LOGARITHMIC DOUBLE PHASE PROBLEMS WITH CRITICAL GROWTH ON THE BOUNDARY

EYLEM ÖZTÜRK AND PATRICK WINKERT

ABSTRACT. In this paper, we study logarithmic double phase problems with critical growth on the boundary of the form

$$-\operatorname{div}\mathcal{L}(u)=-|u|^{p-2}u \quad \text{in } \Omega, \quad \mathcal{L}(u)\cdot\nu=f(x,u)+|u|^{p_*-2}u \quad \text{on } \partial\Omega,$$

where $\operatorname{div}\mathcal{L}$ stands for the logarithmic double phase operator given by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left[\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right]|\nabla u|^{q-2}\nabla u\right),$$

e is Euler's number, $\nu(x)$ is the outer unit normal of Ω at $x\in\partial\Omega$, $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $1<p<N$, $p<q<p_*=\frac{(N-1)p}{N-p}$, $\mu\in L^\infty(\Omega)$ with $\mu\geq 0$, and $f\colon\partial\Omega\times[-\mathcal{K},\mathcal{K}]\to\mathbb{R}$ for some $\mathcal{K}>0$ is a Carathéodory function, just locally defined with a specific behavior near the origin. Using suitable truncation methods and an appropriate auxiliary problem along with an equivalent norm in our function space, we establish the existence of an entire sequence of sign-changing solutions to the above problem, which converges to zero in both the logarithmic Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ and in $L^\infty(\Omega)$.

1. Introduction

Recently, Arora–Crespo-Blanco–Winkert [4] introduced the logarithmic double phase operator defined by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right), \tag{1.1}$$

where $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ with the corresponding logarithmic Musielak-Orlicz Sobolev space generated by the nonlinear function

$$\mathcal{H}_{\log}(x,t)=t^p+\mu(x)t^q\log(e+t) \quad \text{for all } (x,t)\in\overline{\Omega}\times[0,\infty),$$

for $1<p<N$, $p<q$, e is Euler's number and $0\leq\mu(\cdot)\in L^\infty(\Omega)$. The logarithmic double phase structure is introduced in order to model a borderline situation between standard polynomial growth and nearly linear behavior. Compared to the classical double phase density $t^p+\mu(x)t^q$, the additional logarithmic factor produces a growth which is only slightly stronger than the pure q-growth, while still preserving the variational framework. This logarithmic perturbation leads to new analytical features in the associated operator, in particular in the scaling properties and in the control of higher order terms, which cannot be treated by a direct

2020 *Mathematics Subject Classification.* 35A01, 35B40, 35J20, 35J25, 35J62, 35Q74.
Key words and phrases. Critical growth on the boundary, equivalent norm, logarithmic double phase operator, sign-changing solutions, Steklov eigenvalue problem, symmetric mountain pass theorem.

adaptation of the classical double phase theory. The related energy functional of (1.1) is given by

$$u \mapsto \int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x) \frac{|\nabla u|^q}{q} \log(e + |\nabla u|) \right) \mathrm{d}x, \tag{1.2}$$

which has been studied for special cases in recent years. Here we mention the works by Baroni–Colombo–Mingione [7] and De Filippis–Mingione [11] related to local Hölder continuity of the gradient of local minimizers of functionals like (1.2). The functional studied in [11] has its origin from functionals with nearly linear growth given by

$$u \mapsto \int_\Omega |\nabla u| \log(1 + |\nabla u|) \, \mathrm{d}x, \tag{1.3}$$

see the studies in the papers by Fuchs–Mingione [14] and Marcellini–Papi [23]. We note that (1.3) arises in the context of plasticity models with logarithmic hardening; see, for instance, Seregin–Frehse [32] and Fuchs–Seregin [15]. Furthermore, the celebrated work of Marcellini [22] covers, as a particular case, functionals containing a logarithmic term of the form

$$u \mapsto \int_\Omega (1 + |\nabla u|^2)^{\frac{p}{2}} \log(1 + |\nabla u|) \, \mathrm{d}x.$$

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, in this paper we consider logarithmic double phase problems with critical growth on the boundary given by

$$-\operatorname{div} \mathcal{L}(u) = -|u|^{p-2}u \quad \text{in } \Omega, \quad \mathcal{L}(u) \cdot \nu = f(x,u) + |u|^{p_*-2}u \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\nu(x)$ is the outer unit normal of Ω at $x \in \partial\Omega$ and $\operatorname{div} \mathcal{L}$ stands for the logarithmic double phase operator given in (1.1). We assume the following hypotheses on the data of problem (1.4):

(C1) $1 < p < N$, $p < q < p_* = \frac{(N-1)p}{N-p}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$;

(C2) $f \colon \partial\Omega \times [-\mathcal{K}, \mathcal{K}] \to \mathbb{R}$ is a Carathéodory function for $\mathcal{K} > 0$ with $f(x,0) = 0$ and $f(x,\cdot)$ is odd for a.a. $x \in \partial\Omega$;

(C3) there exists $\mathcal{T} \in L^\infty(\partial\Omega)$ such that

$$|f(x,s)| \leq \mathcal{T}(x) \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } |s| \leq \mathcal{K};$$

(C4) there exists $\xi \in \left(1, \min\{p, \frac{p^2}{N-p} \cdot \frac{N-1}{N} + 1\}\right)$ such that

$$\lim_{s \to 0} \frac{f(x,s)}{|s|^{\xi-2}s} = 0 \quad \text{uniformly for a.a. } x \in \partial\Omega;$$

(C5)

$$\lim_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} = +\infty \quad \text{uniformly for a.a. } x \in \partial\Omega.$$

We say that $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ is a weak solution of problem (1.4) if for every test function $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$, the following holds:

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u + \mu(x) \left(\log(e + |\nabla u|) + \frac{|\nabla u|}{q(e + |\nabla u|)} \right) |\nabla u|^{q-2}\nabla u \right] \cdot \nabla v \, \mathrm{d}x$$
$$+ \int_\Omega |u|^{p-2}uv \, \mathrm{d}x = \int_{\partial\Omega} \left(f(x,u) + |u|^{p_*-2}u \right) v \, \mathrm{d}\sigma.$$

Our main result is the following theorem.

Theorem 1.1. *Let the conditions* (C1)–(C5) *be satisfied. Then, problem* (1.4) *possesses a sequence* $\{z_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *of sign-changing solutions such that* $\|z_n\| \to 0$ *in* $W^{1,\mathcal{H}_{\log}}(\Omega)$ *and* $\|z_n\|_\infty \to 0$ *in* $L^\infty(\Omega)$ *as* $n \to \infty$.

We emphasize that the nonlinear Neumann boundary condition of (1.4) reflects the combined influence of a locally defined Carathéodory function $f(x,\cdot)$ and a critical growth term of the form

$$u \mapsto |u|^{p_*-2}u, \quad p_* := \frac{(N-1)p}{N-p}, \quad 1 < p < N.$$

The principal analytical challenge lies in handling this critical boundary term, whose presence causes a loss of compactness along with the appearance of the logarithmic double phase operator with logarithm perturbation. In order to overcome these difficulties, we first study an appropriate auxiliary problem, constructed via suitable truncation functions to ensure coercivity by using properties of the Steklov eigenvalue problem of the p-Laplacian. We then establish the existence of extremal constant-sign solutions to the auxiliary problem, which in turn allows us to apply Kajikiya's symmetric mountain pass theorem [18]. Our contribution extends the work of Liu–Papageorgiou [21] from the double phase setting to the logarithmic double phase framework with critical nonlinear boundary growth, while also relaxing the structural hypotheses in [21]. For related developments, we refer to Carranza–Pimenta–Vetro–Winkert [9] and Papageorgiou–Vetro–Winkert [28].

As noted at the beginning of the Introduction, the logarithmic double phase operator (1.1) is a recent development, and the literature on problems involving this operator remains scarce. The first contribution appears in the work by Arora–Crespo-Blanco–Winkert [4], who studied the problem

$$-\operatorname{div}\mathcal{L}(u) = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where $\operatorname{div}\mathcal{L}$ is given by (1.1) but with variable exponents and $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with subcritical growth and prescribed behavior both at infinity and near the origin. Under the additional assumption $q+1 < p^*$, the authors established the existence of a least energy sign-changing solution by minimizing the associated energy functional over the corresponding Nehari manifold of (1.5), see also a related work by the same authors [3] concerning optimal growth conditions to (1.1). The operator (1.1) also features in the work by Vetro–Winkert [37], where the authors established boundedness, closedness, and compactness of the solution set to the problem

$$-\operatorname{div}\mathcal{L}(u) = f(x,u,\nabla u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with $\operatorname{div}\mathcal{L}$ as in (1.1) but involving variable exponents, and $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ being a convection term subject to very mild structural conditions. Moreover, in [36], Vetro investigated a Kirchhoff-type problem driven by the same operator.

Very recently, Borer–Gasiński–Stapenhorst–Winkert [8] studied least energy sign changing solutions of the problem

$$-\operatorname{div}\mathcal{L}(u) + |u|^{p(x)-2}u = f(x,u) \quad \text{in } \Omega, \quad \mathcal{K}(u)\cdot\nu = g(x,u) - |u|^{p(x)-2}u \quad \text{on } \partial\Omega,$$

where $\operatorname{div}\mathcal{L}$ denotes the logarithmic double phase operator given in (1.1) but with variable exponents while $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ as well as $g\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions having subcritical growths and a certain behavior both at infinity and near the origin. As a result of independent interest, the authors in [8] also proved boundedness results for such type of equations which we used in our paper and they showed the existence of an equivalent norm in the space $W^{1,\mathcal{H}_{\log}}(\Omega)$ even for variable exponents given as

$$\|u\|_{1,\mathcal{H}_{\log}}^{\circ} = \inf \left\{ \lambda > 0 \colon \int_\Omega \left(\left| \frac{\nabla u}{\lambda} \right|^{p(x)} + \mu(x) \left| \frac{\nabla u}{\lambda} \right|^{q(x)} \log \left(e + \frac{|\nabla u|}{\lambda} \right) \right) \mathrm{d}x \right.$$
$$\left. + \int_\Omega \omega_1(x) \left| \frac{u}{\lambda} \right|^{\zeta_1(x)} \mathrm{d}x + \int_{\partial\Omega} \omega_2(x) \left| \frac{u}{\lambda} \right|^{\zeta_2(x)} \mathrm{d}\sigma \leq 1 \right\},$$

where the exponents $1 \leq \zeta_1(\cdot), \zeta_2(\cdot) \in C(\overline{\Omega})$ are allowed to be critical with respect to the exponent $1 < p(\cdot) \in C(\overline{\Omega})$. We are going to use this equivalent norm in our paper as well. A different form of a logarithmic double phase operator, distinct from (1.1), was introduced by Vetro–Zeng [38], who investigated the existence and uniqueness of solutions to equations driven by

$$u \mapsto \Delta_{\mathcal{H}_L} u = \operatorname{div} \left(\frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|} \nabla u \right), \quad u \in W_0^{1,\mathcal{H}_L}(\Omega),$$

where $\mathcal{H}_L \colon \Omega \times [0,\infty) \to [0,\infty)$ is defined by

$$\mathcal{H}_L(x,t) = (t^p + \mu(x) t^q) \log(e+t),$$

with $1 < p < q$. Moreover, we refer to the paper by Tran–Nguyen [35] who showed existence results for equations involving (1.1) when $p = q$.

In addition, we refer to various studies that investigate logarithmic perturbations appearing on the right-hand side in the setting of Schrödinger equations and p-Laplace type problems as well as double phase operators without logarithm. We refer to the works by Alves–de Morais Filho [1], Alves–Ji [2], Bahrouni–Fiscella–Winkert [5, 6], Figueiredo–Montenegro–Stapenhorst [12, 13], Montenegro–de Queiroz [25], Shuai [33], and Squassina–Szulkin [34], see also the references therein.

The paper is organized as follows. Section 2 presents a review of the properties of logarithmic Musielak-Orlicz Sobolev spaces and the logarithmic double phase operator (1.1). Additionally, we summarize the main results concerning the eigenvalue problem for the p-Laplacian with Steklov boundary conditions. In Section 3, we focus on an auxiliary problem, proving the existence of extremal constant-sign solutions, and subsequently apply the results of Kajikiya [18] to establish the proof of Theorem 1.1.

2. Mathematical background

This section provides an overview of the key properties of logarithmic Musielak-Orlicz Sobolev spaces, the associated logarithmic double phase operator, and several tools required for the upcoming sections. We refer to the recent work by Arora–Crespo-Blanco–Winkert [4] as well as the monographs by Harjulehto–Hästö [16] and Papageorgiou–Winkert [29], see also the paper by Crespo-Blanco–Gasiński–Harjulehto–Winkert [10] for the main properties of double phase operators without logarithm.

To this end, by $L^r(\Omega)$ we denote the Lebesgue space with norm $\|\cdot\|_r$ for $1 \leq r \leq \infty$ and $W^{1,r}(\Omega)$ stands for the corresponding Sobolev space equipped with the equivalent norm $\|\cdot\|_{1,r} = (\|\nabla \cdot\|_r^r + \|\cdot\|_r^r)^{\frac{1}{r}}$ for $1 < r < \infty$. For $A \subseteq \Omega$, we denote by $|A|$ the Lebesgue measure of the set A. Moreover, let σ be the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$ and indicate by $L^r(\partial\Omega)$ the boundary Lebesgue space equipped with the norm $\|\cdot\|_{r,\partial\Omega}$ given by

$$\|u\|_{r,\partial\Omega} = \left(\int_{\partial\Omega} |u|^r \,\mathrm{d}\sigma\right)^{\frac{1}{r}} \quad \text{for } u \in L^r(\partial\Omega).$$

Throughout this paper, we avoid explicitly using the trace operator γ and interpret all restrictions of Sobolev functions to the boundary $\partial\Omega$ in the sense of traces.

In the following we suppose that (C1) holds, e stands for Euler's number and we denote by $M(\Omega)$ the set of all measurable functions $u\colon \Omega \to \mathbb{R}$. Given the function $\mathcal{H}_{\log}\colon \overline{\Omega} \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^p + \mu(x) t^q \log(e+t),$$

we are able to introduce the space $L^{\mathcal{H}_{\log}}(\Omega)$ by

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{ u \in M(\Omega)\colon \rho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x < \infty \right\},$$

where $\rho_{\mathcal{H}_{\log}}$ is the related modular function to $\mathcal{H}_{\log}$, equipped with the norm

$$\|u\|_{\mathcal{H}_{\log}} := \inf \left\{ \lambda > 0\colon \rho_{\mathcal{H}_{\log}}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Further, the corresponding logarithmic Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{ u \in L^{\mathcal{H}_{\log}}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}}.$$

We know that both spaces $L^{\mathcal{H}_{\log}}(\Omega)$ and $W^{1,\mathcal{H}_{\log}}(\Omega)$ are separable and reflexive Banach spaces. In what follows, we denote by κ the constant given by

$$\kappa = \frac{e}{e+t_0}, \tag{2.1}$$

where t_0 is the positive number satisfying $t_0 = e\log(e+t_0)$.

From Proposition 3.1 by Borer–Gasiński–Stapenhorst–Winkert [8], we can equip the space $W^{1,\mathcal{H}_{\log}}(\Omega)$ with the equivalent norm

$$\begin{aligned} \|u\| = \inf \Bigg\{ \lambda > 0\colon & \int_\Omega \left(\left|\frac{\nabla u}{\lambda}\right|^p + \mu(x) \left|\frac{\nabla u}{\lambda}\right|^q \log\left(e + \left|\frac{\nabla u}{\lambda}\right|\right)\right) \mathrm{d}x \\ & + \int_\Omega \left|\frac{u}{\lambda}\right|^p \mathrm{d}x \leq 1 \Bigg\}, \end{aligned} \tag{2.2}$$

where the related modular is given by

$$\varrho(u) = \int_\Omega \left(|\nabla u|^p + \mu(x)|\nabla u|^q \log(e+|\nabla u|)\right) \mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x, \tag{2.3}$$

for all $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$.

The modular $\varrho(\cdot)$ in (2.3) and the norm $\|\cdot\|$ in (2.2) are related in the following form, see Borer–Gasiński–Stapenhorst–Winkert [8, Proposition 3.2].

Proposition 2.1. *Let hypotheses (C1) be satisfied, $\lambda > 0$, $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$, and κ as in (2.1). Then the following hold:*

(i) $\|u\| = \lambda$ *if and only if* $\varrho\left(\frac{u}{\lambda}\right) = 1$ *for* $u \neq 0$ *and* $\lambda > 0$;
(ii) $\|u\| < 1$ *(resp.* $= 1$, > 1*) if and only if* $\varrho(u) < 1$ *(resp.* $= 1$, > 1*);*
(iii) $\min\{\|u\|^p, \|u\|^{q+\kappa}\} \leq \varrho(u) \leq \max\{\|u\|^p, \|u\|^{q+\kappa}\}$;
(iv) $\|u\| \to 0$ *if and only if* $\varrho(u) \to 0$;
(v) $\|u\| \to \infty$ *if and only if* $\varrho(u) \to \infty$.

Further, we have the following embedding results, see Arora–Crespo-Blanco–Winkert [4, Proposition 3.7].

Proposition 2.2. *Let hypotheses* (C1) *be satisfied. Then the following hold:*
(i) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *is continuous and* $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^r(\Omega)$ *is compact for all* $1 \leq r < p^*$;
(ii) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ *is continuous and* $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is compact for all* $1 \leq r < p_*$.

The following lemma will be required in subsequent proofs, see Arora–Crespo-Blanco–Winkert [4, Lemma 5.4] for its proof.

Lemma 2.3. *Let* $Q > 1$ *and* $h\colon [0,\infty) \to [0,\infty)$ *given by* $h(t) = \frac{t}{Q(e+t)\log(e+t)}$. *Then* h *attains its maximum value at* t_0 *and the value is* $\frac{\kappa}{Q}$, *where* t_0 *and* κ *are the same as in* (2.1).

Now, let $A\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to W^{1,\mathcal{H}_{\log}}(\Omega)^*$ be the nonlinear operator defined by

$$\begin{aligned}\langle A(u), v\rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\ &\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right]|\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\ &\quad + \int_\Omega |u|^{p-2}uv\,\mathrm{d}x\end{aligned} \tag{2.4}$$

for all $u, v \in W^{1,\mathcal{H}_{\log}}(\Omega)$. The following proposition is taken from Borer–Gasiński–Stapenhorst–Winkert [8, Proposition 3.4].

Proposition 2.4. *Let hypothesis* (C1) *be satisfied. Then, the operator* A *given in* (2.4) *is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone and satisfies the* (S_+)*-property, that is,*

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0,$$

imply $u_n \to u$ *in* $W^{1,\mathcal{H}_{\log}}(\Omega)$.

In order to deal with the logarithm in the operator, we also need the following standard inequality

$$\log(e+xy) \leq \log(e+x) + \log(e+y) \quad \text{for all } x, y > 0. \tag{2.5}$$

Let $C^1(\overline{\Omega})$ be equipped with norm $\|\cdot\|_{C^1(\overline{\Omega})}$ and let $C^1(\overline{\Omega})_+$ be its positive cone defined by

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) \colon u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\},$$

which has a nonempty interior given by

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega})_+ \colon u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

Furthermore, for any $s \in \mathbb{R}$ we put $s_\pm = \max\{\pm s, 0\}$, that is, $s = s_+ - s_-$ and $|s| = s_+ + s_-$. Also, for any function $u\colon \Omega \to \mathbb{R}$ we write $u_\pm(\cdot) = [u(\cdot)]_\pm$.

Next, we want to recall some basic facts about the Steklov eigenvalue problem for the p-Laplacian with $p \in (1,\infty)$ fixed in (C1). This problem is defined by

$$-\Delta_p u = -|u|^{p-2}u \quad \text{in } \Omega, \quad |\nabla u|^{p-2}\nabla u \cdot \nu = \lambda |u|^{p-2}u \quad \text{on } \partial\Omega. \tag{2.6}$$

From Lê [19] we know that problem (2.6) has a smallest eigenvalue λ_1 which is positive, isolated, simple and can be characterized by

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \Big\{ \|\nabla u\|_p^p + \|u\|_p^p \colon \|u\|_{p,\partial\Omega}^p = 1 \Big\}. \tag{2.7}$$

In what follows we denote by u_1 the normalized (i.e., $\|u_1\|_{p,\partial\Omega} = 1$) positive eigenfunction corresponding to λ_1. From the regularity theory of Lieberman [20] and the maximum principle by Pucci–Serrin [31] we know that $u_1 \in \operatorname{int}\big(C^1(\overline{\Omega})_+\big)$.

Finally, we recall some facts about critical point theory. To this end, let X be a Banach space and X^* be its dual space. A functional $\varphi \in C^1(X)$ satisfies the Palais-Smale condition (PS-condition for short), if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence. Further, we define

$$K_\varphi := \{u \in X \colon \varphi'(u) = 0\},$$

being the set of all critical points of φ. Recall that a set $\mathcal{S} \subseteq X$ is called downward directed if for given $u_1, u_2 \in \mathcal{S}$ there exists $u \in \mathcal{S}$ such that $u \le u_1$ and $u \le u_2$. Similarly, $\mathcal{S} \subseteq X$ is called upward directed if for given $v_1, v_2 \in \mathcal{S}$ one can find $v \in \mathcal{S}$ such that $v_1 \le v$ and $v_2 \le v$.

3. Existence of sign-changing solutions

Our analysis starts with a truncated auxiliary problem, which serves to address the critical term in (1.4). For this purpose, let $\Phi \in C^1(\mathbb{R})$ be an even cut-off function with the following properties:

$$\operatorname{supp}\Phi \subseteq [-\mathcal{K}, \mathcal{K}], \quad \Phi\big|_{\left[\frac{-\mathcal{K}}{2}, \frac{\mathcal{K}}{2}\right]} \equiv 1 \quad \text{and} \quad 0 < \Phi \le 1 \quad \text{on } (-\mathcal{K}, \mathcal{K}). \tag{3.1}$$

As a next step, we define $\psi\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\psi(x,s) = \Phi(s)\left(f(x,s) + |s|^{p_*-2}s\right) + (1 - \Phi(s))|s|^{\xi-2}s, \tag{3.2}$$

with ξ as given in assumption (C4). Obviously, the function $\psi\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory type. With view to (3.1) along with (3.2) and (C4) we have the growth

$$|\psi(x,s)| \le C\left(1 + |s|^{\xi-1}\right) \tag{3.3}$$

for a.a. $x \in \partial\Omega$ and for all $s \in \mathbb{R}$ with some constant $C > 0$.

Our next objective is to investigate the solvability of the auxiliary problem.

$$-\operatorname{div}\mathcal{L}(u) = -|u|^{p-2}u \quad \text{in } \Omega, \quad \mathcal{L}(u)\cdot\nu = \psi(x,u) \quad \text{on } \partial\Omega, \tag{3.4}$$

with $\operatorname{div}\mathcal{L}(u)$ being the logarithmic double phase operator given in (1.1). We aim to prove the existence of extremal constant sign solutions to (3.4), which will serve as a foundation for constructing sign-changing solutions to the original problem (1.4). To this end, let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the sets of positive and negative solutions of problem (3.4), respectively. In what follows, we denote by $\Upsilon_\pm \colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ the truncated energy functionals corresponding to (3.4) defined by

$$\begin{aligned} \Upsilon_\pm(u) &= \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q \log(e+|\nabla u|)\right) \mathrm{d}x \\ &\quad + \int_\Omega \frac{1}{p}|u|^p \,\mathrm{d}x - \int_{\partial\Omega} \Psi(x, \pm u_\pm)\,\mathrm{d}\sigma \quad \text{for } u \in W^{1,\mathcal{H}_{\log}}(\Omega), \end{aligned} \tag{3.5}$$

where $\Psi(x,s) = \int_0^s \psi(x,t)\,\mathrm{d}t$. It is easy to see that $\Upsilon_\pm$ are C^1 functionals.

We begin by proving that the sets $\mathcal{S}_\pm$ are nonempty.

Proposition 3.1. *Let hypotheses (C1)–(C5) be satisfied. Then $\mathcal{S}_+$ and $\mathcal{S}_-$ are nonempty subsets in $W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$.*

Proof. First, we prove that $\mathcal{S}_+$ is nonempty. Note that

$$\Upsilon_+(u) \geq \frac{1}{q}\varrho(u) - \int_{\partial\Omega} \Psi(x,u_+)\,\mathrm{d}\sigma.$$

Using this fact together with the growth in (3.3), $\xi < p$ due to (C4) and Proposition 2.1 (iii), we conclude that Υ_+ is coercive. Taking Proposition 2.2 (ii) into account, it follows that $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact for any $1 \leq r < p_*$. This implies that the functional Υ_+ is sequentially weakly lower semicontinuous as well. As a consequence, there exists $u_0 \in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\Upsilon_+(u_0) = \inf\left[\Upsilon_+(u) \colon u \in W^{1,\mathcal{H}_{\log}}(\Omega)\right].$$

We are going to prove that u_0 is nontrivial. Using condition (C5), for each $\delta > 0$, we can find a number $\eta \in \left(0, \min\{\frac{\mathcal{K}}{2}, 1\}\right)$ such that

$$F(x,s) := \int_0^s f(x,t)\,\mathrm{d}t \geq \frac{\delta}{p}|s|^p \quad \text{for all } |s| \leq \eta. \tag{3.6}$$

Now we can choose $t \in (0,1)$ small enough such that $tu_1(x) \in (0,\eta]$ for all $x \in \overline{\Omega}$, where $u_1 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is the L^p-normalized (i.e. $\|u_1\|_{p,\partial\Omega} = 1$) positive eigenfunction corresponding to λ_1 of the Steklov eigenvalue problem of the p-Laplacian given in (2.6). From $tu_1(x) \in (0,\eta]$ for all $x \in \overline{\Omega}$ and $\eta \in \left(0, \min\{\frac{\mathcal{K}}{2}, 1\}\right)$, it follows from (3.1) that

$$\psi(x,tu_1) = f(x,tu_1) + (tu_1)^{p_*-2}tu_1 \geq f(x,tu_1). \tag{3.7}$$

Now, from (2.7), $\|u_1\|_{p,\partial\Omega} = 1$, (2.5), (3.6) and (3.7), we conclude that

$$\begin{aligned} \Upsilon_+(tu_1) &= \int_\Omega \left[\frac{1}{p}|\nabla(tu_1)|^p + \frac{\mu(x)}{q}|\nabla(tu_1)|^q \log(e+t|\nabla u_1|)\right] \mathrm{d}x \\ &\quad + \frac{1}{p}\int_\Omega |(tu_1)|^p \,\mathrm{d}x - \int_{\partial\Omega} \Psi\left(x, tu_1\right)\,\mathrm{d}\sigma \\ &\leq \frac{t^p}{p}\lambda_1 + \frac{t^q \log(e+t)}{q} \int_\Omega \mu(x)|\nabla u_1|^q \,\mathrm{d}x \\ &\quad + \frac{t^q}{q}\int_\Omega \mu(x)|\nabla u_1|^q \log(e+|\nabla u_1|)\,\mathrm{d}x - \frac{t^p}{p}\delta \\ &= \frac{t^p}{p}(\lambda_1 - \delta) + \frac{t^q \log(e+t)}{q} \int_\Omega \mu(x)|\nabla u_1|^q \,\mathrm{d}x \\ &\quad + \frac{t^q}{q}\int_\Omega \mu(x)|\nabla u_1|^q \log(e+|\nabla u_1|)\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

Recall that $p < q$. Thus, if $\delta > \lambda_1$, then (3.8) implies

$$\Upsilon_+(tu_1) < 0 \quad \text{for } t > 0 \text{ sufficiently small.}$$

This proves that $u_0 \neq 0$.

Note that u_0 is the global minimizer of Υ_+, that is, $\Upsilon_+'(u_0) = 0$. Hence, we have

$$\int_\Omega \left(|\nabla u_0|^{p-2}\nabla u_0 + \mu(x)\left(\log(e+|\nabla u_0|) + \frac{|\nabla u_0|}{q(e+|\nabla u_0|)}\right)|\nabla u_0|^{q-2}\nabla u_0 \right)\cdot \nabla v \,\mathrm{d}x$$
$$+ \int_\Omega |u_0|^{p-2}u_0 v\,\mathrm{d}x = \int_{\partial\Omega} \psi(x,(u_0)_+)v\,\mathrm{d}\sigma$$

for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$. Now, choosing $v = -(u_0)_- \in W^{1,\mathcal{H}_{\log}}(\Omega)$ as test function gives $(u_0)_- = 0$. Hence, we have $u_0 \geq 0$ with $u_0 \neq 0$, that is, $u_0 \in W^{1,\mathcal{H}_{\log}}(\Omega$ is a nontrivial positive weak solution of problem (3.4). Furthermore, from Borer–Gasiński–Stapenhorst–Winkert [8, Theorem 4.1], we conclude that $u_0 \in W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$. This shows that $\mathcal{S}_+$ is nonempty. In a similar way, we are able to show the existence of a nontrivial negative weak solution $v_0 \in W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ of problem (3.4) which is the global minimizer of $\Upsilon_- \colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ defined in (3.5). □

Next, we are going to prove that the auxiliary problem (3.4) admits extremal constant sign solutions, that is, there exist a smallest positive solution $u^* \in \mathcal{S}_+$ and a largest negative solution $v^* \in \mathcal{S}_-$.

Proposition 3.2. *Let hypotheses (C1)–(C5) be satisfied. Then we can find elements $u^* \in \mathcal{S}_+$ and $v^* \in \mathcal{S}_-$ such that $u^* \leq u$ for all $u \in \mathcal{S}_+$ and $v^* \geq v$ for all $v \in \mathcal{S}_-$.*

Proof. We only prove the existence of $u^* \in \mathcal{S}_+$, in a similar way one can show the existence of $v^* \in \mathcal{S}_-$. Analogously to the proof of Proposition 7 by Papageorgiou–Rădulescu–Repovš [27, Proposition 7], we can show that the set $\mathcal{S}_+$ is downward directed, which implies, by using Lemma 3.10 of Hu–Papageorgiou [17] that we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ such that $\inf_{n\in\mathbb{N}} u_n = \inf \mathcal{S}_+$. Therefore, due to $u_n \in \mathcal{S}_+$ for all $n \in \mathbb{N}$, it holds

$$\begin{aligned}
&\int_\Omega \Bigg(|\nabla u_n|^{p-2}\nabla u_n \\
&\qquad + \mu(x)\left(\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right)|\nabla u_n|^{q-2}\nabla u_n \Bigg)\cdot \nabla v\,\mathrm{d}x \\
&+ \int_\Omega |u_n|^{p-2}u_n v\,\mathrm{d}x \\
&= \int_{\partial\Omega} \psi(x,u_n)v\,\mathrm{d}\sigma
\end{aligned} \tag{3.9}$$

for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$ and for all $n \in \mathbb{N}$. Next, we set $v = u_n \in W^{1,\mathcal{H}_{\log}}(\Omega)$ in (3.9). Then, taking (3.3) and $0 \leq u_n \leq u_1$ into account, we obtain

$$\varrho(u_n) = \int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^q \log(e+|\nabla u_n|)\,\mathrm{d}x + \int_\Omega |u_n|^p\,\mathrm{d}x < c_1$$

for some $c_1 > 0$ and for all $n \in \mathbb{N}$. From this and Proposition 2.1 (iii), we see that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}_{\log}}(\Omega)$ is bounded.

Now, using assumption (C4) we have $\xi < \frac{p^2}{N-p} \cdot \frac{N-1}{N} + 1$ which is equivalent to $\frac{N}{p}(\xi - 1) < p_*$. So we are able to take a number $t > \frac{N}{p}$ such that $t(\xi - 1) < p_*$. Then, by the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}_{\log}}(\Omega)$ and Proposition 2.2 (ii) we have, for a subsequence if necessary not relabeled, that

$$u_n \rightharpoonup u^* \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad u_n \to u^* \quad \text{in } L^{t(\xi-1)}(\partial\Omega) \tag{3.10}$$

for some $u^* \in W^{1,\mathcal{H}_{\log}}(\Omega)$. Furthermore, from (3.1) and (3.2) as well as (C4) we arrive at

$$|\psi(x,s)| \leq c_2 |s|^{\xi-1} \tag{3.11}$$

for a.a. $x \in \partial\Omega$, for all $s \in \mathbb{R}$ and for some $c_2 > 0$. Now, since $t > \frac{N}{p}$, we can use standard Moser iteration type bootstrap arguments to get from (3.9) and (3.11) the estimate

$$\|u_n\|_\infty \leq B_1 \|u_n\|_{t(\xi-1)}^{B_2} \tag{3.12}$$

for all $n \in \mathbb{N}$ with $t > \frac{N}{p}$ such that $t(\xi - 1) < p_*$ and for some constants $B_1, B_2 > 0$ depending on $N, p, q, \xi, \Omega, \|\mu\|_\infty$ and t. The idea in showing (3.12) is to use the test function $v = u_M^\beta$ with $u_M = \min\{u_n, M\}$, $M > 1$, $\beta \geq 1$ in (3.9) by applying (3.11). Then one sees that the second term with the logarithm is nonnegative and since the embedding $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ is continuous, see Arora–Crespo-Blanco–Winkert [4, Proposition 3.7], we arrive at a standard p-Laplace estimate of the form

$$\beta \int_\Omega u_M^{\beta-1} |\nabla u_M|^p \, dx + \int_\Omega u_n^{p-1} u_M^\beta \, dx \leq \int_{\partial\Omega} u_n^{\xi-1} u_M^\beta \, d\sigma.$$

Now one can proceed in a standard way via bootstrap arguments to show (3.12), as it was done in the works by Lê [19, Theorem 4.3], Perera–Squassina [30, Proposition 2.4] and Marino–Winkert [24, Theorem 3.1].

In the next step, we will show that $u^* \neq 0$. We argue indirectly and assume by contradiction that $u^* = 0$. From (3.10) and (3.12) one has $\|u_n\|_\infty \to 0$ as $n \to +\infty$. This implies the existence of a number $n_0 \in \mathbb{N}$ such that $0 < u_n(x) \leq \eta$ for a.a. $x \in \Omega$ and for all $n \geq n_0$, where $\eta \in \left(0, \min\{\frac{\mathcal{K}}{2}, 1\}\right)$. Thus, with view to (3.1) and (3.2), it follows that

$$\psi(x, u_n(x)) = f(x, u_n(x)) + u_n(x)^{p_*-1} \tag{3.13}$$

for a.a. $x \in \partial\Omega$ and for all $n \geq n_0$. We set $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. This implies that $\|y_n\| = 1$ and $y_n \geq 0$ for all $n \in \mathbb{N}$. Now, we can suppose that

$$y_n \rightharpoonup y \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega) \text{ and } L^p(\partial\Omega)$$

for a subsequence if necessary (not relabeled) and for some $y \in W^{1,\mathcal{H}_{\log}}(\Omega)$ with $y \geq 0$. From $y_n = \frac{u_n}{\|u_n\|}$ we have $u_n = \|u_n\| y_n$. Using this in (3.9) and applying (3.13) leads to

$$\begin{aligned}
&\int_\Omega \Bigg(\|u_n\|^{p-1} |\nabla y_n|^{p-2} \nabla y_n \\
&\qquad + \mu(x) \|u_n\|^{q-1} \left(\log(e + |\nabla u_n|) + \frac{|\nabla u_n|}{q(e + |\nabla u_n|)} \right) |\nabla y_n|^{q-2} \nabla y_n \Bigg) \cdot \nabla v \, \mathrm{d}x \\
&+ \int_\Omega \|u_n\|^{p-1} y_n^{p-1} v \, \mathrm{d}x
\end{aligned}$$

$$= \int_{\partial\Omega} \|u_n\|^{p-1} \left[\frac{f(x,u_n)}{u_n^{p-1}} + u_n^{p_*-p} \right] y_n^{p-1} v \, \mathrm{d}\sigma$$

for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$ and for all $n \geq n_0$. This gives

$$\begin{aligned}
&\int_\Omega \Bigg(|\nabla y_n|^{p-2} \nabla y_n \\
&+ \mu(x) \|u_n\|^{q-p} \left(\log(e + |\nabla u_n|) + \frac{|\nabla u_n|}{q(e+|\nabla u_n|)} \right) |\nabla y_n|^{q-2} \nabla y_n \Bigg) \cdot \nabla v \, \mathrm{d}x \\
&+ \int_\Omega y_n^{p-1} v \, \mathrm{d}x \\
&= \int_{\partial\Omega} \left[\frac{f(x,u_n)}{u_n^{p-1}} + u_n^{p_*-p} \right] y_n^{p-1} v \, \mathrm{d}\sigma
\end{aligned} \tag{3.14}$$

for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$ and for all $n \geq n_0$. Now, recall the elementary inequalities given by

$$\begin{aligned}
\log(e + |\nabla u_n|) &= \log(e + \|u_n\| |\nabla y_n|) \\
&\leq \begin{cases} \log(e + |\nabla y_n|) & \text{if } \|u_n\| < 1, \\ \|u_n\| \log(e + |\nabla y_n|) & \text{if } \|u_n\| \geq 1, \end{cases}
\end{aligned} \tag{3.15}$$

Hence, from (3.15) and Lemma 2.3, we get that the left-hand side of (3.14) is bounded for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$ (similar to the proof of Theorem 4.4 by Arora–Crespo-Blanco–Winkert [4]) and so the same holds for the right-hand side of (3.14). However, taking (C5) into account, it follows that

$$y = 0 \quad \text{and} \quad \frac{f(x,u_n(x))}{u_n(x)^{p-1}} y_n(x)^{p-1} \to 0 \quad \text{for a.a. } x \in \partial\Omega.$$

Now, we take $v = y_n$ in (3.14) and pass to the limit as $n \to +\infty$. This yields

$$\lim_{n\to+\infty} \|\nabla y_n\|_p^p = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|y_n\|_p^p = 0, \tag{3.16}$$

which implies that

$$\nabla y_n(x) \to 0 \quad \text{for a.a. } x \in \Omega \tag{3.17}$$

for a subsequence if necessary, not relabeled.

For $1 < s < q$ we set

$$g_n(x) := \mu(x) |\nabla y_n(x)|^q \log\big(e + |\nabla y_n(x)|\big) \geq 0,$$

$$g_{n,s}(x) := \mu(x)^{\frac{1}{s}} |\nabla y_n(x)|^{\frac{q}{s}} \log\big(e + |\nabla y_n(x)|\big)^{\frac{1}{s}} \geq 0.$$

Then, from (3.17) and Proposition 2.1 (iii), we have

$$g_n(x) \to 0 \quad \text{for a.a. } x \in \Omega \tag{3.18}$$

and

$$\sup_{n\in\mathbb{N}} \int_\Omega g_n \, \mathrm{d}x \leq \sup_{n\in\mathbb{N}} \varrho(y_n) \leq 1. \tag{3.19}$$

Since $|\Omega| < \infty$, by Chacon's biting lemma (see Papageorgiou–Winkert [29, Theorem 4.1.24]) there exist a subsequence $\{g_n\}_{n\in\mathbb{N}}$ (not relabeled) and measurable sets $E_m \subset \Omega$ with $|E_m| \to 0$ as $m \to \infty$ such that for every fixed $m \in \mathbb{N}$, the family

$\{g_n\}_{n\in\mathbb{N}}$ is uniformly integrable in $\Omega \setminus E_m$ Hence, using this with (3.18) as well as (3.19), by Vitali's convergence theorem,

$$\int_{\Omega\setminus E_m} g_n \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty \tag{3.20}$$

for every fixed m.

We have that $\{g_{n,s}(\cdot)\}_{n\in\mathbb{N}} \subset L^s(\Omega)$ is bounded by (3.19). Hence, since $s > 1$,

$$\{g_{n,s}(\cdot)\}_{n\in\mathbb{N}} \quad \text{is uniformly integrable.} \tag{3.21}$$

Claim: For $s \to 1^+$, it holds

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s}(x) \,\mathrm{d}x \to \sup_{n\in\mathbb{N}} \int_{E_m} g_n(x) \,\mathrm{d}x.$$

First note that for all $\ell \in \mathbb{N}$ there exists a number $n_\ell \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_n(x) \,\mathrm{d}x - \frac{1}{\ell} \leq \int_{E_m} g_{n_\ell}(x) \,\mathrm{d}x$$

Observe that

$$\int_{E_m} g_{n,s}(x) \,\mathrm{d}x \to \int_{E_m} g_n(x) \,\mathrm{d}x$$

as $s \to 1^+$. Hence, we can find $s_\ell > 1$ with $s_\ell \to 1^+$ such that

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_n(x) \,\mathrm{d}x - \frac{1}{2\ell} \leq \int_{E_m} g_{n,s_\ell}(x) \,\mathrm{d}x \leq \sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s_\ell}(x) \,\mathrm{d}x.$$

This implies

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_n(x) \,\mathrm{d}x \leq \liminf_{s\to 1^+} \sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s}(x) \,\mathrm{d}x. \tag{3.22}$$

On the other hand, for $\delta > 0$, let $n_\ell \in \mathbb{N}$ be such that

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s}(x) \,\mathrm{d}x \leq \delta + \int_{E_m} g_{n_\ell,s}(x) \,\mathrm{d}x.$$

Therefore, we conclude that

$$\limsup_{s\to 1^+} \sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s}(x) \,\mathrm{d}x \leq \int_{E_m} g_{n_\ell}(x) \,\mathrm{d}x \leq \sup_{n\in\mathbb{N}} \int_{E_m} g_n(x) \,\mathrm{d}x. \tag{3.23}$$

Combining (3.22) and (3.23) proves the Claim.

From the Claim, for given $\varepsilon > 0$, let $s \in (1, q)$ be small such that

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_n(x) \,\mathrm{d}x \leq \frac{\varepsilon}{2} + \sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s}(x) \,\mathrm{d}x. \tag{3.24}$$

On the other hand, due to (3.21), for $m \in \mathbb{N}$ large enough, we have

$$\sup_{n\in\mathbb{N}} \int_{E_m} g_{n,s}(x) \,\mathrm{d}x \leq \frac{\varepsilon}{2}. \tag{3.25}$$

Combining (3.24) and (3.25) yields

$$\int_{E_m} g_n(x) \,\mathrm{d}x \leq \varepsilon,$$

which means that

$$\int_{E_m} g_n(x) \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.26}$$

Hence, from (3.20) and (3.26) we conclude that

$$\int_\Omega g_n(x)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.27}$$

Consequently, from (3.16) and (3.27), it follows that

$$\varrho(y_n) \to 0 \quad \text{as } n \to \infty. \tag{3.28}$$

From Proposition 2.1 (iv) we know that (3.28) is equivalent to $\|y_n\| \to 0$. But by construction we have $\|y_n\| = 1$ for all $n \in \mathbb{N}$ which leads to a contradiction. Hence, u^* is nontrivial and $u^* \in \mathcal{S}_+$ is the smallest positive solution of (3.4). ☐

From condition (C5) we can assume, without any loss of generality, that

$$\frac{f(x,s)}{|s|^{p-2}s} > 0 \qquad \text{for a.a. } x \in \partial\Omega \text{ and for all } |s| \le \mathcal{K},$$

due to the fact that $f(x,\cdot)$ is only locally defined. Then we have, for a.a. $x \in \partial\Omega$, that

$$f(x,s) > 0 \quad \text{for all } 0 < s \le \mathcal{K} \quad \text{and} \quad f(x,s) < 0 \quad \text{for all } -\mathcal{K} \le s < 0. \tag{3.29}$$

Let u^* and v^* be the extremal constant sign solutions from Proposition 3.2 and consider the order interval

$$[v^*,u^*] := \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon v^*(x) \le u(x) \le u^*(x) \text{ for a.a. } x \in \Omega\right\}$$

We define the truncation function $\psi^* \colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\psi^*(x,s) := \begin{cases} \psi(x,v^*(x)) & \text{if } s < v^*(x),\\ \psi(x,s) & \text{if } v^*(x) \le s \le u^*(x),\\ \psi(x,u^*(x)) & \text{if } u^*(x) < s. \end{cases}$$

Furthermore, let $\Upsilon^* \colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ be the truncated C^1-functional defined by

$$\begin{aligned} \Upsilon^*(u) = &\int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q \log(e+|\nabla u|)\right]\mathrm{d}x \\ &+ \int_\Omega \frac{1}{p}|u|^p\,\mathrm{d}x - \int_{\partial\Omega} \Psi^*(x,u)\,\mathrm{d}\sigma, \end{aligned}$$

for all $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$, where $\Psi^*(x,s) = \int_0^s \psi^*(x,t)\,\mathrm{d}t$.

Now, we will show that the critical points of Υ^* belong to the order interval $[v^*,u^*]$, that is,

$$K_{\Upsilon^*} = \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon (\Upsilon^*)'(u) = 0\right\} \subseteq [v^*,u^*]. \tag{3.30}$$

To this end, let $u \in K_{\Upsilon^*} \setminus \{v^*,u^*\}$, that is, for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$, we have

$$\begin{aligned} &\int_\Omega \Bigg(|\nabla u|^{p-2}\nabla u \\ &+ \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\Bigg)\cdot\nabla v\,\mathrm{d}x \\ &+ \int_\Omega |u|^{p-2}uv\,\mathrm{d}x \\ &= \int_{\partial\Omega} \psi^*(x,u)v\,\mathrm{d}\sigma. \end{aligned} \tag{3.31}$$

We first take $v=(u-u^*)_+ \in W^{1,\mathcal{H}_{\log}}(\Omega)$ in (3.31). Then, using the fact that u^* solves (3.4), we obtain

$$
\begin{aligned}
&\langle A(u),(u-u^*)_+\rangle\\
&=\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-u^*)_+\,\mathrm{d}x\\
&\quad+\int_\Omega \mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\cdot\nabla(u-u^*)_+\,\mathrm{d}x\\
&\quad+\int_\Omega |u|^{p-2}u(u-u^*)_+\,\mathrm{d}x\\
&=\int_{\partial\Omega}\psi^*(x,u)(u-u^*)_+\,\mathrm{d}\sigma\\
&=\int_{\partial\Omega}\psi(x,u^*)(u-u^*)_+\,\mathrm{d}\sigma\\
&=\int_\Omega |\nabla u^*|^{p-2}\nabla u^*\cdot\nabla(u-u^*)_+\,\mathrm{d}x\\
&\quad+\int_\Omega \mu(x)\left(\log(e+|\nabla u^*|)+\frac{|\nabla u^*|}{q(e+|\nabla u^*|)}\right)|\nabla u^*|^{q-2}\nabla u^*\cdot\nabla(u-u^*)_+\,\mathrm{d}x\\
&\quad+\int_\Omega |u^*|^{p-2}u^*(u-u^*)_+\,\mathrm{d}x\\
&=\langle A(u^*),(u-u^*)_+\rangle.
\end{aligned}
$$

From this, we conclude that

$$\langle A(u)-A(u^*),(u-u^*)_+\rangle=0.$$

Since the operator $A\colon W^{1,\mathcal{H}_{\log}}(\Omega)\to W^{1,\mathcal{H}_{\log}}(\Omega)^*$ as defined in (2.4) is strictly monotone by Proposition 2.4, we get that $u\le u^*$. Similarly, if we take $v=(v^*-u)_+\in W^{1,\mathcal{H}_{\log}}(\Omega)$ in (3.31), we can show that $v^*\le u$. This proves (3.30).

Before we can prove Theorem 1.1, we need first the following proposition. In the following, we denote by V a finite dimensional subspace of $W^{1,\mathcal{H}_{\log}}(\Omega)\cap L^\infty(\Omega)$.

Proposition 3.3. *Let hypotheses (C1)–(C5) be satisfied. Then, there exists a number $\zeta_V>0$ such that*

$$\sup\left[\Upsilon^*(v)\colon v\in V,\ \|v\|=\zeta_V\right]<0.$$

Proof. Recall that since V has finite dimension, all norms on V are equivalent. Thus, there exists a number $\zeta_V>0$ such that

$$v\in V \quad\text{and}\quad \|v\|\le\zeta_V \quad\text{imply}\quad |v(x)|\le\eta \quad\text{for a.a.}\, x\in\Omega,$$

where $\eta\in\left(0,\min\{\frac{\mathcal{K}}{2},1\}\right)$ is the same from the proof of Proposition 3.1. Now, using (3.1) and $\eta<\frac{\mathcal{K}}{2}$, it follows that $\Phi(v(x))=1$ for a.a. $x\in\Omega$. From this, $v\in V$ with $\|v\|\le\zeta_V$, we see that

$$
\psi^*(x,v(x))=\begin{cases}
f(x,v^*(x))+|v^*(x)|^{p_*-2}v^*(x) & \text{if } v(x)<v^*(x),\\
f(x,v(x))+|v(x)|^{p_*-2}v(x) & \text{if } v^*(x)\le v(x)\le u^*(x),\\
f(x,u^*(x))+|u^*(x)|^{p_*-2}u^*(x) & \text{if } u^*(x)<v(x).
\end{cases}
$$

Next, we define $f^*\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f^*(x,v(x)) = \begin{cases} f(x,v^*(x)) & \text{if } v(x) < v^*(x), \\ f(x,v(x)) & \text{if } v^*(x) \leq v(x) \leq u^*(x), \\ f(x,u^*(x)) & \text{if } u^*(x) < v(x). \end{cases}$$

Then, for $F^*(x,s) := \int_0^s f^*(x,t)\,\mathrm{d}t$, $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$ and $v < v^*$, we deduce that

$$\begin{aligned} F^*(x,v) &= \int_0^{v^*} f^*(x,s)\,\mathrm{d}s + \int_{v^*}^{v} f^*(x,s)\,\mathrm{d}s = \int_0^{v^*} f(x,s)\,\mathrm{d}s + \int_{v^*}^{v} f(x,v^*)\,\mathrm{d}s \\ &= F(x,v^*) + f(x,v^*)(v - v^*). \end{aligned}$$

Taking (3.29) into account, we see that $f(x,v^*) < 0$ for a.a. $x \in \partial\Omega$. This implies $f(x,v^*)(v - v^*) > 0$ for a.a. $x \in \partial\Omega$ which gives

$$\begin{aligned} F(x,v) - F^*(x,v) &= F(x,v) - F(x,v^*) + f(x,v^*)(v^* - v) \\ &\leq F(x,v) - F(x,v^*). \end{aligned}$$

Similarly, if $u^* < v$, we have

$$F^*(x,v) = F(x,u^*) + f(x,u^*)(v - u^*).$$

Then, because of $f(x,u^*)(u^* - v) < 0$ for a.a. $x \in \partial\Omega$ again due to (3.29), this leads to

$$\begin{aligned} F(x,v) - F^*(x,v) &= F(x,v) - F(x,u^*) + F(x,u^*)(u^* - v) \\ &\leq F(x,v) - F(x,u^*). \end{aligned}$$

For this reason, we can write

$$\begin{aligned} \Upsilon^*(v) &= \int_\Omega \left[\frac{1}{p}|\nabla v|^p + \frac{\mu(x)}{q}|\nabla v|^q \log(e + |\nabla v|)\right] \mathrm{d}x \\ &\quad + \int_\Omega \frac{1}{p}|v|^p\,\mathrm{d}x - \int_{\partial\Omega} \Psi^*(x,v)\,\mathrm{d}\sigma \\ &= \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e + |\nabla v|)\,\mathrm{d}x + \int_\Omega \frac{1}{p}|v|^p\,\mathrm{d}x \\ &\quad - \int_{\{x\in\partial\Omega\colon\, v(x)<v^*(x)\}} \left(F^*(x,v) + \frac{1}{p_*}|v^*|^{p_*}\right) \mathrm{d}\sigma \\ &\quad - \int_{\{x\in\partial\Omega\colon\, v^*(x)\leq v(x)\leq u^*(x)\}} \left(F(x,v) + \frac{1}{p_*}|v|^{p_*}\right) \mathrm{d}\sigma \\ &\quad - \int_{\{x\in\partial\Omega\colon\, u^*(x)<v(x)\}} \left(F^*(x,v) + \frac{1}{p_*}|u^*|^{p_*}\right) \mathrm{d}\sigma \\ &\leq \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e + |\nabla v|)\,\mathrm{d}x + \int_\Omega \frac{1}{p}|v|^p\,\mathrm{d}x \\ &\quad - \int_{\{x\in\partial\Omega\colon\, v(x)<v^*(x)\}} F^*(x,v)\,\mathrm{d}\sigma \\ &\quad - \int_{\{x\in\partial\Omega\colon\, v^*(x)\leq v(x)\leq u^*(x)\}} F(x,v)\,\mathrm{d}\sigma \\ &\quad - \int_{\{x\in\partial\Omega\colon\, u^*(x)<v(x)\}} F^*(x,v)\,\mathrm{d}\sigma \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{p}|v|^p \,\mathrm{d}x - \int_{\partial\Omega} F(x,v)\,\mathrm{d}\sigma \\
&\quad + \int_{\{x\in\partial\Omega\colon\, v(x)<v^*(x)\}} \left(F(x,v)-F^*(x,v)\right)\,\mathrm{d}\sigma \\
&\quad + \int_{\{x\in\partial\Omega\colon\, u^*(x)<v(x)\}} \left(F(x,v)-F^*(x,v)\right)\,\mathrm{d}\sigma \\
&\le \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{p}|v|^p \,\mathrm{d}x - \int_{\partial\Omega} F(x,v)\,\mathrm{d}\sigma \\
&\quad + \int_{\{x\in\partial\Omega\colon\, v(x)<v^*(x)\}} \left(F(x,v)-F(x,v^*)\right)\,\mathrm{d}\sigma \\
&\quad + \int_{\{x\in\partial\Omega\colon\, u^*(x)<v(x)\}} \left(F(x,v)-F(x,u^*)\right)\,\mathrm{d}\sigma.
\end{aligned}
$$

From hypothesis (C5), for each $\delta>0$, there exists $\eta\in\left(0,\min\{\frac{\mathcal{K}}{2},1\}\right)$ such that

$$
F(x,s)\ge \frac{\delta}{p}|s|^p \quad \text{for all } |s|\le \eta, \tag{3.32}
$$

see (3.6). Then we can take $\zeta_V>0$ sufficiently small such that

$$
\begin{aligned}
&\int_{\{x\in\partial\Omega\colon\, v(x)<v^*(x)\}} \left(F(x,v)-F(x,v^*)\right)\,\mathrm{d}\sigma \\
&\quad + \int_{\{x\in\partial\Omega\colon\, u^*(x)<v(x)\}} \left(F(x,v)-F(x,u^*)\right)\,\mathrm{d}\sigma < \eta^p.
\end{aligned} \tag{3.33}
$$

Combining (3.32) and (3.33) with the calculations above, we get

$$
\begin{aligned}
\Upsilon^*(v) &\le \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{p}|v|^p \,\mathrm{d}x - \frac{\delta}{p}\int_{\partial\Omega} |v|^p\,\mathrm{d}\sigma + \eta^p \\
&= \frac{1}{p}\|v\|_{1,p}^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x - \frac{\delta}{p}\|v\|_{p,\partial\Omega}^p + \eta^p.
\end{aligned} \tag{3.34}
$$

Note that Proposition 2.1 (iii) gives the following inequalities

$$
\int_\Omega \mu(x)|\nabla v|^q \log(e+|\nabla v|)\,\mathrm{d}x \le \varrho(v) \le \max\{\|v\|^p,\ \|v\|^{q+\kappa}\}. \tag{3.35}
$$

Now, using the fact that all norms on V are equivalent and applying (3.35) in (3.34), we see from (3.34) that there exist positive constants c_1, c_2, c_3, independent of η, such that

$$
\Upsilon^*(v) \le c_1\|v\|_\infty^p + c_2\max\{\|v\|_\infty^p,\ \|v\|_\infty^{q+\kappa}\} - \delta c_3\|v\|_\infty^p + \eta^p.
$$

Then, for $v\in V$ with $\|v\|=\zeta_V$ along with the equivalence of the norms on V, it follows, due to $\eta<1$, that

$$
\Upsilon^*(v) \le c_1\eta^p + c_2\max\{\eta^p,\eta^{q+\kappa}\} - \delta c_3\eta^p + \eta^p
$$

$$= (c_1 + c_2 - \delta c_3 + 1)\, \eta^p.$$

Now, if we take $\delta > \frac{c_1+c_2+1}{c_3}$, we obtain $\Upsilon^*(v) < 0$ for all $v \in V$ with $\|v\| = \zeta_V$. □

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Clearly, $\Upsilon^*\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ is even and coercive, and in addition, it is bounded from below. Furthermore, due to Proposition 5.1.15 by Papageorgiou–Rădulescu–Repovš [26], the functional Υ^* satisfies the PS-condition as well. From these facts and Proposition 3.3 we are now in the position to apply Theorem 1 of Kajikiya [18]. This yields a sequence $\{z_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ such that

$$z_n \in K_{\Upsilon^*} \subseteq [v^*, u^*], \quad z_n \neq 0, \quad \Upsilon^*(z_n) \leq 0 \quad \text{for all } n \in \mathbb{N}$$

and

$$\|z_n\| \to 0 \quad \text{as } n \to +\infty. \tag{3.36}$$

We point out that v^* and u^* are the extremal constant sign solutions of (3.4) obtained in Proposition 3.2. Moreover, as $z_n \in K_{\Upsilon^*} \subseteq [v^*, u^*]$ and $z_n \neq 0$ for all $n \in \mathbb{N}$, we have that z_n is a sign-changing solution of problem (3.4) for all $n \in \mathbb{N}$. Recall that from (3.12), we have

$$\|z_n\|_\infty \leq B_1 \|z_n\|_{t(\xi-1)}^{B_2}$$

for all $n \in \mathbb{N}$ with $t > \frac{N}{p}$ such that $t(\xi - 1) < p_*$ and for some constants $B_1, B_2 > 0$ depending on $N, p, q, \xi, \Omega, \|\mu\|_\infty$ and t. Hence, using (3.36), it follows that $\|z_n\|_\infty \to 0$ as $n \to +\infty$. In particular, we can find a number $n_0 \in \mathbb{N}$ such that $|z_n(x)| \leq \frac{\mathcal{K}}{2}$ for a.a. $x \in \Omega$ and for all $n \geq n_0$. From this we conclude that $\Phi(z_n(x)) = 1$ for a.a. $x \in \Omega$ and for all $n \geq n_0$, see (3.1). Using this fact and (3.2) we know that z_n is a sign-changing solution of our original problem (1.4) for all $n \geq n_0$. □

References

[1] C.O. Alves, D.C. de Morais Filho, *Existence and concentration of positive solutions for a Schrödinger logarithmic equation*, Z. Angew. Math. Phys. **69** (2018), no. 6, Paper No. 144, 22 pp.

[2] C.O. Alves, C. Ji, *Existence and concentration of positive solutions for a logarithmic Schrödinger equation via penalization method*, Calc. Var. Partial Differential Equations **59** (2020), no. 1, Paper No. 21, 27 pp.

[3] R. Arora, Á. Crespo-Blanco, P. Winkert, *Logarithmic double phase problems with generalized critical growth*, NoDEA Nonlinear Differential Equations Appl. **32** (2025), no. 5, Paper No. 98, 74 pp.

[4] R. Arora, Á. Crespo-Blanco, P. Winkert, *On logarithmic double phase problems*, J. Differential Equations **433** (2025), Paper No. 113247, 60 pp.

[5] A. Bahrouni, A. Fiscella, P. Winkert, *Critical logarithmic double phase equations with sign-changing potentials in $\mathbb{R}^N$*, J. Math. Anal. Appl. **547** (2025), no. 2, Paper No. 129311, 24 pp.

[6] A. Bahrouni, A. Fiscella, P. Winkert, *Gain and loss on critical logarithmic double phase equations*, NoDEA Nonlinear Differential Equations Appl. **33** (2026), no. 1, Paper No. 27, 20 pp.

[7] P. Baroni, M. Colombo, G. Mingione, *Non-autonomous functionals, borderline cases and related function classes*, St. Petersburg Math. J. **27** (2016), 347–379.

[8] F. Borer, L. Gasiński, M.F. Stapenhorst, P. Winkert, *Least energy sign-changing solution for logarithmic double phase problems with nonlinear boundary condition*, Calc. Var. Partial Differential Equations **64** (2025), no. 8, Paper No. 258, 38 pp.

[9] Y.B.C. Carranza, M.T.O. Pimenta, F. Vetro, P. Winkert, *On critical logarithmic double phase problems with locally defined perturbation*, J. Math. Anal. Appl. **555** (2026), no. 1, Paper No. 130047, 18 pp.

[10] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, J. Differential Equations **323** (2022), 182–228.

[11] C. De Filippis, G. Mingione, *Regularity for double phase problems at nearly linear growth*, Arch. Ration. Mech. Anal. **247** (2023), no. 5, Paper No. 85, 50 pp.

[12] G.M. Figueiredo, M. Montenegro, M.F. Stapenhorst, *A singular Liouville equation on planar domains*, Math. Nachr. **296** (2023), no. 10, 4569–4609.

[13] G.M. Figueiredo, M. Montenegro, M.F. Stapenhorst, *A log-exp elliptic equation in the plane*, Discrete Contin. Dyn. Syst. **42** (2022), no. 1, 481–504.

[14] M. Fuchs, G. Mingione, *Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth*, Manuscripta Math. **102** (2000), no. 2, 227–250.

[15] M. Fuchs, G. Seregin, "Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids", Springer-Verlag, Berlin, 2000.

[16] P. Harjulehto, P. Hästö, "Orlicz Spaces and Generalized Orlicz Spaces", Springer, Cham, 2019.

[17] S. Hu, N.S. Papageorgiou, "Handbook of Multivalued Analysis. Vol. I", Kluwer Academic Publishers, Dordrecht, 1997.

[18] R. Kajikiya, *A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations*, J. Funct. Anal. **225** (2005), no. 2, 352–370.

[19] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. **64** (2006), no. 5, 1057–1099.

[20] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[21] Z. Liu, N.S. Papageorgiou, *Asymptotically vanishing nodal solutions for critical double phase problems*, Asymptot. Anal. **124** (2021), no. 3-4, 291–302.

[22] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.

[23] P. Marcellini, G. Papi, *Nonlinear elliptic systems with general growth*, J. Differential Equations **221** (2006), no. 2, 412–443.

[24] G. Marino, P. Winkert, *Moser iteration applied to elliptic equations with critical growth on the boundary*, Nonlinear Anal. **180** (2019), 154–169.

[25] M. Montenegro, O.S. de Queiroz, *Existence and regularity to an elliptic equation with logarithmic nonlinearity*, J. Differential Equations **246** (2009), no. 2, 482–511.

[26] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, "Nonlinear Analysis–Theory and Methods", Springer, Cham, 2019.

[27] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Positive solutions for perturbations of the Robin eigenvalue problem plus an indefinite potential*, Discrete Contin. Dyn. Syst. **37** (2017), no. 5, 2589–2618.

[28] N.S. Papageorgiou, F. Vetro, P. Winkert, *Sign changing solutions for critical double phase problems with variable exponent*, Z. Anal. Anwend. **42** (2023), no. 1-2, 235–251.

[29] N.S. Papageorgiou, P. Winkert, "Applied Nonlinear Functional Analysis", Second Revised Edition, De Gruyter, Berlin, 2024.

[30] K. Perera, M. Squassina, *Existence results for double-phase problems via Morse theory*, Commun. Contemp. Math. **20** (2018), no. 2, 1750023, 14 pp.

[31] P. Pucci, J. Serrin, "The Maximum Principle", Birkhäuser Verlag, Basel, 2007.

[32] G.A. Seregin, J. Frehse, *Regularity of solutions to variational problems of the deformation theory of plasticity with logarithmic hardening*, in: "Proceedings of the St. Petersburg Mathematical Society, Vol. V", Amer. Math. Soc., Providence, RI **193**, 1999, 127–152.

[33] W. Shuai, *Multiple solutions for logarithmic Schrödinger equations*, Nonlinearity **32** (2019), no. 6, 2201–2225.

[34] M. Squassina, A. Szulkin, *Multiple solutions to logarithmic Schrödinger equations with periodic potential*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 585–597.

[35] M.-P. Tran, T.-N. Nguyen, *Existence of weak solutions to borderline double-phase problems with logarithmic convection terms*, J. Math. Anal. Appl. **546** (2025), no. 1, Paper No. 129185, 22 pp.

[36] F. Vetro, *Kirchhoff problems with logarithmic double phase operator: Existence and multiplicity results*, Asymptot. Anal. **143** (2025), no. 3, 913–926.
[37] F. Vetro, P. Winkert, *Logarithmic double phase problems with convection: existence and uniqueness results*, Commun. Pure Appl. Anal. **23** (2024), no. 9, 1325–1339.
[38] C. Vetro, S. Zeng, *Regularity and Dirichlet problem for double-phase energy functionals of different power growth*, J. Geom. Anal. **34** (2024), no. 4, Paper No. 105, 27 pp.

(E. Öztürk) Department of Mathematics, Hacettepe University, 06800 Beytepe, Ankara, Türkiye
Email address: `eozturk@hacettepe.edu.tr`

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
Email address: `winkert@math.tu-berlin.de`