations coupled with Born-Infeld theory with Berestycki-Lions conditions on* $\mathbb{R}^3$, Electron. Res. Arch. **32** (2024), no. 4, 2363–2379.

[9] D. Fortunato, L. Orsina, L. Pisani, *Born-Infeld type equations for electrostatic fields*, J. Math. Phys. **43** (2002), no. 11, 5698–5706.

[10] C.-M. He, L. Li, S.-J. Chen, D. O'Regan, *Ground state solution for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical exponents*, Anal. Math. Phys. **12** (2022), no. 2, Paper No. 48, 17 pp.

[11] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 4, 787–809.

[12] M.-J. Liao, L. Li, S.-J. Chen, *Existence of solutions for Klein-Gordon-Born-Infeld system with steep potential well*, Differ. Equ. Dyn. Syst. (2025), https://doi.org/10.1007/s12591-025-00736-y

[13] J. Liu, X. Liu, Z.-Q. Wang, *Multiple mixed states of nodal solutions for nonlinear Schrödinger systems*, Calc. Var. Partial Differential Equations **52** (2015), no. 3-4, 565–586.

[14] Z. Liu, Z.-Q. Wang, J. Zhang, *Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 3, 775–794.

[15] D. Mugnai, *Coupled Klein-Gordon and Born-Infeld-type equations: looking for solitary waves*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **460** (2004), no. 2045, 1519–1527.

[16] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. **60** (1985), no. 4, 558–581.

[17] K. Teng, *Existence and multiplicity of solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory on bounded domain*, Differ. Equ. Appl. **4** (2012), no. 3, 445–457.

[18] K. Teng, K. Zhang, *Existence of solitary wave solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical Sobolev exponent*, Nonlinear Anal. **74** (2011), no. 12, 4241–4251.

[19] L. Wang, C. Xiong, D. Zhang, *Multiple solutions for nonhomogeneous Klein-Gordon equation with sign-changing potential coupled with Born-Infeld theory*, J. Appl. Anal. Comput. **14** (2024), no. 1, 84–105.

[20] L. Wen, X. Tang, S. Chen, *Infinitely many solutions and least energy solutions for Klein-Gordon equation coupled with Born-Infeld theory*, Complex Var. Elliptic Equ. **64** (2019), no. 12, 2077–2090.

[21] Y. Yu, *Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **27** (2010), no. 1, 351–376.

[22] Q. Zhang, *Sign-changing solutions for a kind of Klein-Gordon-Maxwell system*, J. Math. Phys. **62** (2021), no. 9, Paper No. 091507, 9 pp.

[23] Z. Zhang, J. Liu, *Existence and multiplicity of sign-changing solutions for Klein-Gordon equation coupled with Born-Infeld theory with subcritical exponent*, Qual. Theory Dyn. Syst. **22** (2023), no. 1, Paper No. 7, 29 pp.

[24] X.-J. Zhong, C.-L. Tang, *Ground state sign-changing solutions for a Schrödinger-Poisson system with a 3-linear growth nonlinearity*, J. Math. Anal. Appl. **455** (2017), no. 2, 1956–1974.

[25] W. Zou, M. Schechter, "Critical Point Theory and Its Applications", Springer, New York, 2006.

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