MULTIPLE NORMALIZED SOLUTIONS FOR L^2-MASS SUPERCRITICAL CHOQUARD EQUATIONS CONCENTRATING AT A POTENTIAL WELL

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ABSTRACT. In this paper we investigate the multiplicity of normalized solutions to the Choquard equation

$$-\varepsilon^2\Delta u+V(x)u=\lambda u+\varepsilon^{-\alpha}(I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

subject to the prescribed mass constraint

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a^2\varepsilon^N,$$

where $\varepsilon, a>0$, $\alpha\in(0,N)$, $N\geq 3$, I_α denotes the Riesz potential, and the parameter $\lambda\in\mathbb{R}$ arises as a Lagrange multiplier associated with the mass constraint. The nonlinearity $f\colon\mathbb{R}\to\mathbb{R}$ has L^2-supercritical growth, while the potential $V\colon\mathbb{R}^N\to\mathbb{R}$ is assumed to be a C^1-function. Under suitable assumptions on V and f, we relate the number of normalized solutions to the topology of the set where the potential V attains its minimum. The proof relies on Ljusternik-Schnirelmann theory. Our results improve and complement earlier works by Li–Ye (J. Math. Phys. **55** (2014), no. 12, 121501, 19 pp.) and Moroz–Van Schaftingen (Calc. Var. Partial Differential Equations **52** (2015), no. 1-2, 199–235) by establishing multiplicity of normalized solutions concentrating at the potential well V.

1. INTRODUCTION AND MAIN RESULTS

We consider the following Choquard-type equation:

$$-\Delta u+V(x)u=\lambda u+(I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\alpha\in(0,N)$, $N\geq 3$, $\lambda\in\mathbb{R}$ is an unknown parameter, I_α denotes the Riesz potential defined by

$$I_\alpha(x):=\frac{A_\alpha}{|x|^{N-\alpha}}, \quad x\in\mathbb{R}^N\backslash\{0\}, \quad \text{where} \quad A_\alpha:=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha}$$

and Γ denotes the Gamma function. Equation (1.1) is usually referred to as the nonlinear Choquard equation and has been extensively studied over the past decades. In the physical case $N=3$, $\alpha=2$, $V\equiv 1$, $\lambda=0$ and $f(s)=s$, equation (1.1) reduces to the well-known Choquard-Pekar equation which goes back to the work of Pekar [37] in 1954 on the quantum theory of a polaron at rest. The Choquard equation reappeared in 1976 in the work of Choquard on the modeling of an electron trapped in its own hole, arising from a certain approximation of the Hartree-Fock theory for

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a one-component plasma discussed by Lieb [25]. For further physical background on Choquard-type equations we refer to the papers by Moroz–Penrose–Tod [32], Penrose [38], and the references therein. When looking for solutions to (1.1), two distinct approaches can be considered. One may fix the frequency $\lambda \in \mathbb{R}$, or treat it as an unknown parameter while prescribing the mass. The former case is usually referred to as the fixed frequency problem. In the latter case, it is called the prescribed mass problem, where $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier associated with the mass constraint.

In recent decades the fixed frequency problem has been extensively studied. Lieb [25] and Lions [27] initiated a systematic study of equation (1.1). In particular, Moroz–Van Schaftingen [33] proved the existence of a positive ground state solution to the equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

and established regularity and positivity of the ground states. They also obtained qualitative properties and decay asymptotics of the ground states. In [35], Moroz–Van Schaftingen studied semi-classical states for the Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

In that work they used variational methods together with a novel nonlocal penalization technique. Under almost necessary conditions on the nonlinearity F in the spirit of Berestycki–Lions [6], Moroz–Van Schaftingen [33] investigated the existence of ground states for (1.1) with $V(x) \equiv 0$ and $\lambda = -1$. Moreover, if F is even and monotone on $(0,\infty)$, then the ground state has constant sign and is radially symmetric. Further results on existence, nonexistence, non-degeneracy, multiplicity, and asymptotic behavior of solutions to the Choquard equations (1.1), (1.2), and (1.3) can be found in the works of Clapp–Salazar [10], Gao–Yang [13], Ghimenti–Van Schaftingen [14], Li–Ma [21], Moroz–Van Schaftingen [34, 35], and the references therein.

From the perspective of physics, considering equation (1.1) with a prescribed mass $\int_{\mathbb{R}^N} u^2\,\mathrm{d}x = a^2 > 0$ and with $\lambda \in \mathbb{R}$ appearing as a Lagrange multiplier is particularly significant and challenging. The prescribed mass represents the power in nonlinear optics or the total number of atoms in a Bose-Einstein condensation. Besides its importance in physics, the fixed mass problem is mathematically more difficult than the fixed frequency problem. We note that there has been increasing research on normalized solutions of the Choquard equation (1.1) in recent years. It is worth emphasizing that in the study of normalized solutions to the following Choquard equation

$$-\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

three exponents play an important role and must be distinguished: the Hardy-Littlewood-Sobolev lower critical exponent $\underline{p}$, the Hardy-Littlewood-Sobolev upper critical exponent $\overline{p}$, and the L^2-critical exponent p^*, given by

$$\underline{p} := \frac{N+\alpha}{N}, \quad \overline{p} := \begin{cases} +\infty & \text{if } N = 1,2, \\ \frac{N+\alpha}{N-2} & \text{if } N \geq 3, \end{cases} \quad p^* := \frac{N+\alpha+2}{N}.$$

Since for $p < p^*$ the functional $J\colon H^1(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,\mathrm{d}x,$$

is bounded from below on the sphere

$$S_a = \left\{ u \in H^1(\mathbb{R}^N) \colon \|u\|_2 = a \right\},$$

while for $p > p^*$ the functional J is no longer bounded from below on S_a, see Luo [30] and Ye [43] for more details. In particular, Ye [43] obtained a ground state solution for (1.4) by considering a minimizer of J constrained on S_a when $\underline{p} < p < p^*$.

When mixed nonlinearities are considered, there are also many interesting results for the following equation

$$-\Delta u = \lambda u + \gamma(I_\alpha * |u|^p)|u|^{p-2}u + \mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Li [20] proved the existence and orbital stability of ground states for normalized solutions of (1.5) when $N \geq 3$, $\gamma = 1$, $2 < q < 2 + \frac{4}{N}$, and $p = \overline{p}$. Later, for the L^2-critical or L^2-supercritical perturbation term $\mu|u|^{q-2}u$, Li [19] studied the existence, nonexistence, and symmetry of normalized ground states for (1.5) by virtue of the mountain pass theorem, the Pohozaev constraint method, the Schwartz symmetrization rearrangement technique, and polarization methods. In [45], Yao–Chen–Rădulescu–Sun investigated the existence of normalized solutions for (1.5) with $N \geq 3$, $\gamma = -1$, $2 < q < 2^* := \frac{2N}{N-2}$, and $p = \underline{p}$. Moreover, Ye–Shen–Yang [44] and Shang–Ma [39] studied the following upper critical Choquard equation with nonlocal perturbation

$$-\Delta u = \lambda u + \mu(I_\alpha * |u|^p)|u|^{p-2}u + (I_\alpha * |u|^{\overline{p}})|u|^{\overline{p}-2}u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where $\underline{p} < p < p^*$. They proved that equation (1.6) admits normalized ground states and mountain-pass type solutions. Furthermore, Cao–Jia–Luo [7] investigated the equation

$$-\Delta u = \lambda u + \mu(|x|^{-\alpha} * |u|^2)u + (|x|^{-\beta} * |u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where $N \geq 3$. Under different assumptions on α and β, they established the existence of normalized solutions for (1.7) when $0 < \alpha < \beta < \min\{4, N\}$, and further analyzed their asymptotic behavior. Later, Jia–Luo [17] studied the normalized solutions of (1.7) with $N \geq 5$, $0 < \alpha < 4$, and $\beta = 4$.

For the Choquard-type equation with a more general nonlinearity

$$-\Delta u = \lambda u + (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

Li–Ye [23] proved, by using the minimax method together with the concentration compactness principle, that equation (1.8) possesses at least one normalized solution under a set of assumptions on f. In particular, when f takes the special form

$$f(s) = C_1|s|^{r-2}s + C_2|s|^{p-2}s,$$

their result requires that $p^* < r \leq p < \overline{p}$. Later, for $N \geq 1$, Bartsch–Liu–Liu [5] further obtained a normalized ground state and infinitely many normalized solutions to (1.8). As far as we know, external potentials V inevitably arise in many problems coming from practical physical models. Accordingly, many authors introduced such external potentials into Choquard-type equations, namely

$$-\Delta u + V(x)u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.9}$$

and investigated the existence of normalized solutions for (1.9). For the mass subcritical case, we refer to Cao–Wang–Zou [8] and Ye [43], while for the mass critical case we mention the work of Li–Xiang–Zeng [22]. In these works, the presence of a trapping potential allows one to obtain the compactness properties required in the variational analysis. In [29], Long–Li–Rong proved that the Choquard-type equation admits a positive normalized ground state solution via comparison arguments under suitable assumptions on V. Besides, Ao–Zhao–Zou [1] considered a class of nonlinear Choquard equations with more general nonlinearities and proved the compactness of every minimizing sequence together with the existence of normalized solutions for the equation

$$-\Delta u+(V+\lambda)u=(I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

under appropriate conditions on V and F, where $f=F'$. Under suitable assumptions on the potential V, Meng–Wang [31] demonstrated the multiplicity of normalized solutions for the Choquard-type equation in the mass subcritical case. For further results we refer to the works of Gao–He [12], Guo–Zhang [16] and Long–Feng [28], as well as the references therein. Moreover, we mention the following recent results on normalized solutions for different classes of problems, see the papers by Chen–Tang [11], Jin–Yang–Zhou [18], Liang–Ma–Shi–Song [24], Wei–Song [42] and the references therein.

Motivated by the aforementioned works, in this paper we are concerned with multiplicity of solutions to the equation

$$-\varepsilon^2\Delta u+V(x)u=\varepsilon^{-\alpha}(I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.10}$$

with the prescribed mass constraint

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=\varepsilon^N a^2, \tag{1.11}$$

where $\varepsilon, a>0$. The main purpose of this paper is to establish the multiplicity of normalized solutions for the problem (1.10)-(1.11) when the nonlinearity has L^2-supercritical growth. In particular, we relate the number of normalized solutions to the topology of the set where the potential V attains its minimum. This will be achieved by applying the Ljusternik-Schnirelmann theory. Recall that Alves–Thin [3, 4] first studied the multiplicity of normalized solutions for the Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u=\lambda u+f(u) \quad \text{in } \mathbb{R}^N,$$

by means of the Ljusternik-Schnirelmann category theory. In their results the nonlinearity f is a continuous function with L^2-subcritical growth satisfying the following assumptions:

- (C$_1$) The function $f\colon \mathbb{R}\to\mathbb{R}$ is continuous, odd and $\lim_{s\to 0}\frac{|f(s)|}{|s|^{q_0-1}}=\alpha>0$ for some $q_0\in(2,2+\frac{4}{N})$.
- (C$_2$) There exist constants $c_1,c_2>0$ and $p\in(2,2+\frac{4}{N})$ such that
$$|f(s)|\le c_1+c_2|s|^{p-1} \quad \text{for all } s\in\mathbb{R}.$$
- (C$_3$) There is $q\in[q_0,2+\frac{4}{N})$ such that the function $s\mapsto f(s)/s^{q-1}$ is increasing on $(0,+\infty)$.

In [4], the potential V is assumed to satisfy the local condition:

- (A$_1$) There exists a bounded open set $\Lambda\subset\mathbb{R}^N$ such that
$$\min_{x\in\bar{\Lambda}}V(x)<\min_{x\in\partial\Lambda}V(x)=V_\infty.$$

While in [3], the function V is supposed to verify the global condition:

(A_2) $V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ with $V(0) = 0$ such that

$$0 = \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to+\infty} V(x) = V_\infty.$$

We note that, since the nonlinearity f is L^2-subcritical growth, the energy of the normalized solutions obtained in [3, 4] are negative. Recently, the main results and methods in [3, 4], have been extended to the Choquard equation by Wu–He [41], and to the logarithmic Schrödinger equation by Alves–Ji [2].

In this paper we impose the following assumptions on f and V:

(f_1) $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and odd.

(f_2) There exist some numbers $r, p \in \mathbb{R}$ satisfying

$$\frac{N+\alpha+4}{N} < r \le p < \frac{N+\alpha}{N-2},$$

such that

$$0 < rF(s) \le f(s)s \le pF(s) \quad \text{for all } s \in \mathbb{R}\setminus\{0\}.$$

(f_3) The function

$$\tilde{F}(s) := f(s)s - \frac{N+\alpha}{N}F(s)$$

is of class C^1 and the map

$$s \mapsto \frac{\widetilde{F}(s)}{|s|^{(N+\alpha+2)/N}}$$

is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$.

(V_1) $V \in C^1(\mathbb{R}^N)$,

$$V(0) = \inf_{x\in\mathbb{R}^N} V(x) < \lim_{|x|\to+\infty} V(x) = \sup_{x\in\mathbb{R}^N} V(x) = 0,$$

and there exists

$$\sigma_1 \in \left(0, \frac{Nr-N-\alpha-4}{Nr-N-\alpha}\right)$$

such that

$$\left|\int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x\right| \le \sigma_1 \|\nabla u\|_2^2, \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

(V_2) Set $W(x) := \frac{1}{2}\langle \nabla V(x), x\rangle$ with $W(0) = 0$ and $\lim_{|x|\to+\infty} W(x) = 0$. There exists

$$\sigma_2 \in \left(0, \min\left\{\frac{N+\alpha-p(N-2)}{2p}, \frac{(1-\sigma_1)(rN-N-\alpha)}{2} - 1\right\}\right)$$

such that

$$\left|\int_{\mathbb{R}^N} W(x)|u|^2 \, \mathrm{d}x\right| \le \sigma_2 \|\nabla u\|_2^2, \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

(V_3) The gradient $\nabla W(x)$ exists a.e. in $\mathbb{R}^N$ and coincides with the weak gradient of $W(x)$. Define

$$Y(x) := \frac{rN+2-N-\alpha}{2} W(x) + \langle \nabla W(x), x\rangle,$$

and assume $\lim_{|x|\to+\infty} Y(x) = 0$. There exists

$$\sigma_3 \in \left(0, \frac{rN-(N+\alpha+2)}{2}\right)$$

such that

$$\left|\int_{\mathbb{R}^N} Y_+(x)|u|^2\,\mathrm{d}x\right| \leq \sigma_3\|\nabla u\|_2^2, \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

From conditions (V$_1$)–(V$_3$), we see that $\|V\|_{L^\infty} := \sup_{x\in\mathbb{R}^N}|V(x)| < \infty$. In order to relate the number of solutions to (1.10)-(1.11) with the topology of the set of minima of the potential V, we introduce the set of global minima of V given by

$$M = \left\{x \in \mathbb{R}^N \colon V(x) = V(0) = \inf_{x\in\mathbb{R}^N} V(x)\right\}.$$

Note that M is compact. For any $\delta > 0$, we define

$$M_\delta = \left\{x \in \mathbb{R}^N \colon \operatorname{dist}(x, M) \leq \delta\right\}.$$

Now we can state our main result.

Theorem 1.1. *Assume $N \geq 3$, the nonlinearity f satisfies* (f$_1$)–(f$_3$), *and the potential V satisfies* (V$_1$)–(V$_3$). *Then, for any $\delta > 0$, there exist $\varepsilon_0 > 0$ and $V_* > 0$ such that if $\|V\|_{L^\infty} < V_*$, problem* (1.10)-(1.11) *admits at least $\operatorname{cat}_{M_\delta}(M)$ pairs of weak solutions $(u_j, \lambda_j) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ for all $\varepsilon \in (0, \varepsilon_0)$, with $\int_{\mathbb{R}^N}|u_j|^2\,\mathrm{d}x = a^2\varepsilon^N$ and $\lambda_j < 0$.*

By the change of variables $z = \varepsilon x$, problem (1.10)-(1.11) is equivalent to the problem

$$\begin{cases} -\Delta u + V(\varepsilon x)u = \lambda u + (I_\alpha * F(u))f(u), & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x = a^2. \end{cases} \tag{1.12}$$

To obtain solutions of (1.12) by the constrained mass method, we search for critical points of the energy functional $J_\varepsilon \colon H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}V(\varepsilon x)|u|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x$$

restricted to the sphere

$$S_a = \left\{u \in H^1(\mathbb{R}^N) \colon \|u\|_2 = a\right\},$$

where $\|\cdot\|_q$ denotes the usual norm in $L^q(\mathbb{R}^N)$ for $q \in (1, +\infty)$.

Recall that if Y is a closed subset of a topological space X, the Ljusternik-Schnirelmann category $\operatorname{cat}_X(Y)$ is the smallest number of closed and contractible sets in X that cover Y. If $X = Y$, we write $\operatorname{cat}(X)$. For more details we refer to the monograph by Willem [40].

Remark 1.2.

(i) *To the best of our knowledge, there are few results on the multiplicity of normalized solutions for the Choquard equation* (1.10) *obtained via the Ljusternik-Schnirelmann theory, except for the work by Wu–He* [41]. *In that paper the nonlinearity f has L^2-subcritical growth and the potential V satisfies the local condition* (A$_1$). *In the present paper we complement the* *results of Wu–He [41] by considering nonlinearities f satisfying* (f_1)–(f_3) *and potentials V satisfying* (V_1)–(V_3)*. In particular, the energies of the normalized solutions obtained for problem* (1.10)-(1.11) *are positive.*

(ii) *Our results improve those of Li–Ye [23] by establishing the multiplicity of concentrating normalized solutions in the presence of a nonconstant potential V. Moreover, our work extends the results of Moroz–Van Schaftingen [35] in the sense that we investigate the multiplicity of normalized solutions concentrating at the potential well V.*

(iii) *As will be seen in the sequel, several new difficulties arise in our analysis. In particular, the presence of a nonlinearity f with L^2-supercritical growth implies that the functional associated with* (1.10)-(1.11) *is unbounded from below on S_a. Consequently, the direct minimization method cannot be applied. To overcome this difficulty, we shall introduce a new constraint set.*

This paper is organized as follows. In Section 2, we study the autonomous problem associated with (1.10)-(1.11). In Section 3, for the nonautonomous case, we analyze the Palais-Smale condition on the Pohozaev manifold $P_{\varepsilon,a}$ (see (3.7)) for the energy functional J_ε and establish several technical lemmas that are useful for proving a multiplicity result. In Section 4, we prove that J_ε possesses at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $P_{\varepsilon,a}$ by applying the Ljusternik-Schnirelmann category theory.

2. The autonomous case

In this section, we first study the autonomous problem

$$-\Delta u+\mu u=\lambda u+(I_\alpha*F(u))f(u)\quad\text{in }\mathbb{R}^N,\tag{2.1}$$

under the normalization constraint

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a^2,\tag{2.2}$$

where $a>0$, $V(0)\le\mu\le 0$ is fixed, $\alpha\in(0,N)$, $N\ge 3$, I_α is the Riesz potential, $\lambda\in\mathbb{R}$ appears as an unknown Lagrange multiplier, and $f\colon\mathbb{R}\to\mathbb{R}$ fulfills (f_1)–(f_3).

It is easy to see that normalized solutions to (2.1)-(2.2) correspond to critical points of the energy functional

$$J_\mu(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{\mu}{2}\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u)\,\mathrm{d}x,$$

restricted to the constraint

$$S_a=\left\{u\in H^1(\mathbb{R}^N)\colon\|u\|_2=a\right\}.$$

The main result of this section is stated as follows.

Theorem 2.1. *Assume that f satisfies* (f_1)–(f_3)*. Then, for any $a>0$, problem* (2.1)-(2.2) *admits a solution $(u,\lambda)\in H^1(\mathbb{R}^N)\times\mathbb{R}$ for every fixed $V(0)\le\mu\le 0$.*

In what follows we present several lemmas that will be useful in the proof of Theorem 2.1. One of the main tools in our analysis is the Hardy-Littlewood-Sobolev inequality which can be found in the book by Lieb–Loss [26, Theorem 4.3].

Lemma 2.2. *Let $r,t>1$ and $\alpha\in(0,N)$ satisfy $\frac{1}{r}+\frac{1}{t}=1+\frac{\alpha}{N}$. Assume that $f\in L^r(\mathbb{R}^N)$ and $h\in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(r,t,\alpha,N)$, independent of f and h, such that*

$$\iint_{\mathbb{R}^{2N}}\frac{f(x)h(y)}{|x-y|^{N-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\le C(r,t,\alpha,N)\|f\|_r\|h\|_t. \tag{2.3}$$

Moreover, if $r=t=\frac{2N}{N+\alpha}$, then

$$C(r,t,\alpha,N)=\pi^{\frac{N-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{-\frac{\alpha}{N}}.$$

In this case equality in (2.3) holds if and only if $f=Ch$ and

$$h(x)=A(\gamma^2+|x-b|^2)^{-\frac{N+\alpha}{2}}$$

for some $A\in\mathbb{C}$, $\gamma\in\mathbb{R}\setminus\{0\}$, and $b\in\mathbb{R}^N$.

We also recall the well-known Gagliardo-Nirenberg inequality, see Nirenberg [36].

Lemma 2.3. *Let $N\ge 3$. For any $u\in H^1(\mathbb{R}^N)$ we have*

$$\|u\|_t^t\le C\|u\|_2^{(1-\beta)t}\|\nabla u\|_2^{\beta t},\quad \textit{with}\quad \beta=\frac{N(t-2)}{2t}, \tag{2.4}$$

for some positive constant $C=C(N,t)>0$, where $2<t<2^*$.

Lemma 2.4. *Let $N\ge 3$ and assume that f satisfies (f_1)–(f_3). If $u\in H^1(\mathbb{R}^N)$ is a weak solution of (2.1), then $u\in\mathcal{P}$, where*

$$\mathcal{P}:=\left\{u\in H^1(\mathbb{R}^N)\colon P(u)=0\right\}$$

and

$$P(u):=\|\nabla u\|_2^2+\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u)\,\mathrm{d}x-\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))f(u)u\,\mathrm{d}x.$$

Proof. Let $u\in H^1(\mathbb{R}^N)$ be a weak solution of (2.1). Then we have

$$\|\nabla u\|_2^2+(\mu-\lambda)\|u\|_2^2-\int_{\mathbb{R}^N}(I_\alpha*F(u))f(u)u\,\mathrm{d}x=0. \tag{2.5}$$

Moreover, by the Pohozaev identity, we also deduce that

$$\frac{N-2}{2}\|\nabla u\|_2^2+\frac{N}{2}(\mu-\lambda)\|u\|_2^2-\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u)\,\mathrm{d}x=0.$$

Combining this equality with (2.5) and eliminating the unknown parameter λ, we obtain

$$\|\nabla u\|_2^2+\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u)\,\mathrm{d}x-\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))f(u)u\,\mathrm{d}x=0.$$

Hence, $u\in\mathcal{P}$. □

We now recall the fiber map $(t\star u)(x):=t^{\frac{N}{2}}u(tx)$ for $(t,u)\in\mathbb{R}^+\times H^1(\mathbb{R}^N)$, which preserves the L^2-norm. Define

$$J_{\mu,u}(t):=J_\mu(t\star u).$$

A direct computation shows that

$$(J_{\mu,u})'(t)=\frac{1}{t}P(t\star u). \tag{2.6}$$

For $a > 0$, we set

$$\mathcal{P}_a := S_a \cap \mathcal{P}.$$

Lemma 2.5. *Let $u \in S_a$. Then, $t \in \mathbb{R}^+$ is a critical point for $J_{\mu,u}(t) = J_\mu(t \star u)$ if and only if $t \star u \in \mathcal{P}_a$.*

The proof of Lemma 2.5 follows directly from (2.6).

Lemma 2.6. *Let u be a critical point of J_μ restricted to $\mathcal{P}_a$. If $(J_{\mu,u})''(1) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that*

$$J'_\mu(u) - \lambda u = 0 \quad \textit{in } H^{-1}(\mathbb{R}^N).$$

Proof. Let u be a critical point of $J_\mu(u)$ restricted to $\mathcal{P}_a$. Then, by the Lagrange multiplier rule, there exist $\lambda, \lambda_0 \in \mathbb{R}$ such that

$$J'_\mu(u) - \lambda u - \lambda_0 P'(u) = 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{2.7}$$

We claim that $\lambda_0 = 0$. To this end, suppose that u satisfies (2.7) and u must satisfy the corresponding Pohozaev identity

$$(\psi_u)'(1) := \left.\frac{\mathrm{d}\psi(t \star u)}{\mathrm{d}t}\right|_{t=1} = 0,$$

where $\psi(u) := J_\mu(u) - \frac{1}{2}\lambda\|u\|_2^2 - \lambda_0 P(u)$ is the corresponding energy functional of (2.7). Indeed, we observe that

$$\begin{aligned}\psi_u(t) := \psi(t \star u) &= J_\mu(t \star u) - \frac{1}{2}\lambda\|u\|_2^2 - \lambda_0 P(t \star u)\\ &= J_{\mu,u}(t) - \frac{1}{2}\lambda\|u\|_2^2 - \lambda_0 t (J_{\mu,u})'(t).\end{aligned}$$

Consequently,

$$(\psi_u)'(t) := \frac{d\psi(t \star u)}{dt} = (1 - \lambda_0)(J_{\mu,u})'(t) - \lambda_0 t (J_{\mu,u})''(t).$$

Evaluating at $t = 1$, we obtain

$$\begin{aligned}0 = (\psi_u)'(1) &= (1 - \lambda_0)(J_{\mu,u})'(1) - \lambda_0 (J_{\mu,u})''(1)\\ &= (1 - \lambda_0)P(u) - \lambda_0 (J_{\mu,u})''(1)\\ &= -\lambda_0 (J_{\mu,u})''(1).\end{aligned}$$

Since $(J_{\mu,u})''(1) \neq 0$, it follows that $\lambda_0 = 0$. Therefore,

$$J'_\mu(u) - \lambda u = 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

□

Lemma 2.7. *Let $N \geq 3$ and assume that f satisfies* (f$_1$)–(f$_3$). *Then, for any $a > 0$, there exists $\delta_a > 0$ such that*

$$\inf_{u \in \mathcal{P}_a} \|\nabla u\|_2 \geq \delta_a.$$

Proof. Since $u \in \mathcal{P}_a$, we have the Pohozaev identity

$$\|\nabla u\|_2^2 = \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x.$$

Using assumption (f$_2$), the Hardy-Littlewood-Sobolev inequality (2.3) (see Lemma 2.2), the Gagliardo-Nirenberg inequality (2.4) (see Lemma 2.3) and the Sobolev embedding inequality, we deduce that there exists a constant $C>0$ such that

$$\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x \leq C\left(\|\nabla u\|_2^{rN-N-\alpha}+\|\nabla u\|_2^{pN-N-\alpha}\right), \tag{2.8}$$

which means that

$$\|\nabla u\|_2^2 \leq C\left(\|\nabla u\|_2^{rN-N-\alpha}+\|\nabla u\|_2^{pN-N-\alpha}\right).$$

Since $p \geq r > \frac{N+\alpha+2}{N}$, we have

$$pN-N-\alpha \geq rN-N-\alpha > 2,$$

which implies that there exists $\delta_a>0$ such that $\|\nabla u\|_2 \geq \delta_a > 0$. □

Lemma 2.8. *Assume that $N\geq 3$ and f satisfies* (f$_1$)–(f$_3$). *Then the functional J_μ restricted to $\mathcal{P}_a$ is coercive, that is,*

$$\lim_{\substack{u\in\mathcal{P}_a,\\ \|\nabla u\|_2\to+\infty}} J_\mu(u)=+\infty.$$

Proof. Let $u\in\mathcal{P}_a$. By assumption (f$_2$), we have that

$$\|\nabla u\|_2^2=\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x \geq \frac{rN-N-\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x.$$

Since $r>\frac{N+\alpha+2}{N}$, it follows that, as $\|\nabla u\|_2\to+\infty$,

$$\begin{aligned}J_\mu(u)&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{\mu}{2}\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x\\ &\geq\left(\frac{1}{2}-\frac{1}{rN-N-\alpha}\right)\|\nabla u\|_2^2+\frac{\mu}{2}\|u\|_2^2\to+\infty.\end{aligned}$$

Hence,

$$\lim_{\substack{u\in\mathcal{P}_a,\\ \|\nabla u\|_2\to+\infty}} J_\mu(u)=+\infty.$$

□

Lemma 2.9. *Assume that $N\geq 3$ and f satisfies* (f$_1$)–(f$_3$). *Then $(J_{\mu,u})''(1)<0$ for any $u\in\mathcal{P}_a$ and consequently $\mathcal{P}_a$ is a natural constraint for J_μ restricted to S_a.*

Proof. A direct computation shows that

$$\begin{aligned}&(J_{\mu,u})''(t)\\ &=\|\nabla u\|_2^2+\frac{N(N+\alpha+1)}{2t^{N+\alpha+2}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))\widetilde{F}(t^{\frac{N}{2}}u)\,\mathrm{d}x\\ &\quad-\frac{N^2}{4t^{N+\alpha+2}}\int_{\mathbb{R}^N}\left((I_\alpha * \widetilde{F}(t^{\frac{N}{2}}u))f(t^{\frac{N}{2}}u)t^{\frac{N}{2}}u+(I_\alpha * F(t^{\frac{N}{2}}u))\widetilde{F}'(t^{\frac{N}{2}}u)t^{\frac{N}{2}}u\right)\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}(J_{\mu,u})''(1)&=\|\nabla u\|_2^2+\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\left[(N+\alpha+1)\widetilde{F}(u)-\frac{N}{2}\widetilde{F}'(u)u\right]\mathrm{d}x\\ &\qquad-\frac{N^2}{4}\int_{\mathbb{R}^N}(I_\alpha * \widetilde{F}(u))f(u)u\,\mathrm{d}x.\end{aligned}$$

For any $u \in \mathcal{P}_a$, we have the identity

$$\|\nabla u\|_2^2 = \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u)\, \mathrm{d}x. \tag{2.9}$$

Using (2.9) together with (f_2) and (f_3), we deduce that

$$\begin{aligned}
(J_{\mu,u})''(1) &= \|\nabla u\|_2^2 + \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) \left[(N+\alpha+1)\widetilde{F}(u) - \frac{N}{2}\widetilde{F}'(u)u\right] \mathrm{d}x \\
&\quad - \frac{N^2}{4} \int_{\mathbb{R}^N} (I_\alpha * \widetilde{F}(u)) f(u)u\, \mathrm{d}x \\
&\le \|\nabla u\|_2^2 - \frac{rN^2}{4} \int_{\mathbb{R}^N} (I_\alpha * \widetilde{F}(u))F(u)\, \mathrm{d}x \\
&\quad + \frac{N}{2}\left((N+\alpha+1) - \frac{N+\alpha+2}{2}\right) \int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u)\, \mathrm{d}x \\
&= \|\nabla u\|_2^2 + \left((N+\alpha+1) - \frac{N+\alpha+2}{2} - \frac{rN}{2}\right) \|\nabla u\|_2^2 \\
&= \frac{(N+\alpha+2) - rN}{2} \|\nabla u\|_2^2.
\end{aligned}$$

Since $r > \frac{N+\alpha+2}{N}$ and by Lemma 2.7, we have

$$(J_{\mu,u})''(1) = \frac{(N+\alpha+2) - rN}{2}\delta_a^2 < 0. \tag{2.10}$$

Therefore, by Lemma 2.6, the set $\mathcal{P}_a$ is a natural constraint for J_μ restricted to S_a. □

Lemma 2.10. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_a$ be a bounded sequence such that $J_\mu(u_n)$ approaches a possible critical value. Then, by an argument similar to that in Lemma 2.6, we can deduce that*

$$J'_\mu(u_n) - \lambda_n u_n \to 0 \quad \textit{in } H^{-1}(\mathbb{R}^N)$$

for some bounded sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_a$ be such that $J_\mu(u_n)$ approaches a possible critical value. Then, by the Lagrange multiplier rule, there exist $\lambda_n, \lambda_n^\star \in \mathbb{R}$ such that, as $n \to +\infty$,

$$J'_\mu(u_n) - \lambda_n u_n - \lambda_n^\star P'(u_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

By Lemma 2.6, we have

$$\lambda_n^\star (J_{\mu,u_n})''(1) \to 0.$$

By Lemma 2.7 and (2.10), it follows that

$$(J_{\mu,u_n})''(1) \le \frac{(N+\alpha+2) - rN}{2}\delta_a^2 < 0.$$

Hence, $\lambda_n^\star \to 0$. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, we deduce that, as $n \to +\infty$,

$$J'_\mu(u_n) - \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

□

Lemma 2.11. *Assume that $N \geq 3$ and f satisfies* (f$_1$)–(f$_3$). *Then, for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u \star u \in \mathcal{P}_a$. Moreover, $t_u > 0$ is the unique critical point of the functional $J_{\mu,u}$,and satisfies*

$$J_\mu(t_u \star u) = \max_{t>0} J_\mu(t \star u).$$

Proof. Let $u \in S_a$. Since $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we have $\|\nabla u\|_2 > 0$. A direct computation shows that

$$\begin{aligned}(J_{\mu,u})'(t) &= t\|\nabla u\|_2^2 - \frac{N}{2t^{N+\alpha+1}} \int_{\mathbb{R}^N} (I_\alpha * F(t^{\frac{N}{2}}u))\widetilde{F}(t^{\frac{N}{2}}u)\,\mathrm{d}x \\ &= t\left(\|\nabla u\|_2^2 - \frac{N}{2}\psi(t)\right),\end{aligned}$$

where

$$\psi(t) = \int_{\mathbb{R}^N} \left(I_\alpha * \frac{F(t^{\frac{N}{2}}u)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}} \right) \frac{\widetilde{F}(t^{\frac{N}{2}}u)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x.$$

For any $s \in \mathbb{R} \setminus \{0\}$, assumptions (f$_2$) and (f$_3$) imply that the functions

$$\frac{F(ts)}{t^r} \quad \text{and} \quad \frac{\widetilde{F}(ts)}{t^{(N+\alpha+2)/N}}$$

are nondecreasing for $t \in (0, +\infty)$. Moreover, since $r > \frac{N+\alpha+2}{N}$ and

$$\frac{F(ts)}{t^{(N+\alpha+2)/N}} = \frac{F(ts)}{t^r} t^{r-\frac{N+\alpha+2}{N}},$$

we deduce that $\frac{F(ts)}{t^{(N+\alpha+2)/N}}$ is strictly increasing in $t \in (0, +\infty)$. Hence $t \mapsto \psi(t)$ is strictly increasing for $t > 0$, and therefore there is at most one $t > 0$ such that $(J_{\mu,u})'(t) = 0$. From (f$_2$), for $s \in \mathbb{R}$, we have

$$t^p F(s) \leq F(ts) \leq t^r F(s), \quad \text{if } 0 \leq t \leq 1, \tag{2.11}$$

$$t^r F(s) \leq F(ts) \leq t^p F(s), \quad \text{if } t > 1. \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$\lim_{t\to 0} \frac{F(ts)}{t^{(N+\alpha+2)/N}} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{F(ts)}{t^{(N+\alpha+2)/N}} = +\infty. \tag{2.13}$$

Since

$$\left(r - \frac{N+\alpha}{N}\right) F(s) \leq \widetilde{F}(s) \leq \left(p - \frac{N+\alpha}{N}\right) F(s)$$

and $(N + \alpha + 2)/N < r \leq p$, we also have

$$\lim_{t\to 0} \frac{\widetilde{F}(ts)}{t^{(N+\alpha+2)/N}} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{\widetilde{F}(ts)}{t^{(N+\alpha+2)/N}} = +\infty. \tag{2.14}$$

From (2.13) and (2.14), we deduce that $\psi(t) \to 0$ as $t \to 0$ by Lebesgue's dominated convergence theorem. Hence, $(J_{\mu,u})'(t) > 0$ for $t > 0$ sufficiently small. So, there exists $t_1 > 0$ such that $t \mapsto J_{\mu,u}(t)$ is increasing on $(0, t_1)$.

On the other hand, by Fatou's Lemma, we have $\psi(t) \to +\infty$ as $t \to +\infty$. Hence there exists $t_2 \geq t_1$ such that

$$J_{\mu,u}(t_2) = \max_{t>0} J_{\mu,u}(t).$$

It follows that $(J_{\mu,u})'(t_2)=0$, and therefore $t_2\star u\in\mathcal{P}_a$ by Lemma 2.5. Assume by contradiction that there exists another $t_3>0$ such that $t_3\star u\in\mathcal{P}_a$. Then, by Lemma 2.9, both t_2 and t_3 are strict local maxima of $J_{\mu,u}(\cdot)$. Without loss of generality suppose that $t_3>t_2$. Then there exists some $t_4\in(t_2,t_3)$ such that

$$J_{\mu,u}(t_4)=\min_{t\in[t_2,t_3]}J_{\mu,u}(t).$$

Hence, t_4 is a local minimum of $J_{\mu,u}(t)$, so $(J_{\mu,u})'(t_4)=0$. Thus $t_4\star u\in\mathcal{P}_a$ and $(J_{\mu,t_4\star u})''(1)=(J_{\mu,u})''(t_4)\geq 0$, which contradicts Lemma 2.9. Therefore the critical point is unique. □

Lemma 2.12. *The following minimax structure holds*

$$\Upsilon_{\mu,a}:=\inf_{u\in\mathcal{P}_a}J_\mu(u)=\inf_{u\in S_a}\max_{t>0}J_\mu(t\star u).$$

Proof. Let $u\in\mathcal{P}_a$. By Lemma 2.11, we have

$$J_\mu(u)=J_{\mu,u}(1)=\max_{t>0}J_\mu(t\star u)\geq\inf_{u\in S_a}\max_{t>0}J_\mu(t\star u),$$

which implies

$$\inf_{u\in\mathcal{P}_a}J_\mu(u)\geq\inf_{u\in S_a}\max_{t>0}J_\mu(t\star u). \tag{2.15}$$

Conversely, let $u\in S_a$. By Lemma 2.11, there exists t_u such that $t_u\star u\in\mathcal{P}_a$ and $J_\mu\left(t_u\star u\right)=\max\limits_{t>0}J_\mu(t\star u)$. Therefore,

$$\inf_{u\in\mathcal{P}_a}J_\mu(u)\leq J_\mu\left(t_u\star u\right)=\max_{t>0}J_\mu(t\star u),$$

which yields

$$\inf_{u\in\mathcal{P}_a}J_\mu(u)\leq\inf_{u\in S_a}\max_{t>0}J_\mu(t\star u). \tag{2.16}$$

Combining (2.15) and (2.16), we conclude that

$$\inf_{u\in\mathcal{P}_a}J_\mu(u)=\inf_{u\in S_a}\max_{t>0}J_\mu(t\star u).$$

□

Corollary 2.13. *Assume that $N\geq 3$ and f satisfies* (f$_1$)–(f$_3$), *$V(0)\leq\mu_1<\mu_2\leq 0$, and there exists some $V_*>0$ such that $|V(0)|<V_*$. Then, $0<\Upsilon_{\mu_1,a}<\Upsilon_{\mu_2,a}$.*

Proof. Let $u_0\in\mathcal{P}_a$ be such that $J_{\mu_2}(u_0)=\Upsilon_{\mu_2,a}$. Then, we have

$$\Upsilon_{\mu_1,a}\leq J_{\mu_1}(u_0)<J_{\mu_2}(u_0)=\Upsilon_{\mu_2,a}.$$

Similar to the proof of Lemma 2.8, we can derive the estimate

$$\begin{aligned}J_{\mu_1}(u)&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{\mu_1}{2}\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha*F(u))F(u)\,\mathrm{d}x\\&\geq\left(\frac{1}{2}-\frac{1}{rN-N-\alpha}\right)\|\nabla u\|_2^2+\frac{\mu_1}{2}\|u\|_2^2.\end{aligned}$$

From $r>\frac{N+\alpha+2}{N}$ and Lemma 2.7, we obtain

$$\left(\frac{1}{2}-\frac{1}{rN-N-\alpha}\right)\|\nabla u\|_2^2\geq\left(\frac{1}{2}-\frac{1}{rN-N-\alpha}\right)\delta_a^2>0.$$

Hence, we infer that

$$\begin{aligned} J_{\mu_1}(u) &\geq \left(\frac{1}{2} - \frac{1}{rN-N-\alpha}\right)\delta_a^2 + \frac{V(0)}{2}a^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{rN-N-\alpha}\right)\delta_a^2 - \frac{\|V\|_{L^\infty}}{2}a^2 > 0, \end{aligned}$$

under the condition

$$\|V\|_{L^\infty} < V_* := \frac{2\delta_a^2}{a^2}\left(\frac{1}{2} - \frac{1}{rN-N-\alpha}\right).$$

So, we have $\Upsilon_{\mu_1,a} > 0$. □

Lemma 2.14. *Assume that* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_a$ *is a minimizing sequence for* $\Upsilon_{\mu,a}$. *Then there exist a sequence* $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ *and constants* $R>0$, $\kappa>0$ *such that*

$$\int_{B_R(y_n)} |u_n|^2 \,\mathrm{d}x \geq \kappa.$$

Proof. Assume by contradiction that the statement is false. Then, by the Lions vanishing lemma, we obtain

$$\int_{\mathbb{R}^N} |u_n|^p \,\mathrm{d}x \to 0, \quad \text{as } n\to+\infty, \text{ for } 2<p<2^*.$$

Using condition (f$_2$), the Hardy-Littlewood-Sobolev inequality (2.3) and Lebesgue's dominated convergence theorem, we deduce

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)\,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n\,\mathrm{d}x \to 0,$$

as $n\to+\infty$. Since $u_n \in \mathcal{P}_a$, we obtain

$$\begin{aligned} 0 = P(u_n) &= \|\nabla u_n\|_2^2 + \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)\,\mathrm{d}x \\ &\quad - \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n\,\mathrm{d}x \\ &= \|\nabla u_n\|_2^2 + o_n(1). \end{aligned} \tag{2.17}$$

However, by Lemma 2.7, there exists $\delta>0$ such that $\delta < \|\nabla u_n\|_2^2$, which contradicts (2.17). Hence the result follows. □

Lemma 2.15. *Fix* $\mu\in\mathbb{R}$ *and let* $0<a_1<a_2<+\infty$. *Then,* $\Upsilon_{\mu,a_2} < \frac{a_2^2}{a_1^2}\Upsilon_{\mu,a_1}$. *In particular, if* $\mu\equiv 0$, *then* $\Upsilon_{0,a_2} < \Upsilon_{0,a_1}$.

Proof. Let $0<a_1<a_2<+\infty$ and set $\theta = \frac{a_2}{a_1} > 1$. For any $u_1 \in S_{a_1}$, define $u_2(x) = u_1(\theta^{-\frac{2}{N}}x)$. Then,

$$\|\nabla u_2\|_2^2 = \theta^{2-\frac{4}{N}}\|\nabla u_1\|_2^2, \quad \|u_2\|_2^2 = \theta^2\|u_1\|_2^2 = a_2^2,$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_2))F(u_2)\,\mathrm{d}x = \theta^{2+\frac{2\alpha}{N}}\int_{\mathbb{R}^N} (I_\alpha * F(u_1))F(u_1)\,\mathrm{d}x.$$

By Lemma 2.11, there exists $t_{u_2}>0$ such that $t_{u_2}\star u_2 \in \mathcal{P}_{a_2}$ and

$$\max_{t>0} J_\mu(t\star u_2) = J_\mu(t_{u_2}\star u_2).$$

Hence,

$$\begin{aligned}
\Upsilon_{\mu,a_2} &\le \max_{t>0} J_\mu(t\star u_2)\\
&= J_\mu(t_{u_2}\star u_2)\\
&= \frac{1}{2}t_{u_2}^2\|\nabla u_2\|_2^2+\frac{\mu}{2}\|u_2\|_2^2-\frac{1}{2t_{u_2}^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}^{\frac{N}{2}}u_2))F(t_{u_2}^{\frac{N}{2}}u_2)\,\mathrm{d}x\\
&= \frac{1}{2}t_{u_2}^2\theta^{2-\frac{4}{N}}\|\nabla u_1\|_2^2+\frac{\mu}{2}\theta^2\|u_1\|_2^2\\
&\quad -\frac{1}{2t_{u_2}^{N+\alpha}}\theta^{2+\frac{2\alpha}{N}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}^{\frac{N}{2}}u_1))F(t_{u_2}^{\frac{N}{2}}u_1)\,\mathrm{d}x\\
&= \theta^2 J_\mu(t_{u_2}\star u_1)+\frac{1}{2}(\theta^{2-\frac{4}{N}}-\theta^2)\|\nabla(t_{u_2}\star u_1)\|_2^2\\
&\quad -\frac{1}{2}(\theta^{2+\frac{2\alpha}{N}}-\theta^2)\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}\star u_1))F(t_{u_2}\star u_1)\,\mathrm{d}x\\
&< \theta^2 J_\mu(t_{u_2}\star u_1)\\
&\le \theta^2\max_{t>0} J_\mu(t\star u_1).
\end{aligned}$$

Since $u_1\in S_{a_1}$ is arbitrary, we have

$$\Upsilon_{\mu,a_2}<\theta^2\inf_{u_1\in S_{a_1}}\max_{t>0} J_\mu\left(t\star u_1\right)=\frac{a_2^2}{a_1^2}\Upsilon_{\mu,a_1}.$$

If $\mu\equiv 0$, let again $0<a_1<a_2<+\infty$ and set $\theta=\frac{a_2}{a_1}>1$. For $u_1\in S_{a_1}$, define $u_2(x)=\theta^{\frac{2-N}{2}}u_1(\theta^{-1}x)$. Then,

$$\|\nabla u_2\|_2^2=\|\nabla u_1\|_2^2,\quad \|u_2\|_2^2=\theta^2\|u_1\|_2^2=a_2^2,$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_2))F(u_2)\,\mathrm{d}x=\theta^{N+\alpha}\int_{\mathbb{R}^N}(I_\alpha * F(\theta^{\frac{2-N}{2}}u_1))F(\theta^{\frac{2-N}{2}}u_1)\,\mathrm{d}x.$$

Using Lemma 2.11 again, we can find $t_{u_2}>0$ such that $t_{u_2}\star u_2\in\mathcal{P}_{a_2}$ and

$$\max_{t>0} J_0(t\star u_2)=J_0(t_{u_2}\star u_2).$$

Therefore

$$\begin{aligned}
\Upsilon_{0,a_2} &\le \max_{t>0} J_0(t\star u_2)\\
&= J_0(t_{u_2}\star u_2)\\
&= \frac{1}{2}t_{u_2}^2\|\nabla u_2\|_2^2-\frac{1}{2t_{u_2}^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}^{\frac{N}{2}}u_2))F(t_{u_2}^{\frac{N}{2}}u_2)\,\mathrm{d}x\\
&= \frac{1}{2}t_{u_2}^2\|\nabla u_1\|_2^2-\frac{1}{2t_{u_2}^{N+\alpha}}\theta^{N+\alpha}\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}^{\frac{N}{2}}\theta^{\frac{2-N}{2}}u_1))F(t_{u_2}^{\frac{N}{2}}\theta^{\frac{2-N}{2}}u_1)\,\mathrm{d}x\\
&< \frac{1}{2}t_{u_2}^2\|\nabla u_1\|_2^2-\frac{1}{2t_{u_2}^{N+\alpha}}\theta^{N+\alpha-p(N-2)}\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}^{\frac{N}{2}}u_1))F(t_{u_2}^{\frac{N}{2}}u_1)\,\mathrm{d}x\\
&< \frac{1}{2}t_{u_2}^2\|\nabla u_1\|_2^2-\frac{1}{2t_{u_2}^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{u_2}^{\frac{N}{2}}u_1))F(t_{u_2}^{\frac{N}{2}}u_1)\,\mathrm{d}x\\
&= J_0(t_{u_2}\star u_1)\\
&\le \max_{t>0} J_0(t\star u_1).
\end{aligned}$$

Since $u_1 \in S_{a_1}$ is arbitrary, we have $\Upsilon_{0,a_2} < \Upsilon_{0,a_1}$. $\square$

Proof of Theorem 2.1. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_a$ be such that $J_\mu(u_n) \to \Upsilon_{\mu,a}$. Let $\tilde{u}_n(x) := u_n(x+y_n)$, where $\{y_n\}_{n\in\mathbb{N}}$ is the sequence given by Lemma 2.14. Clearly, $J_\mu(\tilde{u}_n) \to \Upsilon_{\mu,a}$ and $\tilde{u}_n \in \mathcal{P}_a$. By Lemma 2.8, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, up to a subsequence, we may assume that there exists $\tilde{u} \in H^1(\mathbb{R}^N)$ and such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} & \text{in } H^1(\mathbb{R}^N), \\ \tilde{u}_n \to \tilde{u} & \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \le q < 2^*, \\ \tilde{u}_n \to \tilde{u} & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

We claim that $\tilde{u} \not\equiv 0$. Otherwise, a direct computation gives

$$\int_{B_R(0)} |\tilde{u}_n(x)|^2 \,\mathrm{d}x = \int_{B_R(0)} |u_n(x+y_n)|^2 \,\mathrm{d}x = \int_{B_R(y_n)} |u_n(x)|^2 \,\mathrm{d}x = o_n(1),$$

which contradicts Lemma 2.14. By Lemma 2.10, we know that

$$\begin{aligned} o_n(1) &= \left\langle J'_\mu(u_n) - \lambda_n u_n, u_n \right\rangle \\ &= \|\nabla u_n\|_2^2 + (\mu - \lambda_n)\|u_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n \,\mathrm{d}x. \end{aligned}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, by assumption (f$_2$), the Hardy-Littlewood-Sobolev inequality (2.3), the Gagliardo-Nirenberg inequality (2.4) and the Sobolev embedding inequality, we deduce that

$$\begin{aligned} |\mu - \lambda_n| a^2 &\le \|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n \,\mathrm{d}x + o_n(1) \\ &\le \|\nabla u_n\|_2^2 + C_1 (\|\nabla u_n\|_2^{rN-N-\alpha} + \|\nabla u_n\|_2^{pN-N-\alpha}) + o_n(1) \\ &\le C_2, \end{aligned}$$

for some $C_1, C_2 > 0$. Therefore $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence if necessary, we may assume that $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$.

Again by Lemma 2.10, we have

$$J'_\mu(u_n) - \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Thus, for any $\phi \in H^1(\mathbb{R}^N)$, we get

$$\begin{aligned} &\left\langle J'_\mu(\tilde{u}_n) - \lambda_n \tilde{u}_n, \phi \right\rangle \\ &= \int_{\mathbb{R}^N} \nabla \tilde{u}_n \cdot \nabla \phi \,\mathrm{d}x + (\mu - \lambda_n) \int_{\mathbb{R}^N} \tilde{u}_n \phi \,\mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u}_n)) f(\tilde{u}_n) \phi \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \nabla u_n(x) \cdot \nabla \phi(x - y_n) \,\mathrm{d}x + (\mu - \lambda_n) \int_{\mathbb{R}^N} u_n(x) \phi(x - y_n) \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * F(u_n(x))) f(u_n(x)) \phi(x - y_n) \,\mathrm{d}x \\ &= \left\langle J'_\mu(u_n) - \lambda_n u_n, \phi(\cdot - y_n) \right\rangle \\ &= o_n(1), \end{aligned}$$

which implies that

$$J'_\mu(\tilde{u}_n) - \lambda_n \tilde{u}_n \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{2.18}$$

For any $\varphi \in H^1(\mathbb{R}^N)$, by the definition of weak convergence, it follows that

$$\int_{\mathbb{R}^N} \nabla \tilde{u}_n \cdot \nabla \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^N} \nabla \tilde{u} \cdot \nabla \varphi \,\mathrm{d}x, \quad \text{as } n \to +\infty. \tag{2.19}$$

Since $\lambda_n \to \lambda$, we also have

$$\lambda_n \int_{\mathbb{R}^N} \tilde{u}_n \varphi \,\mathrm{d}x \to \lambda \int_{\mathbb{R}^N} \tilde{u} \varphi \,\mathrm{d}x, \quad \text{as } n \to +\infty. \tag{2.20}$$

By the Sobolev embedding theorem, $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$. Using (f$_2$), the sequence $\{F(\tilde{u}_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Hence, $F(\tilde{u}_n) \rightharpoonup F(\tilde{u})$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Since the Riesz potential defines a continuous linear map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ into $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we get $I_\alpha * (F(\tilde{u}_n)) \rightharpoonup I_\alpha * (F(\tilde{u}))$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Again by (f$_2$), the sequence $\{f(\tilde{u}_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$ and $f(\tilde{u}_n) \to f(\tilde{u})$ in $L^{\frac{2N}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u}_n)) f(\tilde{u}_n) \varphi \,\mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u})) f(\tilde{u}) \varphi \,\mathrm{d}x \to 0.$$

Moreover, by Hölder's inequality, the sequence $\{(I_\alpha * F(\tilde{u}_n)) f(\tilde{u}_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, because

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(I_\alpha * F(\tilde{u}_n)) f(\tilde{u}_n)|^{\frac{2N}{N+2}} \,\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^N} |I_\alpha * F(\tilde{u}_n)|^{\frac{2N}{N-\alpha}} \,\mathrm{d}x \right)^{\frac{N-\alpha}{N+2}} \left(\int_{\mathbb{R}^N} |f(\tilde{u}_n)|^{\frac{2N}{\alpha+2}} \,\mathrm{d}x \right)^{\frac{\alpha+2}{N+2}}.
\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we conclude that

$$(I_\alpha * F(\tilde{u}_n)) f(\tilde{u}_n) \rightharpoonup (I_\alpha * F(\tilde{u})) f(\tilde{u}) \quad \text{in } L^{\frac{2N}{N+2}}(\mathbb{R}^N). \tag{2.21}$$

Consequently, for all $\varphi \in H^1(\mathbb{R}^N)$, by virtue of (2.18), (2.19), (2.20), and (2.21), we obtain

$$\begin{aligned}
o_n(1) &= \langle J'_\mu(\tilde{u}_n) - \lambda_n \tilde{u}_n, \varphi \rangle \\
&= \int_{\mathbb{R}^N} \nabla \tilde{u}_n \cdot \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} \mu \tilde{u}_n \varphi \,\mathrm{d}x - \lambda_n \int_{\mathbb{R}^N} \tilde{u}_n \varphi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u}_n)) f(\tilde{u}_n) \varphi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla \tilde{u} \cdot \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} \mu \tilde{u} \varphi \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \tilde{u} \varphi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u})) f(\tilde{u}) \varphi \,\mathrm{d}x + o_n(1).
\end{aligned}$$

Hence, $\tilde{u}$ satisfies

$$-\Delta \tilde{u} + \mu \tilde{u} = \lambda \tilde{u} + (I_\alpha * F(\tilde{u})) f(\tilde{u}) \quad \text{in } \mathbb{R}^N. \tag{2.22}$$

Testing (2.22) with $\tilde{u} \in H^1(\mathbb{R}^N)$, we have

$$\|\nabla \tilde{u}\|_2^2 + (\mu - \lambda) \|\tilde{u}\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u})) f(\tilde{u}) \tilde{u} \,\mathrm{d}x = 0. \tag{2.23}$$

Moreover, by the Pohozaev identity, we also deduce that

$$\frac{N-2}{2}\|\nabla\tilde{u}\|_2^2+\frac{N}{2}(\mu-\lambda)\|\tilde{u}\|_2^2-\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}))F(\tilde{u})\,\mathrm{d}x=0. \tag{2.24}$$

Combining this identity with (2.23), we get

$$\|\nabla\tilde{u}\|_2^2=\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}))\widetilde{F}(\tilde{u})\,\mathrm{d}x,$$

that is, $P(\tilde{u})=0$. Moreover, by (2.23), (2.24) and (f_2) we have

$$\begin{aligned}(\mu-\lambda)\|u\|_2^2&=\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}))\left(\frac{N+\alpha}{2}F(\tilde{u})-\frac{N-2}{2}f(\tilde{u})\tilde{u}\right)\mathrm{d}x\\&\geq\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}))\left(\frac{N+\alpha}{2p}-\frac{N-2}{2}\right)f(\tilde{u})\tilde{u}\,\mathrm{d}x\\&>0,\end{aligned}$$

which implies $\mu-\lambda>0$, and so $\lambda<\mu\leq 0$.

We next prove that $\|\tilde{u}\|_2=a$. Assume by contradiction that $\|\tilde{u}\|_2=l\neq a$. Since $\tilde{u}_n\rightharpoonup\tilde{u}$ in $L^2(\mathbb{R}^N)$ and $\|\tilde{u}_n\|_2=a$, we necessarily have $l\in(0,a)$. Set $v_n=\tilde{u}_n-\tilde{u}$ By the Brezis-Lieb splitting lemma, we have

$$\begin{aligned}\|\nabla\tilde{u}_n\|_2^2&=\|\nabla v_n\|_2^2+\|\nabla\tilde{u}\|_2^2+o_n(1),\\\|\tilde{u}_n\|_2^2&=\|v_n\|_2^2+\|\tilde{u}\|_2^2+o_n(1).\end{aligned}$$

Arguing as in the proof of Lemma 2.17 in Li–Ye [23], we also have

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}_n))F(\tilde{u}_n)\,\mathrm{d}x&=\int_{\mathbb{R}^N}(I_\alpha*F(v_n))F(v_n)\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}))F(\tilde{u})\,\mathrm{d}x+o_n(1).\end{aligned}$$

Furthermore, by Lemma 2.15, we have that

$$\begin{aligned}\Upsilon_{\mu,a}=J_\mu(\tilde{u}_n)+o_n(1)&=J_\mu(v_n)+J_\mu(\tilde{u})+o_n(1)\\&\geq\Upsilon_{\mu,\|v_n\|_2}+\Upsilon_{\mu,\|\tilde{u}\|_2}+o_n(1)\\&\geq\frac{\|v_n\|_2^2}{a^2}\Upsilon_{\mu,a}+\Upsilon_{\mu,\|\tilde{u}\|_2}+o_n(1).\end{aligned}\tag{2.25}$$

Let $h_n^2=\|v_n\|_2^2$. Passing to a subsequence, we may assume that $h_n^2\to h^2$. Since $\|\tilde{u}\|_2^2=l^2$, we have $a^2=h^2+l^2$. Letting $n\to+\infty$ in (2.25), we get

$$\Upsilon_{\mu,a}\geq\frac{h^2}{a^2}\Upsilon_{\mu,a}+\Upsilon_{\mu,\|\tilde{u}\|_2}.$$

Since $l\in(0,a)$, Lemma 2.15 yields

$$\Upsilon_{\mu,a}>\frac{h^2}{a^2}\Upsilon_{\mu,a}+\frac{l^2}{a^2}\Upsilon_{\mu,a}=\left(\frac{h^2}{a^2}+\frac{l^2}{a^2}\right)\Upsilon_{\mu,a}=\Upsilon_{\mu,a},$$

a contradiction. Therefore, $\|\tilde{u}\|_2=l=a$. It follows that $\tilde{u}_n\to\tilde{u}$ in $L^2(\mathbb{R}^N)$. By interpolation, we have that

$$\tilde{u}_n\to\tilde{u}\quad\text{in } L^q(\mathbb{R}^N),\quad\text{for all } 2\leq q<2^*.$$

Consequently,

$$\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}_n))F(\tilde{u}_n)\,\mathrm{d}x\to\int_{\mathbb{R}^N}(I_\alpha*F(\tilde{u}))F(\tilde{u})\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u}_n))f(\tilde{u}_n)\tilde{u}_n \, \mathrm{d}x \to \int_{\mathbb{R}^N} (I_\alpha * F(\tilde{u}))f(\tilde{u})\tilde{u} \, \mathrm{d}x.$$

Since $P(\tilde{u}_n) = P(\tilde{u}) = 0$, we deduce that $\|\nabla \tilde{u}_n\|_2^2 \to \|\nabla \tilde{u}\|_2^2$. Therefore, $\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^N)$. This completes the proof. ☐

3. THE NONAUTONOMOUS CASE

In this section, we consider the nonautonomous Choquard problem

$$-\Delta u + V(\varepsilon x)u = \lambda u + (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

subject to the normalization constraint $\int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a^2$, where $\varepsilon, a > 0$ and $\lambda \in \mathbb{R}$ is an unknown parameter appearing as a Lagrange multiplier. We will prove some properties of the functional J_ε given by

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 \, \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) \, \mathrm{d}x$$

on the constraint set S_a.

Lemma 3.1. *Assume that $N \geq 3$, f satisfies* (f$_1$)–(f$_3$) *and V satisfies* (V$_1$)–(V$_3$). *If $u \in H^1(\mathbb{R}^N)$ is a weak solution of* (3.1), *then $u \in \mathcal{P}_\varepsilon$, where*

$$\mathcal{P}_\varepsilon := \big\{u \in H^1(\mathbb{R}^N) \colon P_\varepsilon(u) = 0\big\}$$

and

$$P_\varepsilon(u) := \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u^2 \, \mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u) \, \mathrm{d}x.$$

Proof. Assume that $u \in H^1(\mathbb{R}^N)$ is a weak solution of (3.1). Then it holds that

$$\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 \, \mathrm{d}x = \lambda \|u\|_2^2 + \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)u \, \mathrm{d}x. \tag{3.2}$$

Additionally, by the Pohozaev identity, we deduce that

$$\begin{aligned} &\frac{N-2}{2}\|\nabla u\|_2^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 \, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(\varepsilon x), \varepsilon x\rangle u^2 \, \mathrm{d}x \\ &= \frac{N}{2}\lambda \|u\|_2^2 + \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) \, \mathrm{d}x. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we conclude that

$$\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u^2 \, \mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u) \, \mathrm{d}x = 0.$$

Thus, $u \in \mathcal{P}_\varepsilon$. ☐

We introduce the scaling map

$$(t \star u)(x) := t^{\frac{N}{2}} u(tx), \quad \text{for } (t, u) \in \mathbb{R}^+ \times H^1(\mathbb{R}^N),$$

which preserves the L^2-norm of u. For every $u \in H^1(\mathbb{R}^N)$, define the associated fiber map

$$J_{\varepsilon,u}(t) := J_\varepsilon(t \star u).$$

A straightforward computation yields

$$(J_{\varepsilon,u})'(t) = \frac{1}{t}P_\varepsilon(t \star u). \tag{3.4}$$

For $a > 0$, we set

$$\mathcal{P}_{\varepsilon,a} := S_a \cap \mathcal{P}_\varepsilon.$$

Lemma 3.2. *Let $u \in S_a$. Then $t \in \mathbb{R}^+$ is a critical point of $J_{\varepsilon,u}(t) = J_\varepsilon(t \star u)$ if and only if $t \star u \in \mathcal{P}_{\varepsilon,a}$.*

Proof. The result follows directly from (3.4). □

Lemma 3.3. *For any critical point of $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$, if $(J_{\varepsilon,u})''(1) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that*

$$J'_\varepsilon(u) - \lambda u = 0 \quad \textit{in } H^{-1}(\mathbb{R}^N).$$

Proof. Let u be a critical point of $J_\varepsilon(u)$ restricted to $\mathcal{P}_{\varepsilon,a}$. By the Lagrange multiplier rule, there exist $\lambda, \lambda_0 \in \mathbb{R}$ such that

$$J'_\varepsilon(u) - \lambda u - \lambda_0 P'_\varepsilon(u) = 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{3.5}$$

It suffices to show $\lambda_0 = 0$. For this purpose, let u be a solution to (3.5). Then, u must satisfy the corresponding Pohozaev identity

$$(\phi_u)'(1) := \frac{\mathrm{d}\phi(t \star u)}{\mathrm{d}t}\bigg|_{t=1} = 0,$$

where $\phi(u) := J_\varepsilon(u) - \frac{1}{2}\lambda\|u\|_2^2 - \lambda_0 P_\varepsilon(u)$ is the corresponding energy functional of (3.5). In fact, we observe that

$$\begin{aligned}\phi_u(t) := \phi(t \star u) &= J_\varepsilon(t \star u) - \frac{1}{2}\lambda\|u\|_2^2 - \lambda_0 P_\varepsilon(t \star u) \\ &= J_{\varepsilon,u}(t) - \frac{1}{2}\lambda\|u\|_2^2 - \lambda_0 t (J_{\varepsilon,u})'(t).\end{aligned}$$

Hence,

$$(\phi_u)'(t) := \frac{d\phi(t \star u)}{dt} = (1 - \lambda_0)(J_{\varepsilon,u})'(t) - \lambda_0 t (J_{\varepsilon,u})''(t).$$

Evaluating at $t = 1$, we obtain

$$\begin{aligned}0 = (\phi_u)'(1) &= (1 - \lambda_0)(J_{\varepsilon,u})'(1) - \lambda_0 (J_{\varepsilon,u})''(1) \\ &= (1 - \lambda_0) P_\varepsilon(u) - \lambda_0 (J_{\varepsilon,u})''(1) \\ &= -\lambda_0 (J_{\varepsilon,u})''(1).\end{aligned}$$

Due to $(J_{\varepsilon,u})''(1) \neq 0$, we get $\lambda_0 = 0$, which implies that

$$J'_\varepsilon(u) - \lambda u = 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

□

Lemma 3.4. *Assume that $N \geq 3$, f satisfies* (f$_1$)–(f$_3$) *and V satisfies* (V$_1$)–(V$_3$)*. Then, for any $a > 0$, there exists $\widetilde{\delta}_a > 0$ such that*

$$\inf_{u \in \mathcal{P}_{\varepsilon,a}} \|\nabla u\|_2 \geq \widetilde{\delta}_a.$$

Proof. Since $u \in \mathcal{P}_{\varepsilon,a}$, we have

$$\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)|u|^2 \,\mathrm{d}x = \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x. \tag{3.6}$$

By assumption (V_2), for any $u \in H^1(\mathbb{R}^N)$, one has

$$\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)|u|^2 \,\mathrm{d}x \geq (1-\sigma_2)\|\nabla u\|_2^2,$$

Combining this estimate with (2.8) and (3.6), we obtain

$$\begin{aligned}(1-\sigma_2)\|\nabla u\|_2^2 &\leq \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x \\ &\leq C\left(\|\nabla u\|_2^{rN-N-\alpha} + \|\nabla u\|_2^{pN-N-\alpha}\right).\end{aligned}$$

Since

$$pN-N-\alpha \geq rN-N-\alpha > 2 \quad \text{and} \quad 1-\sigma_2>0,$$

it follows that there exists $\widetilde{\delta}_a > 0$ such that

$$\|\nabla u\|_2 \geq \widetilde{\delta}_a.$$

□

We decompose$\mathcal{P}_{\varepsilon,a}$ into the disjoint union

$$\mathcal{P}_{\varepsilon,a} = \mathcal{P}^+_{\varepsilon,a} \cup \mathcal{P}^0_{\varepsilon,a} \cup \mathcal{P}^-_{\varepsilon,a}, \tag{3.7}$$

where

$$\begin{aligned}\mathcal{P}^+_{\varepsilon,a} &:= \{u \in \mathcal{P}_{\varepsilon,a} \colon (J_{\varepsilon,u})''(1) > 0\},\\ \mathcal{P}^0_{\varepsilon,a} &:= \{u \in \mathcal{P}_{\varepsilon,a} \colon (J_{\varepsilon,u})''(1) = 0\},\\ \mathcal{P}^-_{\varepsilon,a} &:= \{u \in \mathcal{P}_{\varepsilon,a} \colon (J_{\varepsilon,u})''(1) < 0\}.\end{aligned}$$

Lemma 3.5. *Assume that $N \geq 3$, f satisfies* (f_1)–(f_3) *and V satisfies* (V_1)–(V_3). *Then, $\mathcal{P}^-_{\varepsilon,a} = \mathcal{P}_{\varepsilon,a}$ is closed in $H^1(\mathbb{R}^N)$ and it is a natural constraint for $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$.*

Proof. For any $u \in \mathcal{P}_{\varepsilon,a}$, we have

$$\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)|u|^2\,\mathrm{d}x = \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x. \tag{3.8}$$

In view of (3.8), and assumptions (f_2), (f_3), (V_3), together with Lemma 3.4, we obtain

$$\begin{aligned}&(J_{\varepsilon,u})''(1)\\ &= \|\nabla u\|_2^2 + \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\left[(N+\alpha+1)\widetilde{F}(u) - \frac{N}{2}\widetilde{F}'(u)u\right]\mathrm{d}x\\ &\quad - \frac{N^2}{4}\int_{\mathbb{R}^N}(I_\alpha * \widetilde{F}(u))f(u)u\,\mathrm{d}x + \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^N}\langle \nabla W(\varepsilon x), \varepsilon x\rangle u^2\,\mathrm{d}x\\ &\leq \|\nabla u\|_2^2 - \frac{rN^2}{4}\int_{\mathbb{R}^N}(I_\alpha * \widetilde{F}(u))F(u)\,\mathrm{d}x + \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x\\ &\quad + \frac{N}{2}\left((N+\alpha+1) - \frac{N+\alpha+2}{2}\right)\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^N}\langle \nabla W(\varepsilon x), \varepsilon x\rangle u^2\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \|\nabla u\|_2^2 + \frac{N}{2}\left((N+\alpha+1) - \frac{N+\alpha+2}{2} - \frac{rN}{2}\right)\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\langle \nabla W(\varepsilon x), \varepsilon x\rangle u^2\,\mathrm{d}x \\
&= \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\langle \nabla W(\varepsilon x), \varepsilon x\rangle u^2\,\mathrm{d}x \\
&\quad + \left((N+\alpha+1) - \frac{N+\alpha+2}{2} - \frac{rN}{2}\right)(\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)|u|^2\,\mathrm{d}x) \\
&= \frac{(N+\alpha+2)-rN}{2}\|\nabla u\|_2^2 + \frac{rN+2-N-\alpha}{2}\int_{\mathbb{R}^N} W(\varepsilon x)|u|^2\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\langle \nabla W(\varepsilon x), \varepsilon x\rangle u^2\,\mathrm{d}x \\
&= \frac{(N+\alpha+2)-rN}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} Y(\varepsilon x)u^2\,\mathrm{d}x \\
&\leq \left(\frac{(N+\alpha+2)-rN}{2} + \sigma_3\right)\|\nabla u\|_2^2 < 0.
\end{aligned}
$$

Therefore, $\mathcal{P}^+_{\varepsilon,a} = \mathcal{P}^0_{\varepsilon,a} = \emptyset$. Hence, $\mathcal{P}^-_{\varepsilon,a} = \mathcal{P}_{\varepsilon,a}$ is closed in $H^1(\mathbb{R}^N)$. By Lemma 3.3, $\mathcal{P}_{\varepsilon,a}$ is a natural constraint of $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$. ☐

Remark 3.6. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(\mathrm{PS})_c$-sequence of $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$ at level c. Then there exist two sequences $\{\lambda_n\}_{n\in\mathbb{N}}, \{\lambda^\star_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that, as $n \to +\infty$,*

$$
J'_\varepsilon(u_n) - \lambda_n u_n - \lambda^\star_n P'_\varepsilon(u_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).
$$

By Lemma 3.5, we know that $(J_{\varepsilon,u_n})''(1) < 0$. Arguing as in Lemma 3.3, we deduce that, as $n \to +\infty$,

$$
J'_\varepsilon(u_n) - \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).
$$

Lemma 3.7. *Assume that $N \geq 3$, f satisfies (f_1)–(f_3) and V satisfies (V_1)–(V_3). Then, for every $u \in S_a$, there exists a unique $t_u > 0$ such that $t_u \star u \in \mathcal{P}_{\varepsilon,a}$. Moreover, t_u is the unique critical point of the function $J_\varepsilon(t_u \star u)$, and satisfies*

$$
J_{\varepsilon,u}(t_u) = \max_{t>0} J_\varepsilon(t \star u).
$$

Proof. Let $u \in S_a$. Since $u \in H^1(\mathbb{R}^N)$, we have $\|\nabla u\|_2 > 0$. By assumption (V_2) and a direct computation, we obtain

$$
\begin{aligned}
(J_{\varepsilon,u})'(t) &= t\|\nabla u\|_2^2 - \frac{N}{2t^{N+\alpha+1}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))\widetilde{F}(t^{\frac{N}{2}}u)\,\mathrm{d}x \\
&\quad - t^2\int_{\mathbb{R}^N} W(\varepsilon x)u^2(tx)\,\mathrm{d}x \\
&\geq (1-\sigma_2)t\|\nabla u\|_2^2 - \frac{N}{2t^{N+\alpha+1}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))\widetilde{F}(t^{\frac{N}{2}}u)\,\mathrm{d}x \\
&= t\left((1-\sigma_2)\|\nabla u\|_2^2 - \frac{N}{2}g(t)\right),
\end{aligned}
$$

where

$$
g(t) = \int_{\mathbb{R}^N}\left(I_\alpha * \frac{F(t^{\frac{N}{2}}u)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{\widetilde{F}(t^{\frac{N}{2}}u)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x.
$$

By Lemma 2.11, we have $g(t) \to 0$ as $t \to 0$. By (V_2), we have $1-\sigma_2 > 0$. Thus, $(J_{\varepsilon,u})'(t) > 0$ for $t > 0$ sufficiently small. Hence, there exists $t_1 > 0$ such that $J_{\varepsilon,u}(t)$ is increasing on $(0, t_1)$.

On the other hand, by (V_1), we deduce

$$\begin{aligned}
J_{\varepsilon,u}(t) &= \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)(t\star u)^2\,\mathrm{d}x - \frac{1}{2t^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))F(t^{\frac{N}{2}}u)\,\mathrm{d}x\\
&\le \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{\sigma_1}{2}\|\nabla(t\star u)\|_2^2 - \frac{1}{2t^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))F(t^{\frac{N}{2}}u)\,\mathrm{d}x\\
&= \frac{1+\sigma_1}{2}t^2\|\nabla u\|_2^2 - \frac{1}{2t^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))F(t^{\frac{N}{2}}u)\,\mathrm{d}x\\
&= t^2\left(\frac{1+\sigma_1}{2}\|\nabla u\|_2^2 - \frac{1}{2t^{N+\alpha+2}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}u))F(t^{\frac{N}{2}}u)\,\mathrm{d}x\right)\\
&= t^2\left(\frac{1+\sigma_1}{2}\|\nabla u\|_2^2 - \frac{1}{2}h(t)\right),
\end{aligned}$$

where

$$h(t) = \int_{\mathbb{R}^N}\left(I_\alpha * \frac{F(t^{\frac{N}{2}}u)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{F(t^{\frac{N}{2}}u)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x.$$

By Lemma 2.11, we have $h(t) \to +\infty$ as $t \to +\infty$. Hence, there exists $t_2 \ge t_1$ such that

$$J_{\varepsilon,u}(t_2) = \max_{t>0} J_{\varepsilon,u}(t).$$

It is clear that $(J_{\varepsilon,u})'(t_2) = 0$ and $t_2 \star u \in \mathcal{P}_{\varepsilon,a}$ by Lemma 3.2.

Suppose by contradiction that there exists another $t_3 > 0$ such that $t_3 \star u \in \mathcal{P}_{\varepsilon,a}$. Then, by Lemma 3.5, both t_2 and t_3 are strict local maxima of $J_{\varepsilon,u}(t)$. Without loss of generality assume $t_3 > t_2$. Then there exists some $t_4 \in (t_2, t_3)$ such that

$$J_{\varepsilon,u}(t_4) = \min_{t\in[t_2,t_3]} J_{\varepsilon,u}(t),$$

Thus, t_4 is a local minimum of $J_{\varepsilon,u}(t)$. Hence, $(J_{\varepsilon,u})'(t_4) = 0$, which implies $t_4 \star u \in \mathcal{P}_{\varepsilon,a}$. Moreover $(J_{\varepsilon,t_4\star u})''(1) = (J_{\varepsilon,u})''(t_4) \ge 0$, which contradicts Lemma 3.5. Therefore, $t_u = t_2$ is unique. □

Lemma 3.8. *Assume that $N \ge 3$, f satisfies* (f_1)–(f_3) *and V satisfies* (V_1)–(V_3). *Then, the functional $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$ is coercive, that is,*

$$\lim_{\substack{u\in\mathcal{P}_{\varepsilon,a},\\ \|\nabla u\|_2\to+\infty}} J_\varepsilon(u) = +\infty.$$

Proof. For any $u \in \mathcal{P}_{\varepsilon,a}$, by ($\mathrm{V}_2$) and ($\mathrm{f}_2$), we have that

$$\begin{aligned}
(1+\sigma_2)\|\nabla u\|_2^2 &\ge \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x\\
&= \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x\\
&\ge \frac{rN-N-\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x.
\end{aligned}$$

Combining this estimate with assumption (V_1) and the condition $r > \frac{N+\alpha+2}{N}$, we obtain, as $\|\nabla u\|_2 \to +\infty$,

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\,\mathrm{d}x \\ &\geq \frac{1}{2}(1-\sigma_1)\|\nabla u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\,\mathrm{d}x \\ &\geq \left(\frac{1-\sigma_1}{2} - \frac{1+\sigma_2}{rN-N-\alpha}\right)\|\nabla u\|_2^2 \to +\infty. \end{aligned} \tag{3.9}$$

By the assumption

$$\sigma_2 \in \left(0, \min\left\{\frac{N+\alpha-p(N-2)}{2p}, \frac{(1-\sigma_1)(rN-N-\alpha)}{2} - 1\right\}\right),$$

we deduce

$$\frac{1-\sigma_1}{2} - \frac{1+\sigma_2}{rN-N-\alpha} > 0.$$

Hence,

$$\lim_{\substack{u\in\mathcal{P}_{\varepsilon,a},\\ \|\nabla u\|_2\to+\infty}} J_\varepsilon(u) = +\infty.$$

This completes the proof. ☐

Lemma 3.9. *The following minimax characterization holds:*

$$\Upsilon_{\varepsilon,a} := \inf_{u\in\mathcal{P}_{\varepsilon,a}} J_\varepsilon(u) = \inf_{u\in S_a}\max_{t>0} J_\varepsilon(t\star u) > 0.$$

Proof. For any $u \in \mathcal{P}_{\varepsilon,a}$, by Lemma 3.7, we have

$$J_\varepsilon(u) = J_{\varepsilon,u}(1) = \max_{t>0} J_\varepsilon(t\star u) \geq \inf_{u\in S_a}\max_{t>0} J_\varepsilon(t\star u),$$

which implies

$$\inf_{u\in\mathcal{P}_{\varepsilon,a}} J_\varepsilon(u) \geq \inf_{u\in S_a}\max_{t>0} J_\varepsilon(t\star u). \tag{3.10}$$

On the other hand, for any $u \in S_a$, Lemma 3.7 ensures the existence of $t_u > 0$ such that $t_u \star u \in \mathcal{P}_{\varepsilon,a}$ and $J_\varepsilon(t_u\star u) = \max_{t>0} J_\varepsilon(t\star u)$. Therefore,

$$\inf_{u\in\mathcal{P}_{\varepsilon,a}} J_\varepsilon(u) \leq J_\varepsilon(t_u\star u) = \max_{t>0} J_\varepsilon(t\star u),$$

which implies

$$\inf_{u\in\mathcal{P}_{\varepsilon,a}} J_\varepsilon(u) \leq \inf_{u\in S_a}\max_{t>0} J_\varepsilon(t\star u). \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\inf_{u\in\mathcal{P}_{\varepsilon,a}} J_\varepsilon(u) = \inf_{u\in S_a}\max_{t>0} J_\varepsilon(t\star u).$$

Finally, in view of (3.9) and Lemma 3.4, we deduce

$$J_\varepsilon(u) \geq C\widetilde{\delta}_a^2 > 0, \quad \text{for any } u \in \mathcal{P}_{\varepsilon,a}.$$

Hence, we have $\Upsilon_{\varepsilon,a} > 0$. ☐

Next, we establish some properties of $\Upsilon_{\varepsilon,a}$ that will play a crucial role in the sequel.

Lemma 3.10. *There exists* $\varepsilon_0 > 0$ *such that* $\limsup_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,a} \leq \Upsilon_{V(0),a}$ *for all* $\varepsilon \in (0,\varepsilon_0)$.

Proof. Let $u_0 \in \mathcal{P}_a$ be such that $J_{V(0)}(u_0) = \Upsilon_{V(0),a}$. By assumptions (V_1) and (V_2), letting $\varepsilon \to 0^+$, one has

$$W(\varepsilon x)u_0^2 \to 0 \quad \text{and} \quad V(\varepsilon x)u_0^2 \to V(0)u_0^2 \quad \text{a.e. in } \mathbb{R}^N.$$

Using again (V_1) and (V_2), there exists some $k_1 > 0$ such that

$$|W(\varepsilon x)|u_0^2 \leq k_1 u_0^2 \quad \text{and} \quad |V(\varepsilon x)|u_0^2 \leq |V_0|u_0^2.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N} W(\varepsilon x)u_0^2\,\mathrm{d}x = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N} V(\varepsilon x)u_0^2\,\mathrm{d}x = V(0)\|u_0\|_2^2,$$

implying that $P_\varepsilon(u_0) = o(1)$ as $\varepsilon \to 0^+$. By Lemma 3.7, there exists a unique $t_{\varepsilon,u_0} > 0$ such that $t_{\varepsilon,u_0} \star u_0 \in \mathcal{P}_\varepsilon$.

Claim: $\lim_{\varepsilon\to 0^+} t_{\varepsilon,u_0} = 1$.

Indeed, let $t_{\varepsilon_n} := t_{\varepsilon_n,u_0}$. If $\lim_{n\to+\infty} t_{\varepsilon_n} = 0$, by Lemma 3.4, there exists $\delta > 0$ such that

$$0 < \delta < \|\nabla(t_{\varepsilon_n} \star u_0)\|_2^2 = t_{\varepsilon_n}^2\|\nabla u_0\|_2^2 \to 0,$$

which is impossible. For any $t_{\varepsilon_n} \star u_0 \in \mathcal{P}_{\varepsilon_n,a}$, we have

$$\begin{aligned}&t_{\varepsilon_n}^2\|\nabla u_0\|_2^2 - \int_{\mathbb{R}^N} W\left(\frac{\varepsilon_n x}{t_{\varepsilon_n}}\right)|u_0|^2\,\mathrm{d}x\\ &= \frac{N}{2t_{\varepsilon_n}^{N+\alpha}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{\varepsilon_n}^{\frac{N}{2}}u_0))\left(f(t_{\varepsilon_n}^{\frac{N}{2}}u_0)t_{\varepsilon_n}^{\frac{N}{2}}u_0 - \frac{N+\alpha}{N}F(t_{\varepsilon_n}^{\frac{N}{2}}u_0)\right)\mathrm{d}x.\end{aligned} \tag{3.12}$$

By (f_2), we obtain

$$\begin{aligned}&\|\nabla u_0\|_2^2 - \frac{1}{t_{\varepsilon_n}^2}\int_{\mathbb{R}^N} W\left(\frac{\varepsilon_n x}{t_{\varepsilon_n}}\right)|u_0|^2\,\mathrm{d}x\\ &= \frac{N}{2t_{\varepsilon_n}^{N+\alpha+2}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{\varepsilon_n}^{\frac{N}{2}}u_0))\left(f(t_{\varepsilon_n}^{\frac{N}{2}}u_0)t_{\varepsilon_n}^{\frac{N}{2}}u_0 - \frac{N+\alpha}{N}F(t_{\varepsilon_n}^{\frac{N}{2}}u_0)\right)\mathrm{d}x\\ &\geq \frac{N}{2t_{\varepsilon_n}^{N+\alpha+2}}\int_{\mathbb{R}^N}(I_\alpha * F(t_{\varepsilon_n}^{\frac{N}{2}}u_0))\left(rF(t_{\varepsilon_n}^{\frac{N}{2}}u_0) - \frac{N+\alpha}{N}F(t_{\varepsilon_n}^{\frac{N}{2}}u_0)\right)\mathrm{d}x\\ &= \frac{rN-N-\alpha}{2}\int_{\mathbb{R}^N}\left(I_\alpha * \frac{F(t_{\varepsilon_n}^{\frac{N}{2}}u_0)}{(t_{\varepsilon_n}^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{F(t_{\varepsilon_n}^{\frac{N}{2}}u_0)}{(t_{\varepsilon_n}^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x.\end{aligned}$$

If $\lim_{n\to+\infty} t_{\varepsilon_n} = +\infty$, by $\frac{N+\alpha+2}{N} < r$ and Lemma 2.11, we deduce that $\|\nabla u_0\|_2^2 \to +\infty$, which contradicts $u_0 \in H^1(\mathbb{R}^N)$. Therefore there exist constants $t_0 > 0$ and $T_0 < +\infty$ such that

$$0 < t_0 < t_{\varepsilon_n} < T_0.$$

Passing to a subsequence we may assume that $\lim_{n\to+\infty} t_{\varepsilon_n} = T$.

Since $u_0 \in \mathcal{P}_a$, we get

$$\int_{\mathbb{R}^N}|\nabla u_0|^2\,\mathrm{d}x = \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_0))\widetilde{F}(u_0)\,\mathrm{d}x. \tag{3.13}$$

From $0 < t_0 < t_{\varepsilon_n} < T_0$, we have $t_0 < t_{\varepsilon_n} < T_0$ for n sufficiently large, and using (f_2) together with the Hardy-Littlewood-Sobolev inequality(2.3), we deduce that there exists some $C_0 > 0$ such that

$$\begin{aligned}&\frac{N}{2} t_{\varepsilon_n}^{-(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * F(t_{\varepsilon_n}^{\frac{N}{2}} u_0)) \widetilde{F}(t_{\varepsilon_n}^{\frac{N}{2}} u_0)\,\mathrm{d}x \\ &\le t_0^{-(N+\alpha)} C_0 \left(\|T_0^{\frac{N}{2}} u_0\|_{\frac{2Nr}{N+\alpha}}^{2r} + \|T_0^{\frac{N}{2}} u_0\|_{\frac{2Np}{N+\alpha}}^{2p} \right)\end{aligned}$$

and

$$t_{\varepsilon_n}^{-(N+\alpha)} (I_\alpha * F(t_{\varepsilon_n}^{\frac{N}{2}} u_0)) \widetilde{F}(t_{\varepsilon_n}^{\frac{N}{2}} u_0) \to T^{-(N+\alpha)} (I_\alpha * F(T^{\frac{N}{2}} u_0)) \widetilde{F}(T^{\frac{N}{2}} u_0)$$

a.e. in $\mathbb{R}^N$. Thus, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}&\frac{N}{2} \lim_{n\to+\infty} t_{\varepsilon_n}^{-(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * F(t_{\varepsilon_n}^{\frac{N}{2}} u_0)) \widetilde{F}(t_{\varepsilon_n}^{\frac{N}{2}} u_0)\,\mathrm{d}x \\ &= \frac{N}{2} T^{-(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * F(T^{\frac{N}{2}} u_0)) \widetilde{F}(T^{\frac{N}{2}} u_0)\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

Using (3.12), (3.14) and (V_2), we obtain

$$T^2 \|\nabla u_0\|_2^2 = \frac{N}{2} T^{-(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * F(T^{\frac{N}{2}} u_0)) \widetilde{F}(T^{\frac{N}{2}} u_0)\,\mathrm{d}x.$$

Combining this with (3.13) yields

$$\begin{aligned}0 = \int_{\mathbb{R}^N} &\left(\left(I_\alpha * \frac{F(T^{\frac{N}{2}} u_0)}{(T^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}} \right) \frac{\widetilde{F}(T^{\frac{N}{2}} u_0)}{(T^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}} - (I_\alpha * F(u_0)) \widetilde{F}(u_0) \right) \mathrm{d}x \\ &+ o_n(1).\end{aligned} \tag{3.15}$$

By assumptions (f_2) and (f_3), we know that

$$\frac{F(t^{\frac{N}{2}} s)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}} \quad \text{and} \quad \frac{\widetilde{F}(t^{\frac{N}{2}} s)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}$$

are nondecreasing on $(0, +\infty)$. Consequently, (3.15) holds if and only if $T = 1$. Thus, $t_{\varepsilon,u_0} \to 1$ as $\varepsilon \to 0^+$

Finally, we derive that

$$\begin{aligned}\Upsilon_{\varepsilon,a} &\le J_\varepsilon(t_{\varepsilon,u_0} \star u_0) \\ &= J_\varepsilon(u_0) + o_n(1) \\ &= \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u_0|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u_0))F(u_0)\,\mathrm{d}x + o_n(1) \\ &= J_{V(0)}(u_0) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - V(0)) u_0^2\,\mathrm{d}x + o_n(1).\end{aligned}$$

Taking the limit as $\varepsilon \to 0^+$, one has

$$\limsup_{\varepsilon\to0^+} \Upsilon_{\varepsilon,a} \le \limsup_{\varepsilon\to0^+} J_\varepsilon(t_{\varepsilon,u_0} \star u_0) = J_{V(0)}(u_0) = \Upsilon_{V(0),a}.$$

From (V_1) and Corollary 2.13, we know that $\Upsilon_{V(0),a} < \Upsilon_{0,a}$. Thus, there exists $\varepsilon_0 > 0$ such that $\Upsilon_{\varepsilon,a} \le \Upsilon_{0,a}$ for any $\varepsilon \in (0, \varepsilon_0)$. □

Hereafter, we fix

$$\rho = \frac{1}{2}(\Upsilon_{0,a} - \Upsilon_{V(0),a}) > 0.$$

The next two Lemmas will be used to prove that the $(\mathrm{PS})_c$-condition holds for the functional J_ε constrained to $\mathcal{P}_{\varepsilon,a}$ at levels $c < \Upsilon_{V(0),a} + \rho$.

Lemma 3.11. *Fix $\varepsilon \in (0, \varepsilon_0)$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\varepsilon,a}$ be a $(\mathrm{PS})_c$-sequence for $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$ with $c < \Upsilon_{V(0),a} + \rho$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then $u \neq 0$ and $J_\varepsilon(u) \geq 0$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\varepsilon,a}$ be a $(\mathrm{PS})_c$-sequence for $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$ at level c. Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\varepsilon,a}$, Lemma 3.8 implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence if necessary, we may assume that there exists $u \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } H^1(\mathbb{R}^N), \\ u_n \to u & \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \leq q < 2^*, \\ u_n \to u & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.16}$$

We claim that u is nontrivial. Suppose by contradiction that $u = 0$. By (V_1), for any $\zeta > 0$, there exists $R_1 > 0$ such that

$$|V(x)| < \zeta_1, \quad \text{for all } |x| \geq R_1.$$

Thus, by (3.16) and the arbitrariness of ζ_1, we deduce that

$$\begin{aligned} \Upsilon_{\varepsilon,a} &= J_\varepsilon(u_n) \\ &= J_0(u_n) + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2 \,\mathrm{d}x \\ &= J_0(u_n) + \frac{1}{2}\int_{B_{R_1/\varepsilon}(0)} V(\varepsilon x)|u_n|^2 \,\mathrm{d}x + \frac{1}{2}\int_{B^c_{R_1/\varepsilon}(0)} V(\varepsilon x)|u_n|^2 \,\mathrm{d}x \\ &\geq J_0(u_n) + \frac{1}{2}\int_{B_{R_1/\varepsilon}(0)} V(\varepsilon x)|u_n|^2 \,\mathrm{d}x - \frac{\zeta_1}{2}\int_{B^c_{R_1/\varepsilon}(0)} |u_n|^2 \,\mathrm{d}x \\ &= J_0(u_n) + o_n(1). \end{aligned} \tag{3.17}$$

On the other hand, by (V_2), for any $\zeta_2 > 0$, there exists $R_2 > 0$ such that

$$|W(x)| \leq \zeta_2, \quad \text{for all } |x| \geq R_2.$$

Hence, by (3.16) and the arbitrariness of ζ_2, we have

$$\int_{\mathbb{R}^N} W(\varepsilon x)|u_n|^2 \,\mathrm{d}x = \int_{B^c_{R_2/\varepsilon}(0)} W(\varepsilon x)|u_n|^2 \,\mathrm{d}x + \int_{B_{R_2/\varepsilon}(0)} W(\varepsilon x)|u_n|^2 \,\mathrm{d}x = o_n(1).$$

Since $u_n \in \mathcal{P}_{\varepsilon,a}$, it follows that $P(u_n) = o_n(1)$. By Lemma 2.11, for each n there exists a unique $t_{u_n} > 0$ such that $t_{u_n} \star u_n \in \mathcal{P}_a$. Arguing as in the proof of Lemma 3.10, we obtain

$$t_{u_n} = 1 + o_n(1). \tag{3.18}$$

Therefore, by (3.17), (3.18), and the assumption $c < \Upsilon_{V(0),a} + \rho$, we have

$$\begin{aligned}\Upsilon_{V(0),a} + \rho &> c \\ &= J_\varepsilon(u_n) + o_n(1) \\ &\geq J_0(u_n) + o_n(1) \\ &= J_0\left(t_{u_n} \star u_n\right) + o_n(1) \\ &\geq \Upsilon_{0,a} + o_n(1).\end{aligned}$$

Letting $n \to +\infty$, we get

$$\Upsilon_{V(0),a} + \rho \geq \Upsilon_{0,a},$$

which contradicts the definition of ρ. Hence, $u \neq 0$.

By Remark 3.6, we know that

$$\langle J_\varepsilon'(u_n) - \lambda_n u_n, u_n\rangle = o_n(1).$$

The boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$ indicates that $\{\lambda_n\}_{n\in\mathbb{N}}$ is also a bounded sequence in $\mathbb{R}$. Indeed,

$$\begin{aligned}|\lambda_n| &= \frac{1}{\|u_n\|_2^2}\left|\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)u_n\,\mathrm{d}x\right| \\ &\leq \frac{1}{a^2}\left(\|\nabla u_n\|_2^2 + |V(0)|\|u_n\|_2^2 + C(\|\nabla u_n\|_2^{rN-N-\alpha} + \|\nabla u_n\|_2^{pN-N-\alpha})\right),\end{aligned}$$

which implies that $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence if necessary, we may assume $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$. For any $\varphi \in H^1(\mathbb{R}^N)$, we easily get

$$\lambda_n \int_{\mathbb{R}^N} u_n\varphi\,\mathrm{d}x \to \lambda\int_{\mathbb{R}^N} u\varphi\,\mathrm{d}x. \tag{3.19}$$

Moreover, by weak convergence,

$$\int_{\mathbb{R}^N} \nabla u_n\cdot\nabla\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi\,\mathrm{d}x. \tag{3.20}$$

By the Sobolev embedding theorem, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)\cap L^{2^*}(\mathbb{R}^N)$. By (f_2), the sequence $\{F(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, and hence

$$F(u_n) \rightharpoonup F(u) \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Since the Riesz potential is a continuous linear map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we have

$$I_\alpha * (F(u_n)) \rightharpoonup I_\alpha * (F(u)) \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Again by (f_2), the sequence $\{f(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$, and

$$f(u_n) \to f(u) \quad \text{in } L^{\frac{2N}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^N).$$

Then, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * F(u))f(u)\varphi\,\mathrm{d}x \to 0.$$

Note that $\{(I_\alpha * F(u_n))f(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, because by Hölder's inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |(I_\alpha * F(u_n))f(u_n)|^{\frac{2N}{N+2}}\,\mathrm{d}x\\ &\le \left(\int_{\mathbb{R}^N} |I_\alpha * F(u_n)|^{\frac{2N}{N-\alpha}}\,\mathrm{d}x\right)^{\frac{N-\alpha}{N+2}}\left(\int_{\mathbb{R}^N} |f(u_n)|^{\frac{2N}{\alpha+2}}\,\mathrm{d}x\right)^{\frac{\alpha+2}{N+2}}.\end{aligned}$$

Then, since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we deduce that

$$(I_\alpha * F(u_n))f(u_n) \rightharpoonup (I_\alpha * F(u))f(u) \quad \text{in } L^{\frac{2N}{N+2}}(\mathbb{R}^N). \tag{3.21}$$

Consequently, for any $\varphi \in H^1(\mathbb{R}^N)$, by virtue of (3.19), (3.20), and (3.21), it follows

$$\begin{aligned}o_n(1) &= \langle J_\varepsilon'(u_n) - \lambda_n u_n, \varphi\rangle\\ &= \int_{\mathbb{R}^N} \nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)u_n\varphi\,\mathrm{d}x - \lambda_n\int_{\mathbb{R}^N} u_n\varphi\,\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)\varphi\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N} \nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)u\varphi\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} u\varphi\,\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * F(u))f(u)\varphi\,\mathrm{d}x + o_n(1)\\ &= \langle J_\varepsilon'(u) - \lambda u, \varphi\rangle + o_n(1).\end{aligned}\tag{3.22}$$

Hence, u is a nontrivial weak solution of

$$-\Delta u + V(\varepsilon x)u = \lambda u + (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N. \tag{3.23}$$

Since $u \in H^1(\mathbb{R}^N)$ is a nontrivial weak solution of (3.23), it satisfies the corresponding Pohozaev identity $P_\varepsilon(u) = 0$, where

$$P_\varepsilon(u) = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x. \tag{3.24}$$

Using (f_2), (V_2), and (3.24), we deduce

$$\begin{aligned}(1+\sigma_2)\,\|\nabla u\|_2^2 &\ge \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u^2\,\mathrm{d}x\\ &= \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\widetilde{F}(u)\,\mathrm{d}x\\ &\ge \frac{rN-N-\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x.\end{aligned}$$

Combining this with (V_1) and $\frac{N+\alpha+2}{N} < r$, implies that

$$\begin{aligned}
J_\varepsilon(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x \\
&\geq \frac{1-\sigma_1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x \\
&\geq \frac{1-\sigma_1}{2}\|\nabla u\|_2^2 - \frac{1+\sigma_2}{rN-N-\alpha}\|\nabla u\|_2^2 \qquad (3.25)\\
&= \left(\frac{1-\sigma_1}{2} - \frac{1+\sigma_2}{rN-N-\alpha}\right)\|\nabla u\|_2^2 \\
&> 0.
\end{aligned}$$

This completes the proof. □

Lemma 3.12. *Fix $\varepsilon \in (0,\varepsilon_0)$. Assume that $N \geq 3$, f satisfies (f_1)–(f_3) and V satisfies (V_1)–(V_3). For any $a > 0$, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\varepsilon,a}$ be a $(\mathrm{PS})_c$-sequence for $J_\varepsilon|_{\mathcal{P}_{\varepsilon,a}}$ at level $c < \Upsilon_{V(0),a} + \rho$. Then there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, and a function $u \in H^1(\mathbb{R}^N)$ with $u \neq 0$ such that*

$$-\Delta u + V(\varepsilon x)u = \lambda u + (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N.$$

Moreover, one of the following alternatives holds.

(i) *Either $u_n \to u$ in $H^1(\mathbb{R}^N)$;*

(ii) *or there exists $k_0 \in \mathbb{N}^+$, sequences $\{y_n^k\} \subset \mathbb{R}^N$ with $|y_n^k| \to \infty$ as $n \to +\infty$ for each $1 \leq k \leq k_0$, and nontrivial solutions $\omega^1, \ldots, \omega^{k_0}$ of the problem*

$$-\Delta u = \lambda u + (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \qquad (3.26)$$

such that

$$c = J_\varepsilon(u) + \sum_{k=1}^{k_0} J_0(\omega^k),$$

$$\left\| u_n - u - \sum_{k=1}^{k_0}\omega^k(-y_n^k)\right\| = o_n(1),$$

$$\|\nabla u_n\|_2^2 \to \|\nabla u\|_2^2 + \sum_{k=1}^{k_0}\|\nabla\omega^k\|_2^2.$$

Here

$$J_\varepsilon(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x$$

and

$$J_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,\mathrm{d}x.$$

Proof. By Lemma 3.11, there exists $u \in H^1(\mathbb{R}^N)$ with $u \neq 0$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } H^1(\mathbb{R}^N), \\ u_n \to u & \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \leq q < 2^*, \\ u_n \to u & \text{a.e. in } \mathbb{R}^N. \end{cases} \qquad (3.27)$$

and u is a nontrivial solution of

$$-\Delta u + V(\varepsilon x)u = \lambda u + (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N. \tag{3.28}$$

Step 1. Set $u_n^1 := u_n - u$. Then we have

$$\begin{cases} u_n^1 \rightharpoonup 0 & \text{in } H^1(\mathbb{R}^N), \\ u_n^1 \to 0 & \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } 1 \le q < 2^*, \\ u_n^1 \to 0 & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

Set

$$L := \liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^1|^2 \, \mathrm{d}x$$

If $L = 0$, we claim that $u_n \to u$ in $H^1(\mathbb{R}^N)$. By Lions' vanishing Lemma, $u_n^1 \to 0$ in $L^p(\mathbb{R}^N)$ for every $p \in (2, 2^*)$. By the Brezis-Lieb splitting Lemma, we have

$$\|\nabla u_n\|_2^2 = \|\nabla u_n^1\|_2^2 + \|\nabla u\|_2^2 + o_n(1). \tag{3.29}$$

Moreover, by (f$_2$), (f$_3$), and Lemma 2.3 of Long–Feng [28], we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n \, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} (I_\alpha * F(u_n^1))f(u_n^1)u_n^1 \, \mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)u \, \mathrm{d}x + o_n(1) \\ &= \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)u \, \mathrm{d}x + o_n(1). \end{aligned} \tag{3.30}$$

Since $u \in H^1(\mathbb{R}^N)$ is a nontrivial solution of (3.28), it satisfies the corresponding Pohozaev identity $P_\varepsilon(u) = 0$, where

$$P_\varepsilon(u) = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u^2 \, \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u) \, \mathrm{d}x. \tag{3.31}$$

Since $u_n \in \mathcal{P}_{\varepsilon,a}$, we have

$$\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x)u_n^2 \, \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))\widetilde{F}(u_n) \, \mathrm{d}x = 0. \tag{3.32}$$

By (V$_2$) and $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} W(\varepsilon x)(u_n^2 - u^2) \, \mathrm{d}x = o_n(1). \tag{3.33}$$

By Lemma 2.17 of Li–Ye [23] and Lemma 2.3 of Long–Feng [28], we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))\widetilde{F}(u_n) \, \mathrm{d}x &= \int_{\mathbb{R}^N} (I_\alpha * F(u_n^1))\widetilde{F}(u_n^1) \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u) \, \mathrm{d}x + o_n(1) \\ &= \int_{\mathbb{R}^N} (I_\alpha * F(u))\widetilde{F}(u) \, \mathrm{d}x + o_n(1). \end{aligned} \tag{3.34}$$

Using (3.31), (3.32), (3.33), and (3.34), we deduce that $\|\nabla u_n\|_2^2 \to \|\nabla u\|_2^2$. It follows from (V$_1$) and (3.27) that

$$\int_{\mathbb{R}^N} V(\varepsilon x)|u_n^1|^2 \, \mathrm{d}x = o_n(1). \tag{3.35}$$

By (3.22) in Lemma 3.11, we know that

$$J_\varepsilon'(u) - \lambda u = 0 \quad \text{in } H^{-1}(\mathbb{R}^N), \tag{3.36}$$

where λ is the limit of the Lagrange multiplier sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ from Lemma 3.11. Consequently, by combining (3.29), (3.30), (3.35) and (3.36), we get

$$\begin{aligned}\left\langle J_0'(u_n^1) - \lambda u_n^1, u_n^1 \right\rangle &= \left\langle J_\varepsilon'(u_n^1) - \lambda u_n^1, u_n^1 \right\rangle - \int_{\mathbb{R}^N} V(\varepsilon x)|u_n^1|^2 \,\mathrm{d}x\\ &= \left\langle J_\varepsilon'\left(u_n^1\right) - \lambda u_n^1, u_n^1 \right\rangle + o_n(1)\\ &= \langle J_\varepsilon'\left(u_n\right) - \lambda u_n, u_n \rangle - \langle J_\varepsilon'(u) - \lambda u, u \rangle + o_n(1)\\ &= o_n(1),\end{aligned}$$

which implies that

$$\|\nabla u_n^1\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u_n^1)) f(u_n^1) u_n^1 \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} |u_n^1|^2 \,\mathrm{d}x = o_n(1). \tag{3.37}$$

Since (f$_2$) and $u_n^1 \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ imply

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n^1)) f(u_n^1) u_n^1 \,\mathrm{d}x = o_n(1),$$

it follows from (3.37) and $\|\nabla u_n\|_2^2 \to \|\nabla u\|_2^2$ that

$$\lambda \int_{\mathbb{R}^N} |u_n^1|^2 \,\mathrm{d}x = o_n(1). \tag{3.38}$$

Since $u_n \in \mathcal{P}_{\varepsilon,a}$ and $\langle J_\varepsilon'(u_n) - \lambda_n u_n, u_n \rangle = o_n(1)$, we have

$$\begin{aligned}-\lambda_n \|u_n\|_2^2 &= \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n \,\mathrm{d}x - \|\nabla u_n\|_2^2\\ &\quad - \int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \,\mathrm{d}x + o_n(1)\\ &\geq \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n \,\mathrm{d}x - \|\nabla u_n\|_2^2 + o_n(1)\\ &= \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n \,\mathrm{d}x - \int_{\mathbb{R}^N} W(\varepsilon x)|u_n|^2 \,\mathrm{d}x\\ &\quad - \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) \widetilde{F}(u_n) \,\mathrm{d}x + o_n(1)\\ &\geq \frac{N + \alpha - p(N-2)}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) F(u_n) \,\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N} W(\varepsilon x)|u_n|^2 \,\mathrm{d}x\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon x) u_n^2 \,\mathrm{d}x &= \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) \widetilde{F}(u_n) \,\mathrm{d}x\\ &\leq \frac{pN - N - \alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) F(u_n) \,\mathrm{d}x.\end{aligned} \tag{3.40}$$

Combining (3.39) and (3.40) yields

$$\begin{aligned}-\lambda_n\|u_n\|_2^2 \geq{}& \frac{N+\alpha-p(N-2)}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N}W(\varepsilon x)|u_n|^2\,\mathrm{d}x\\ \geq{}& \frac{N+\alpha-p(N-2)}{pN-N-\alpha}(\|\nabla u_n\|_2^2-\int_{\mathbb{R}^N}W(\varepsilon x)u_n^2\,\mathrm{d}x)\\ &-\int_{\mathbb{R}^N}W(\varepsilon x)|u_n|^2\,\mathrm{d}x\\ \geq{}& \frac{N+\alpha-p(N-2)}{pN-N-\alpha}(\|\nabla u_n\|_2^2-\sigma_2\|\nabla u_n\|_2^2)-\sigma_2\|\nabla u_n\|_2^2\\ ={}& \left(\frac{N+\alpha-p(N-2)}{pN-N-\alpha}-\frac{2p}{pN-N-\alpha}\sigma_2\right)\|\nabla u_n\|_2^2.\end{aligned} \tag{3.41}$$

By (V$_2$), (3.41) and Lemma 3.4, there exists $\delta>0$ such that

$$-\lambda_n a^2 \geq \delta > 0.$$

Hence, $\lim\limits_{n\to\infty}\lambda_n=\lambda<0$, and so, (3.38) gives $u_n\to u$ in $L^2(\mathbb{R}^N)$. Since $\|\nabla u_n\|_2^2\to\|\nabla u\|_2^2$, we conclude that $u_n\to u$ in $H^1(\mathbb{R}^N)$.

If $L>0$, then there exists a sequence $\{y_n^1\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that

$$\int_{B_1(y_n^1)}|u_n^1|^2\,\mathrm{d}x \geq \frac{L}{2}>0.$$

Set $\omega_n^1(x)=u_n^1(x+y_n^1)$. Then, $\{\omega_n^1\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and we may assume that $\omega_n^1\rightharpoonup\omega^1$ in $H^1(\mathbb{R}^N)$. Clearly, $\omega^1\neq 0$. Otherwise, local compactness would imply

$$o_n(1)=\int_{B_1(0)}|\omega_n^1(x)|^2\,\mathrm{d}x=\int_{B_1(y_n^1)}|u_n^1|^2\,\mathrm{d}x\geq\frac{L}{2}>0,$$

a contradiction.

Arguing as in the proof leading to (3.21), we have that, for all $\varphi\in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n^1))f(u_n^1)\varphi\,\mathrm{d}x=o_n(1) \tag{3.42}$$

and thus,

$$J_\varepsilon'(u_n^1)-\lambda u_n^1\to 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{3.43}$$

By (V$_1$) and similar to the proof of (3.35), we have

$$\int_{\mathbb{R}^N}V(\varepsilon x)u_n^1\varphi(x-y_n^1)\,\mathrm{d}x=o_n(1). \tag{3.44}$$

Hence, by virtue of (3.42), (3.43), and (3.44), for all $\varphi\in H^1(\mathbb{R}^N)$, we get

$$\begin{aligned}&\left\langle J_0'(\omega_n^1)-\lambda\omega_n^1,\varphi\right\rangle\\ &=\int_{\mathbb{R}^N}\nabla\omega_n^1\nabla\varphi\,\mathrm{d}x-\int_{\mathbb{R}^N}(I_\alpha * F(u_n^1))f(u_n^1)\varphi\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}\nabla\omega_n^1\nabla\varphi\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\nabla u_n^1(x+y_n^1)\nabla\varphi\,\mathrm{d}x-\int_{\mathbb{R}^N}(I_\alpha * F(u_n^1(x+y_n^1)))f(u_n^1(x+y_n^1))\varphi\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_{\mathbb{R}^N} \nabla u_n^1(x+y_n^1)\nabla\varphi \,\mathrm{d}x\\
&= \int_{\mathbb{R}^N} \nabla u_n^1(x)\nabla\varphi(x-y_n^1)\,\mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * F(u_n^1(x)))f(u_n^1(x))\varphi(x-y_n^1)\,\mathrm{d}x\\
&\quad - \lambda \int_{\mathbb{R}^N} \nabla u_n^1(x)\varphi(x-y_n^1)\,\mathrm{d}x\\
&= \left\langle J_0'(u_n^1) - \lambda u_n^1, \varphi(x-y_n^1)\right\rangle\\
&= \left\langle J_\varepsilon'(u_n^1) - \lambda u_n^1, \varphi(x-y_n^1)\right\rangle - \int_{\mathbb{R}^N} V(\varepsilon x) u_n^1 \varphi(x-y_n^1)\,\mathrm{d}x\\
&= o_n(1),
\end{aligned}
$$

Therefore, $J_0'(\omega_n^1) - \lambda\omega_n^1 \to 0$ in $H^{-1}(\mathbb{R}^N)$. Thus, the weak convergence implies that

$$J_0'(\omega^1) - \lambda\omega^1 = 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Moreover, since $u_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we have $|y_n^1| \to +\infty$. Otherwise, there would exist $R_0 > 0$ such that $|y_n^1| \le R_0$, and then

$$0 < \frac{L}{2} = \int_{B_1(y_n^1)} |u_n^1(x)|^2 \,\mathrm{d}x \le \int_{B_{1+R_0}(0)} |u_n^1(x)|^2 \,\mathrm{d}x = o_n(1),$$

which is a contradiction.

Step 2. Set $u_n^2 := u_n - u - \omega^1(\cdot - y_n^1)$. By the Brezis-Lieb lemma, we have

$$
\begin{aligned}
\|\nabla u_n^2\|_2^2 &= \|\nabla u_n\|_2^2 - \|\nabla u\|_2^2 - \|\nabla\omega^1\|_2^2 + o_n(1),\\
\|u_n^2\|_2^2 &= \|u_n\|_2^2 - \|u\|_2^2 - \|\omega^1\|_2^2 + o_n(1),\\
J_0(u_n^2) &= c - J_\varepsilon(u) - J_0(\omega^1) + o_n(1),
\end{aligned}
$$

and

$$J_0'(u_n^2) - \lambda u_n^2 \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Let

$$\widetilde{L} := \liminf_{n+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^2|^2 \,\mathrm{d}x.$$

Similarly to Step 1, if $\widetilde{L} = 0$, then $\|u_n^2\|_{H^1(\mathbb{R}^N)} \to 0$, that is,

$$\left\| u_n - u - \omega^1(\cdot - y_n^1)\right\|_{H^1(\mathbb{R}^N)} \to 0.$$

Moreover,

$$\|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla\omega^1\|_2^2 + o_n(1) \quad \text{and} \quad c = J_\varepsilon(u) + J_0(\omega^1).$$

If $\tilde{L} > 0$, then there exists a sequence $\{y_n^2\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n^2)} |u_n^2|^2 \,\mathrm{d}x \ge \frac{\widetilde{L}}{2} > 0$$

In addition, there exists a nontrivial solution $\omega^2 \in H^1(\mathbb{R}^N)$ of (3.26) such that $\omega_n^2(x) = u_n^2(x + y_n^2) \rightharpoonup \omega^2$ in $H^1(\mathbb{R}^N)$. Furthermore, $u_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ implies

$|y_n^2| \to +\infty$. We claim that $|y_n^2 - y_n^1| \to +\infty$. Otherwise, there would exist $\widetilde{R}_0 > 0$ such that $|y_n^2 - y_n^1| < \widetilde{R}_0$, and thus

$$\begin{aligned} 0 < \frac{\widetilde{L}}{2} &\leq \int_{B_1(y_n^2)} |u_n^2|^2 \,\mathrm{d}x = \int_{B_1(y_n^2)} |u_n^1(x) - \omega^1(x - y_n^1)|^2 \,\mathrm{d}x \\ &= \int_{B_1(y_n^2)} |\omega_n^1(x - y_n^1) - \omega^1(x - y_n^1)|^2 \,\mathrm{d}x = \int_{B_1(y_n^2 - y_n^1)} |\omega_n^1 - \omega^1|^2 \,\mathrm{d}x \\ &\leq \int_{B_{1+\widetilde{R}_0}(0)} |\omega_n^1 - \omega^1|^2 \,\mathrm{d}x = o_n(1), \end{aligned}$$

a contradiction.

Step 3. Repeating the above argument, we obtain nontrivial solutions ω^k, $k \geq 1$ of (3.26). Since $V(\varepsilon x) \leq 0$, Lemma 3.11 implies that J_0 is bounded from below by a positive constant on nontrivial solutions, and therefore the iteration stops after finitely many steps. Thus, for some $k_0 \in \mathbb{N}^+$, we obtain nontrivial solutions $\omega^1, \omega^2 \ldots\ldots \omega^{k_0}$ of (3.26) such that

$$c = J_\varepsilon(u) + \sum_{k=1}^{k_0} J_0(\omega^k),$$

$$\left\| u_n - u - \sum_{k=1}^{k_0} \omega^k(-y_n^k) \right\| = o_n(1),$$

$$\|\nabla u_n\|_2^2 \to \|\nabla u\|_2^2 + \sum_{k=1}^{k_0} \|\nabla \omega^k\|_2^2.$$

□

Next, we establish a compactness result for the functional J_ε.

Theorem 3.13. *Assume that $N \geq 3$, f satisfies* (f$_1$)–(f$_3$) *and V satisfies* (V$_1$)–(V$_3$)*. Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, the functional J_ε restricted to $\mathcal{P}_{\varepsilon,a}$ satisfies the* (PS)$_c$*-condition for $c < \Upsilon_{V(0),a} + \rho$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS)$_c$-sequence for J_ε restricted to $\mathcal{P}_{\varepsilon,a}$ with

$$c < \Upsilon_{0,a} + \rho \quad \text{and} \quad 0 < \rho = \frac{1}{2}(\Upsilon_{0,a} - \Upsilon_{V(0),a}).$$

By Lemma 3.12, if case (ii) occurs, then $k_0 \geq 1$ ($k_0 \in \mathbb{N}$), and by (3.25) and (3.26), we have

$$c = J_\varepsilon(u) + \sum_{k=1}^{k_0} J_0(\omega^k) \geq \sum_{k=1}^{k_0} \Upsilon_{0,\|\omega^k\|_2}.$$

Moreover,

$$a^2 = \|u_n\|_2^2 = \|u\|_2^2 + \sum_{k=1}^{k_0} \|\omega^k\|_2^2$$

On the other hand, for each $k \in \{1, \ldots, k_0\}$, Lemma 2.15 yields

$$\Upsilon_{0,\|\omega^k\|_2} > \Upsilon_{0,a},$$

and so,

$$c \geq \sum_{k=1}^{k_0} \Upsilon_{0,\|\omega^k\|_2} > \Upsilon_{0,a},$$

which implies that

$$\frac{1}{2}\Upsilon_{0,a} + \frac{1}{2}\Upsilon_{V(0),a} = \Upsilon_{V(0),a} + \rho > c > \Upsilon_{0,a}.$$

By Corollary 2.13, it is impossible. Hence, $u_n \to u$ in $H^1(\mathbb{R}^N)$. $\square$

4. Multiplicity result

In this section, we study the multiplicity of normalized solutions. Fix $\delta > 0$ and let w be a positive solution of

$$\begin{cases} -\Delta u + V(0)u = \lambda u + (I_\alpha * F(u))f(u), & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a^2, \end{cases}$$

such that $J_{V(0)}(w) = \Upsilon_{V(0),a}$. Let $\eta\colon [0,+\infty) \to [0,1]$ be a smooth cut-off function defined by

$$\eta(t) = \begin{cases} 1, & 0 \leq t \leq \frac{\delta}{2}, \\ 0, & t \geq \delta. \end{cases}$$

For any $y \in M$, define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad \text{and} \quad \tilde{\Psi}_{\varepsilon,y}(x) = a\frac{\Psi_{\varepsilon,y}(x)}{\|\Psi_{\varepsilon,y}\|_2}.$$

We then define $\Phi_\varepsilon\colon M \to \mathcal{P}_{\varepsilon,a}$ by

$$\Phi_\varepsilon(y) = t_{\varepsilon,y} \star \tilde{\Psi}_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 4.1. *The map Φ_ε satisfies*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = \Upsilon_{V(0),a} \quad \textit{uniformly in } y \in M.$$

Proof. Assume by contradiction that there exist $\delta_0 > 0$, a sequence $\{y_n\}_{n\in\mathbb{N}} \subset M$, and $\varepsilon_n \to 0$ such that

$$\left|J_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y_n)\right) - \Upsilon_{V(0),a}\right| \geq \delta_0, \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

It is easy to see that $\Phi_{\varepsilon_n}(y_n) = t_{\varepsilon_n,y_n} \star \tilde{\Psi}_{\varepsilon_n,y_n}$ and $\Phi_{\varepsilon_n}(y_n) \in \mathcal{P}_{\varepsilon_n,a}$. Arguing as in the proof of Lemma 3.10, we show that $\lim\limits_{n\to+\infty} t_{\varepsilon_n,y_n} = 1$. Indeed, if $\lim\limits_{n\to+\infty} t_{\varepsilon_n,y_n} = 0$, then by Lemma 3.4, there exists some $\theta > 0$ such that

$$0 < \theta < \|\nabla\Phi_{\varepsilon_n}(y_n)\|_2^2 = \|\nabla(t_{\varepsilon_n,y_n} \star \tilde{\Psi}_{\varepsilon_n,y_n})\|_2^2 = t^2_{\varepsilon_n,y_n}\|\nabla\tilde{\Psi}_{\varepsilon_n,y_n}\|_2^2 \to 0$$

for all $\Phi_{\varepsilon_n}(y_n) \in \mathcal{P}_{\varepsilon_n,a}$, which is impossible.

For any $\Phi_{\varepsilon_n}(y_n) \in \mathcal{P}_{\varepsilon_n,a}$, we have

$$\begin{aligned}
&t^2_{\varepsilon_n,y_n}\|\nabla\tilde{\Psi}_{\varepsilon_n,y_n}\|_2^2 - \int_{\mathbb{R}^N} W\left(\frac{\varepsilon_n x}{t_{\varepsilon_n,y_n}}\right)|\tilde{\Psi}_{\varepsilon_n,y_n}|^2\,\mathrm{d}x \\
&= \frac{N}{2t^{N+\alpha}_{\varepsilon_n,y_n}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})) \\
&\qquad\qquad \times\left(f(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n} - \frac{N+\alpha}{N}F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})\right)\,\mathrm{d}x.
\end{aligned}\tag{4.2}$$

By (f$_2$), we have

$$\begin{aligned}
&\|\nabla\tilde{\Psi}_{\varepsilon_n,y_n}\|_2^2 - \frac{1}{t^2_{\varepsilon_n,y_n}}\int_{\mathbb{R}^N} W\left(\frac{\varepsilon_n x}{t_{\varepsilon_n,y_n}}\right)|\tilde{\Psi}_{\varepsilon_n,y_n}|^2\,\mathrm{d}x \\
&= \frac{N}{2t^{N+\alpha+2}_{\varepsilon_n,y_n}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})) \\
&\qquad\qquad \times\left(f(t^{\frac{N}{2}}_{\varepsilon_n,y_n}u)t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n} - \frac{N+\alpha}{N}F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})\right)\,\mathrm{d}x \\
&\geq \frac{N}{2t^{N+\alpha+2}_{\varepsilon_n,y_n}}\int_{\mathbb{R}^N}(I_\alpha * F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})) \\
&\qquad\qquad \times\left(rF(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n}) - \frac{N+\alpha}{N}F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})\right)\,\mathrm{d}x \\
&= \frac{rN-N-\alpha}{2}\int_{\mathbb{R}^N}\left(I_\alpha * \frac{F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})}{(t^{\frac{N}{2}}_{\varepsilon_n,y_n})^{\frac{N+\alpha+2}{N}}}\right)\frac{F(t^{\frac{N}{2}}_{\varepsilon_n,y_n}\tilde{\Psi}_{\varepsilon_n,y_n})}{(t^{\frac{N}{2}}_{\varepsilon_n,y_n})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x.
\end{aligned}\tag{4.3}$$

Since

$$|\eta(|\varepsilon_n z|)w(z)|^2 \leq |w(z)|^2 \in L^1(\mathbb{R}^N)$$

and

$$\lim_{n\to+\infty}|\eta(|\varepsilon_n z|)w(z)|^2 = |w(z)|^2, \quad \text{a.e. in } \mathbb{R}^N,$$

Lebesgue's dominated convergence theorem yields

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\eta(|\varepsilon_n z|)w(z)|^2\,dz = \int_{\mathbb{R}^N}|w(z)|^2\,\mathrm{d}z = a^2,$$

which implies that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\Psi_{\varepsilon_n,y_n}(x)|^2\,\mathrm{d}x = a^2. \tag{4.4}$$

Similar to the proof of (4.4), again by Lebesgue's dominated convergence theorem, we can verify that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla\tilde{\Psi}_{\varepsilon_n,y_n}|^2\,\mathrm{d}x = \int_{\mathbb{R}^N}|\nabla w|^2\,\mathrm{d}x, \tag{4.5}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}W\left(\frac{\varepsilon_n x}{t_{\varepsilon_n,y_n}}\right)|\tilde{\Psi}_{\varepsilon_n,y_n}|^2\,\mathrm{d}x = 0 \tag{4.6}$$

and

$$
\begin{aligned}
&\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left(I_\alpha * \frac{F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}\tilde{\Psi}_{\varepsilon_n,y_n})}{(t_{\varepsilon_n,y_n}^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}\tilde{\Psi}_{\varepsilon_n,y_n})}{(t_{\varepsilon_n,y_n}^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x \\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^N}\left(I_\alpha * \frac{F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w)}{(t_{\varepsilon_n,y_n}^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w)}{(t_{\varepsilon_n,y_n}^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\,\mathrm{d}x.
\end{aligned}
\tag{4.7}
$$

If $\lim\limits_{n\to+\infty} t_{\varepsilon_n,y_n} = +\infty$, then by $r > \frac{N+\alpha+2}{N}$, together with (2.13), (4.3), (4.5), (4.6), and (4.7), we would get $\|\nabla w\|_2^2 \to +\infty$, a contradiction. Therefore there exist constants $t_0 > 0$ and $T_0 < +\infty$ such that

$$
0 < t_0 < t_{\varepsilon_n,y_n} < T_0.
$$

Passing to a subsequence if necessary, we may assume that $\lim\limits_{n\to+\infty} t_{\varepsilon_n,y_n} = T$. It remains to prove that $T = 1$. Since $w \in \mathcal{P}_a$, we have

$$
\int_{\mathbb{R}^N} |\nabla w|^2\,\mathrm{d}x = \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(w))\widetilde{F}(w)\,\mathrm{d}x. \tag{4.8}
$$

Because $t_0 < t_{\varepsilon_n,y_n} < T_0$ for large n, assumption (f$_2$), and the Hardy-Littlewood-Sobolev inequality (2.3), imply that there exists $C > 0$ such that

$$
\begin{aligned}
&\frac{N}{2} t_{\varepsilon_n,y_n}^{-(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w))\widetilde{F}(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w)\,\mathrm{d}x \\
&\le t_0^{-(N+\alpha)} C\left(\|T_0^{\frac{N}{2}}w\|_{\frac{2Nr}{N+\alpha}}^{2r} + \|T_0^{\frac{N}{2}}w\|_{\frac{2Np}{N+\alpha}}^{2p}\right)
\end{aligned}
$$

is uniformly bounded, and moreover

$$
\begin{aligned}
&t_{\varepsilon_n,y_n}^{-(N+\alpha)}(I_\alpha * F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w))\widetilde{F}(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w) \\
&\to T^{-(N+\alpha)}(I_\alpha * F(T^{\frac{N}{2}}w))\widetilde{F}(T^{\frac{N}{2}}w), \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
$$

Hence, by Lebesgue's dominated convergence theorem, we have

$$
\begin{aligned}
&\frac{N}{2}\lim_{n\to+\infty} t_{\varepsilon_n,y_n}^{-(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * F(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w))\widetilde{F}(t_{\varepsilon_n,y_n}^{\frac{N}{2}}w)\,\mathrm{d}x \\
&= \frac{N}{2}T^{-(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * F(T^{\frac{N}{2}}w))\widetilde{F}(T^{\frac{N}{2}}w)\,\mathrm{d}x.
\end{aligned}
\tag{4.9}
$$

By using (4.2), (4.5), (4.6) and (4.9), we obtain

$$
T^2\|\nabla w\|_2^2 = \frac{N}{2}T^{-(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * F(T^{\frac{N}{2}}w))\widetilde{F}(T^{\frac{N}{2}}w)\,\mathrm{d}x.
$$

Combining this with (4.8) gives

$$
0 = \int_{\mathbb{R}^N}\left(\left(I_\alpha * \frac{F(T^{\frac{N}{2}}w)}{(T^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{\widetilde{F}(T^{\frac{N}{2}}w)}{(T^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}} - (I_\alpha * F(w))\widetilde{F}(w)\right)\mathrm{d}x. \tag{4.10}
$$

From (f$_2$) and (f$_3$), we know that the functions

$$
\frac{F(t^{\frac{N}{2}}s)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}} \quad\text{and}\quad \frac{\widetilde{F}(t^{\frac{N}{2}}s)}{(t^{\frac{N}{2}})^{\frac{N+\alpha+2}{N}}}
$$

are nondecreasing on $(0,+\infty)$. Hence, (4.10) holds if and only if $T=1$. Consequently, we get

$$\lim_{n\to+\infty} J_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y_n)\right)=\lim_{n\to+\infty} J_{\varepsilon_n}\left(t_{\varepsilon_n,y_n}\star\tilde{\Psi}_{\varepsilon_n,y_n}\right)=J_{V(0)}(w)=\Upsilon_{V(0),a},$$

which contradicts (4.1). This completes the proof. □

For any $\delta>0$, let $R=R(\delta)>0$ be such that $M_\delta\subset B_R(0)$. Define the map $\chi\colon\mathbb{R}^N\to\mathbb{R}^N$ by

$$\chi(x)=\begin{cases} x, & \text{if } |x|<R,\\ \frac{Rx}{|x|}, & \text{if } |x|\geq R.\end{cases}$$

Finally, we introduce the map $\beta_\varepsilon\colon\mathcal{P}_{\varepsilon,a}\to\mathbb{R}^N$ given by

$$\beta_\varepsilon(u)=\frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)|u|^2\,\mathrm{d}x}{a^2}.$$

Lemma 4.2. *The map Φ_ε satisfies*

$$\lim_{\varepsilon\to0}\beta_\varepsilon\left(\Phi_\varepsilon(y)\right)=y,\quad \textit{uniformly in } y\in M.$$

Proof. Suppose by contradiction that there exist $\delta_0>0$, a sequence $\{y_n\}_{n\in\mathbb{N}}\subset M$, and $\varepsilon_n\to0$ such that

$$\left|\beta_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y_n)\right)-y_n\right|\geq\delta_0,\quad \text{for all } n\in\mathbb{N}. \tag{4.11}$$

By the definition of $\Phi_{\varepsilon_n}(y_n)$ and β_{ε_n}, we can see that

$$\begin{aligned}\beta_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y_n)\right)&=\int_{\mathbb{R}^N}\chi(\varepsilon_n x)\frac{|t_{\varepsilon_n,y_n}\star\tilde{\Psi}_{\varepsilon_n,y_n}|^2}{a^2}\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\chi\left(\frac{\varepsilon_n x}{t_{\varepsilon_n,y_n}}\right)\frac{|\Psi_{\varepsilon_n,y_n}(x)|^2}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\chi\left(\frac{\varepsilon_n z+y_n}{t_{\varepsilon_n,y_n}}\right)\frac{|\eta(|\varepsilon_n z|)w(z)|^2}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}\,\mathrm{d}z\\ &=y_n+\frac{\int_{\mathbb{R}^N}\left(\chi\left(\frac{\varepsilon_n z+y_n}{t_{\varepsilon_n,y_n}}\right)-y_n\right)|\eta(|\varepsilon_n z|)w(z)|^2\,\mathrm{d}z}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}+o_n(1).\end{aligned}$$

Since $\{y_n\}_{n\in\mathbb{N}}\subset M$, we have $\operatorname{dist}(y_n,M)=0$. By the definition of M_δ, this implies $y_n\in M_\delta$. Hence, $\{y_n\}_{n\in\mathbb{N}}\subset M_\delta\subset B_R(0)$, which shows that $|y_n|\leq R$. By (4.4), for n sufficiently large, we deduce that

$$\left|\frac{\left(\chi(\frac{\varepsilon_n z+y_n}{t_{\varepsilon_n,y_n}})-y_n\right)|\eta(|\varepsilon_n z|)w(z)|^2}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}\right|\leq\frac{4R}{a^2}|w(z)|^2\in L^1(\mathbb{R}^N)$$

and the right-hand side belongs to $L^1(\mathbb{R}^N)$. By Lemma 4.1, we know that

$$\lim_{n\to+\infty}t_{\varepsilon_n,y_n}=1.$$

Since

$$\lim_{n\to+\infty}\frac{\left(\chi(\frac{\varepsilon_n z+y_n}{t_{\varepsilon_n,y_n}})-y_n\right)|\eta(|\varepsilon_n z|)w(z)|^2}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}=0,\quad \text{a.e. in } \mathbb{R}^N,$$

Lebesgue's dominated convergence theorem yields

$$\lim_{n\to+\infty}\frac{\int_{\mathbb{R}^N}\left(\chi\left(\frac{\varepsilon_n z+y_n}{t_{\varepsilon_n,y_n}}\right)-y_n\right)|\eta(|\varepsilon_n z|)w(z)|^2\,\mathrm{d}z}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}=0.$$

Therefore,

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-y_n|\to 0,$$

which contradicts (4.11). ☐

Lemma 4.3. *Let $\varepsilon_n\to 0$ and let $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{P}_{\varepsilon_n,a}$ be such that $J_{\varepsilon_n}(u_n)\to\Upsilon_{V(0),a}$. Then there exists a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that $v_n(x)=u_n(x+\tilde{y}_n)$ admits a convergent subsequence in $H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n:=\varepsilon_n\tilde{y}_n\to y$ for some $y\in M$.*

Proof. Similar to the proof of Lemma 2.14, there exist $R_0,\kappa>0$ and $\{\tilde{y}_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that

$$\int_{B_{R_0}(\tilde{y}_n)}|u_n|^2\,\mathrm{d}x\geq\kappa,\quad\text{for all } n\in\mathbb{N}.$$

Let $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^N)$. By (V$_2$), for any $\theta>0$, there exists $R>0$ such that

$$|W(x)|\leq\theta,\quad\text{for all } |x|\geq R.$$

Thus, we have

$$\left|\int_{B^c_{R/\varepsilon_n}(0)}W(\varepsilon_n x)(u_n^2-u^2)\,\mathrm{d}x\right|\leq\theta\int_{B^c_{R/\varepsilon_n}(0)}|u_n^2-u^2|\,\mathrm{d}x.$$

By (V$_2$) and $u_n\to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we get

$$\int_{B_{R/\varepsilon_n}(0)}W(\varepsilon_n x)(u_n^2-u^2)\,\mathrm{d}x=o_n(1).$$

Since θ is arbitrary, we deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^N}W(\varepsilon_n x)(u_n^2-u^2)\,\mathrm{d}x\right|&\leq\left|\int_{B^c_{R/\varepsilon_n}(0)}W(\varepsilon_n x)(u_n^2-u^2)\,\mathrm{d}x\right|\\&\quad+\left|\int_{B_{R/\varepsilon_n}(0)}W(\varepsilon_n x)(u_n^2-u^2)\,\mathrm{d}x\right|\\&\leq\theta o_n(1)+o_n(1)=o_n(1).\end{aligned}\tag{4.12}$$

On the other hand, since $|W(\varepsilon_n x)u^2|\in L^1(\mathbb{R}^N)$ and $\lim_{n\to+\infty}|W(\varepsilon_n x)u^2|=0$ for a.a. $x\in\mathbb{R}^N$, Lebesgue's dominated convergence theorem yields

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}W(\varepsilon_n x)u^2\,\mathrm{d}x=0.\tag{4.13}$$

Taking (4.12) and (4.13) into account gives

$$\begin{aligned}&\int_{\mathbb{R}^N}W(\varepsilon_n x)u_n^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^N}W(\varepsilon_n x)(u_n^2-u^2)\,\mathrm{d}x+\int_{\mathbb{R}^N}W(\varepsilon_n x)u^2\,\mathrm{d}x=o_n(1).\end{aligned}\tag{4.14}$$

Set $v_n(x) = u_n(x + \tilde{y}_n)$. Since $u_n \in \mathcal{P}_{\varepsilon_n,a}$ and (4.14), we deduce that

$$\begin{aligned} 0 &= P_{\varepsilon_n}(u_n) \\ &= \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} W(\varepsilon_n x) u_n^2 \,\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u_n))\widetilde{F}(u_n)\,\mathrm{d}x \\ &= \|\nabla v_n\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(v_n))\widetilde{F}(v_n)\,\mathrm{d}x + o_n(1), \end{aligned}$$

which implies that

$$P(v_n) = o_n(1).$$

By Lemma 2.11, there exists a unique t_{v_n} such that $t_{v_n} \star v_n \in \mathcal{P}_a$ and $t_{v_n} = 1 + o_n(1)$. Hence,

$$\Upsilon_{V(0),a} \le J_{V(0)}(t_{v_n} \star v_n) = J_{V(0)}(v_n) + o_n(1) = \Upsilon_{V(0),a}.$$

Arguing as in the proof of Theorem 2.1, we know that $t_{v_n} \star v_n \to v$ in $H^1(\mathbb{R}^N)$ and $t_{v_n} = 1 + o_n(1)$. Hence,

$$\|t_{v_n} \star v_n\|_{H^1(\mathbb{R}^N)} \to \|v\|_{H^1(\mathbb{R}^N)}$$

and

$$\int_{\mathbb{R}^N} |\nabla(t_{v_n} \star v_n)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |t_{v_n} \star v_n|^2 \,\mathrm{d}x = t_{v_n}^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |v_n|^2 \,\mathrm{d}x.$$

As $n \to +\infty$, we have that

$$\|v_n\|_{H^1(\mathbb{R}^N)} \to \|v\|_{H^1(\mathbb{R}^N)},$$

which leads to $v_n \to v$ in $H^1(\mathbb{R}^N)$ and $v \in \mathcal{P}_a$.

Now, we prove that $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$. Suppose by contradiction, up to subsequence, that $|y_n| \to +\infty$. Then, by (V_1) and $|y_n| \to +\infty$, we have

$$\lim_{n\to+\infty} V(\varepsilon_n x + y_n) = 0$$

and therefore

$$\lim_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|v_n|^2 \,\mathrm{d}x = 0.$$

On the other hand, we also have

$$\lim_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x.$$

In addition, by (f_1), (f_2) and $v_n \to v$ in $H^1(\mathbb{R}^N)$, we also deduce that

$$\int_{\mathbb{R}^N} (I_\alpha * F(v_n))F(v_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)\,\mathrm{d}x.$$

Therefore, by $v \in \mathcal{P}_a$, we have

$$\Upsilon_{V(0),a} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)\,\mathrm{d}x \ge \Upsilon_{0,a},$$

which contradicts Corollary 2.13. Thus, $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Then, up to a subsequence, $y_n \to y$ in $\mathbb{R}^N$. Arguing as above, we get

$$\Upsilon_{V(0),a} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(y)|v|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)\,\mathrm{d}x \ge \Upsilon_{V(y),a}.$$

If $V(y) > V(0)$, by Corollary 2.13, we know that

$$\Upsilon_{V(0),a} < \Upsilon_{V(y),a},$$

which is a contradiction. Thus $V(y) \leq V(0)$. In virtue of $V(y) \geq V(0)$ for all $y \in \mathbb{R}^N$, we deduce that $V(y) = 0$. Hence, $y \in M$ and the proof is completed. □

Let $h\colon [0,+\infty) \to [0,+\infty)$ be a positive function satisfying $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Define

$$\tilde{\mathcal{P}}_{\varepsilon,a} = \left\{u \in \mathcal{P}_{\varepsilon,a} \colon J_\varepsilon(u) \leq \Upsilon_{V(0),a} + h(\varepsilon)\right\}.$$

By Lemma 4.1, for some $y \in M$, we deduce that $h(\varepsilon) = \left|J_\varepsilon\left(\Phi_\varepsilon(y)\right) - \Upsilon_{V(0),a}\right|$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $\Phi_\varepsilon(y) \in \tilde{\mathcal{P}}_{\varepsilon,a}$ for all $y \in M$.

Lemma 4.4. *For any $\delta > 0$, let $M_\delta = \left\{x \in \mathbb{R}^N \colon \operatorname{dist}(x, M) \leq \delta\right\}$. Then there holds*

$$\lim_{\varepsilon\to 0} \sup_{u\in\tilde{\mathcal{P}}_{\varepsilon,a}} \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| = 0.$$

Proof. Note that for any $\eta > 0$, there exist $\varepsilon_1 > 0$, such that

$$\sup_{u\in\tilde{\mathcal{P}}_{\varepsilon,a}} \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| < \eta, \quad \varepsilon \in (0, \varepsilon_1).$$

For η given above, there exists $u_\eta \in \tilde{\mathcal{P}}_{\varepsilon,a}$, such that

$$\inf_{z\in M_\delta} |\beta_\varepsilon(u_\eta) - z| > \sup_{u\in\tilde{\mathcal{P}}_{\varepsilon,a}} \inf_{z\in M_\delta} |\beta_\varepsilon(u_\eta) - z| - \frac{\eta}{2},$$

which yields that

$$\eta > \sup_{u\in\tilde{\mathcal{P}}_{\varepsilon,a}} \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| \geq \inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_\eta) - z| > \sup_{u\in\tilde{\mathcal{P}}_{\varepsilon,a}} \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| - \frac{\eta}{2}.$$

Therefore, if we set $\eta = \frac{1}{n}, \varepsilon_n = \frac{1}{n^2}$ and $u_\eta = u_n$, we have

$$\inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| = \sup_{u_n\in\tilde{\mathcal{P}}_{\varepsilon,a}} \inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| + o_n(1).$$

From the above equality, it suffices to find a sequence $\{y_n\}_{n\in\mathbb{N}} \subset M_\delta$ such that

$$\lim_{n\to+\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

As in (4.14), we have $\int_{\mathbb{R}^N} W(\varepsilon x)|u_n|^2 \,\mathrm{d}x = o_n(1)$, and thus $P(u_n) = o_n(1)$. By Lemma 2.11, there exists $t_{u_n} = 1 + o_n(1)$ such that $P(t_{u_n} \star u_n) = 0$, and so

$$\Upsilon_{V(0),a} \leq J_{V(0)}(t_{u_n} \star u_n) = J_{V(0)}(u_n) + o_n(1).$$

Since $u_n \in \tilde{\mathcal{P}}_{\varepsilon_n,a}$, we have

$$\Upsilon_{V(0),a} \leq J_{V(0)}(u_n) + o_n(1) \leq J_{\varepsilon_n}(u_n) \leq \Upsilon_{V(0),a} + h(\varepsilon_n)$$

for all $n \in \mathbb{N}$, which implies that $J_{\varepsilon_n}(u_n) \to \Upsilon_{V(0),a}$. By virtue of Lemma 4.3, there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n\tilde{y}_n \to y$ for some $y \in M$ and $v_n(x) = u_n(x+\tilde{y}_n)$ is strongly convergent to some $v \in H^1(\mathbb{R}^N)$ with $v \neq 0$. Then, $\{y_n\}_{n\in\mathbb{N}} \subset M_\delta$ for n sufficiently large and

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{1}{a^2}\int_{\mathbb{R}^N} (\chi(\varepsilon_n z + y_n) - y_n)|v_n|^2 \,\mathrm{d}z,$$

Then, using Lebesgue's dominated convergence theorem, this leads to

$$\beta_{\varepsilon_n}(u_n) - y_n = \frac{\int_{\mathbb{R}^N} (\chi(\varepsilon_n z + y_n) - y_n)|v_n|^2 \, \mathrm{d}z}{a^2} \to 0.$$

The proof is completed. □

Now we can give the proof of our main result.

Proof of Theorem 1.1. Fix $\varepsilon \in (0, \varepsilon_0)$. Consider the inclusion map $\mathrm{id} \colon M \to M_\delta$. By Lemmas 4.1, 4.2 and 4.4, we can argue as in Section 6 of Cingolani–Lazzo [9] to conclude that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to id.

Let $M, M_\delta, \tilde{\mathcal{P}}_{\varepsilon,a}$ be closed sets with $M \subset M_\delta$. Note that $\mathrm{id} \colon M \to M_\delta$, $\beta_\varepsilon \colon M \to \tilde{\mathcal{P}}_{\varepsilon,a}$ and $\Phi_\varepsilon \colon \tilde{\mathcal{P}}_{\varepsilon,a} \to M_\delta$ are continuous maps. Define $H \colon M \times [0,1] \to M_\delta$ by

$$H(y,t) = t\beta_\varepsilon \circ \Phi_\varepsilon(y) + (1-t)\,\mathrm{id}(y)$$

Then $H(y,0) = \mathrm{id}(y)$ and $H(y,1) = \beta_\varepsilon \circ \Phi_\varepsilon(y)$, so $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion map id. Hence

$$\mathrm{cat}(\tilde{\mathcal{P}}_{\varepsilon,a}) \geq \mathrm{cat}_{M_\delta}(M).$$

On the other hand, by arguments as in Lemma 3.9, we also have that J_ε is bounded from below on $\mathcal{P}_{\varepsilon,a}$ and by Theorem 3.13, we know that the functional J_ε satisfies the $(\mathrm{P\ S})_c$-condition at level

$$c \in (\Upsilon_{V(0),a}, \Upsilon_{V(0),a} + h(\varepsilon)).$$

Therefore, by the Ljusternik-Schnirelmann category (see the books by Ghoussoub [15] and Willem [40]), J_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $\mathcal{P}_{\varepsilon,a}$. This completes the proof. □

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